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EXTREMAL ALGEBRAIC CONNECTIVITIES OF CERTAIN CATERPILLAR CLASSES AND SYMMETRIC CATERPILLARS*

OSCAR ROJO†, LUIS MEDINA†, NAIR M. M. DE ABREU‡, AND CLAUDIA JUSTEL§

Abstract. A caterpillar is a tree in which the removal of all pendant vertices makes it a path. Let $d \geq 3$ and $n \geq 6$ be given. Let P_{d-1} be the path of $d-1$ vertices and S_p be the star of $p+1$ vertices. Let $\mathbf{p} = [p_1, p_2, ..., p_{d-1}]$ such that $p_1 \geq 1, p_2 \geq 1, ..., p_{d-1} \geq 1$. Let $C(\mathbf{p})$ be the caterpillar obtained from the stars $S_{p_1}, S_{p_2}, ..., S_{p_{d-1}}$ and the path P_{d-1} by identifying the root of S_{p_i} with the $i-$vertex of P_{d-1}. Let $n > 2(d-1)$ be given. Let

$$\mathcal{C} = \{C(\mathbf{p}) : p_1 + p_2 + ... + p_{d-1} = n - d + 1\}$$

and

$$\mathcal{S} = \{C(\mathbf{p}) \in \mathcal{C} : p_j = p_{d-j},\ j = 1, 2, \cdots, \lfloor \frac{d-1}{2} \rfloor\}.$$

In this paper, the caterpillars in $\mathcal{C}$ and in $\mathcal{S}$ having the maximum and the minimum algebraic connectivity are found. Moreover, the algebraic connectivity of a caterpillar in $\mathcal{S}$ as the smallest eigenvalue of a 2×2 - block tridiagonal matrix of order $2s \times 2s$ if $d = 2s+1$ or $d = 2s+2$ is characterized.

Key words. Laplacian matrix, Algebraic connectivity, Caterpillar, Bottleneck matrices, Perron branches, Characteristic vertices.

AMS subject classifications. 05C50, 15A48, 05C05.

1. Introduction.

Let $G = (V, E)$ be a simple undirected graph on n vertices. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees. It is known that $L(G)$ is a positive semidefinite matrix and $(0, \mathbf{e})$ is an eigenpair of $L(G)$ where $\mathbf{e}$ is the all ones vector. In [12], some of the many results known for Laplacian matrices are given. Fiedler [5] proves that G is a connected graph if and only if the second smallest Laplacian eigenvalue is positive. This eigenvalue is called the algebraic connectivity of G, it is denoted by $a(G)$ and any of its corresponding eigenvectors

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†Department of Mathematics, Universidad Católica del Norte, Antofagasta, Chile. Work supported by Project Fondecyt 1070537, Chile (orojo@ucn.cl). Part of this research was conducted while O. Rojo was a visitor at the Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile.

‡Universidad Federal de Rio de Janeiro, Rio de Janeiro, Brazil. Work supported by CNPq 300563/94-9, Brazil.

§Instituto Militar de Engenharia, Rio de Janeiro, Brazil.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

is called a Fiedler vector. In [3], a survey of old and new results on the algebraic connectivity of graphs is given.

We recall that a tree is a connected acyclic graph. Let us denote by P_n a path of n vertices and by S_p a star of $p+1$ vertices. Let n and d be fixed. Let $T(a,b,d)$ be the tree with n vertices and diameter d obtained from the path P_{d-1} and the stars S_a and S_b by identifying the pendant vertices in P_{d-1} with the centers of the stars. We may consider $a \leq b$. Observe that $b = n-a-(d-1)$. Then $1 \leq a \leq \frac{1}{2}(n-d+1)$.

The problem of ordering trees by algebraic connectivity is an active area of research. The next theorem due to Grone and Merris solves the problem for trees of order n and diameter $d = 3$.

THEOREM 1.1. [7] *The algebraic connectivity of* $T(a,b,3)$, $b = n-a-2$, *is the unique Laplacian eigenvalue less than* 1 *and it is a strictly decreasing function for* $1 \leq a \leq \frac{1}{2}(n-2)$.

Important contributions to the problem for trees of order n and diameter $d = 4$ are due to X-D Zhang [19, 2007]. Shao et al. [15, 2008] determine the first four trees of order $n \geq 9$ with the smallest algebraic connectivity. Yuan et al. [17, 2007] introduce six classes of trees with n vertices and determine the ordering of those trees by this spectral invariant. Zhang and Liu [18, 2008] find the largest twelve values of algebraic connectivity of trees in a set of trees on $2k+1$ vertices with nearly perfect matching. In [4], Fallat and Kirkland find the tree that minimizes and the tree that maximizes the algebraic connectivity among the trees on n vertices and diameter d :

THEOREM 1.2. [4, Theorem 3.2] *Among all trees on* n *vertices with fixed diameter* d, *the minimum algebraic connectivity is attained by the tree*

$$T\left(\left\lfloor \frac{n-d+1}{2} \right\rfloor, \left\lceil \frac{n-d+1}{2} \right\rceil, d\right).$$

THEOREM 1.3. [4, Theorem 3.8] *Among all trees on* n *vertices with fixed diameter* d, *the maximum algebraic connectivity is attained by the tree* $P_{d,n-d}$ *obtained from the path* P_d *on vertices* $1,2,...,d$ *and the star* S_{n-d} *identifying the root of* S_{n-d} *with the vertex* $\left\lceil \frac{d+1}{2} \right\rceil$ *of the path* P_d.

A caterpillar is a tree in which the removal of all pendant vertices makes it a path. Let $n > 2(d-1)$ be given with $d \geq 3$. Let $\mathbf{p} = [p_1, p_2, ..., p_{d-1}]$. with $p_1 \geq 1, p_2 \geq 1, ..., p_{d-1} \geq 1$. Let $C(\mathbf{p})$ be the caterpillar obtained from the stars $S_{p_1}, S_{p_2}, ..., S_{p_{d-1}}$ and the path P_{d-1} by identifying the root of S_{p_i} with the $i-$vertex of P_{d-1}.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Let

$$\mathcal{C} = \{C(\mathbf{p}) : p_1 + p_2 + ... + p_{d-1} = n - d + 1\}$$

and

$$\mathcal{S} = \left\{ C(\mathbf{p}) \in \mathcal{C} : p_j = p_{d-j},\ j = 1, 2, ..., \left\lfloor \frac{d-1}{2} \right\rfloor \right\}.$$

Clearly any caterpillar in $\mathcal{C}$ is a caterpillar on n vertices with diameter d.

In [2], the authors prove that the caterpillars are the trees having minimum algebraic connectivity among all trees with a given degree sequence.

In this paper, we find the caterpillars in $\mathcal{C}$ and in $\mathcal{S}$ having the maximum and the minimum algebraic connectivity. Moreover, we characterize the algebraic connectivity of a caterpillar in $\mathcal{S}$ as the smallest eigenvalue of a 2×2 - block tridiagonal matrix of order $2s \times 2s$ if $d = 2s + 1$ or $d = 2s + 2$.

2. Basic tools. In this Section, we present some results on the algebraic connectivity of trees which are the basic tools of this paper.

LEMMA 2.1. [8, Corollary 4.2] *Let v be a pendant vertex of the graph $\widetilde{G}$. Let G be the graph obtained from $\widetilde{G}$ by removing v and its edge. Then the eigenvalues of $L(G)$ interlace the eigenvalues of $L\left(\widetilde{G}\right)$.*

From Lemma 2.1, it follows

COROLLARY 2.2. *Let T be a subtree of the tree $\widetilde{T}$. Then*

$$a\left(\widetilde{T}\right) \leq a(T).$$

For the rest of this Section, T is a tree on vertices $1, 2, ..., n$ and

$$\mathbf{f} = \begin{bmatrix} f_1 & f_2 & \cdots & \cdots & f_{n-1} & f_n \end{bmatrix}^T.$$

THEOREM 2.3. [6] *If $\mathbf{f}$ is a Fiedler vector then exactly one of the following two cases occurs:*

(A) No entry of $\mathbf{f}$ is 0. In this case, there is a unique pair of vertices i and j such that i and j are adjacent with $f_i > 0$ and $f_j < 0$.

(B) Some entry of $\mathbf{f}$ is 0. In this case, the subgraph induced by the vertices corresponding to zeros in $\mathbf{f}$ is a connected subgraph. Moreover, there is a unique vertex k such that $f_k = 0$ and k is adjacent to a vertex i with $f_i \neq 0$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

ELA

In [11], Merris proves that the occurrence of (A) or (B) does not depend on the Fiedler vector.

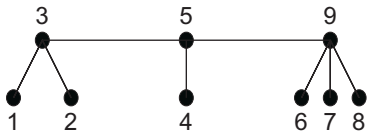
DEFINITION 2.4. 1. T is said *Type II* if (A) holds and the vertices i and j are called the characteristic vertices of T. The edge defined by i and j is called the characteristic edge of T.

2. The tree T is said *Type I* if (B) holds and the vertex k is called the characteristic vertex of T.

3. Let m be a vertex of the tree T. A branch of T at m is any of the connected component of $T - m$ obtained from T removing the vertex m and the edges incident with m.

LEMMA 2.5. [10] *Let L_k be the principal submatrix of the Laplacian matrix $L(T)$ obtained by deleting the $k-$row and $k-$column from $L(T)$. Then the $(i,j)-$entry of L_k^{-1} is equal to the number of edges of T which are on both the path from vertex i to vertex k and the path from vertex j to vertex k.*

EXAMPLE 2.6. Let T be the caterpillar below



Then

$$L_5^{-1} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly the (i,j) entry of L_k^{-1} is positive if and only if the vertices i and j are in the same branch of T at the vertex k. Then there is a labeling of the vertices of T such that L_k^{-1} is similar to a block diagonal matrix in which the number of diagonal blocks is the degree of the vertex k. Moreover, each diagonal block is a positive matrix corresponding to a unique branch at k. This matrix is called the *bottleneck matrix for that branch at k*. In Example 2.6, the bottleneck matrices for the branches at the

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Volume 20, pp. 136-157, February 2010

ELA

vertex 5 are the matrices

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ [1], \ \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

From the Perron-Frobenius Theory for nonnegative matrices, it follows that each bottleneck matrix has a simple eigenvalue equal to its spectral radius, its Perron root.

DEFINITION 2.7. The Perron root of the bottleneck matrix of a branch is called the *Perron root of the branch*. A branch is called a *Perron branch* if its Perron root is equal to the spectral radius of L_k^{-1}.

The notion of Perron branches was first introduced by Kirkland, Neumann and Shader [10].

Let $\rho(A)$ be the spectral radius of a matrix A and let $\mathbf{e}$ be the all ones vector of the appropriate order.

THEOREM 2.8. [10] *Let i and j be adjacent vertices of T. Then T is Type II tree with characteristic vertices i and j if and only if there exists $0 < \gamma < 1$ such that*

$$\rho\left(M - \gamma \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M} - (1-\gamma)\,\mathbf{e}\mathbf{e}^T\right)$$

where M is the bottleneck matrix of the branch at j containing i and $\widehat{M}$ is the bottleneck matrix of the branch at i containing j. Moreover, if this condition holds then

$$\frac{1}{a(T)} = \rho\left(M - \gamma \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M} - (1-\gamma)\,\mathbf{e}\mathbf{e}^T\right)$$

and any eigenvector of $L(T)$ corresponding to $a(T)$ can be permuted such that it has the form $\begin{bmatrix} -\mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$ where $\mathbf{f}_1$ is a Perron vector for $M - \gamma\mathbf{e}\mathbf{e}^T$ and $\mathbf{f}_2$ is a Perron vector for $\widehat{M} - (1-\gamma)\,\mathbf{e}\mathbf{e}^T$ and $\mathbf{e}^T\mathbf{f}_1 = \mathbf{e}^T\mathbf{f}_2$.

COROLLARY 2.9. *T is a Type II tree with characteristic vertices i and j if and only if (1) the vertices i and j are adjacent, (2) the branch at i containing the vertex j is the unique Perron branch at i, and (3) the branch at j containing the vertex i is the unique Perron branch at j.*

THEOREM 2.10. [10] *T is a Type I tree with characteristic vertex k if and only if there are two or more Perron branches of T at k. In this case,*

$$a(T) = \frac{1}{\rho\left(L_k^{-1}\right)}$$

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ELA

and if $\mathbf{f}$ *is an eigenvector corresponding to* $a(T)$ *then* $\mathbf{f}$ *can be permuted such that each of the resulting nonzero subvectors is a Perron vector for the bottleneck matrix of a Perron branch at* k.

COROLLARY 2.11. *T is a Type I tree if and only if there is exactly one vertex at which there are two or more Perron branches. T is a Type II tree if and only if at each vertex there is unique Perron branch.*

3. Extremal algebraic connectivities.

3.1. Characteristic vertices of caterpillars and their Perron branches. Let $v_1, v_2, ..., v_{d-1}$ be the vertices of the path P_{d-1}. We observe that for any tree, no pendant vertex is a characteristic vertex. From this fact and Theorem 2.8, it follows

LEMMA 3.1. *If $C_t \in \mathcal{C}$ is a Type II tree and v_i and v_{i+1} are its characteristic vertices then*

$$\frac{1}{a(C_t)} = \rho\left(M_t - \gamma_t \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_t - (1-\gamma_t)\mathbf{e}\mathbf{e}^T\right) \tag{3.1}$$

where M_t is the bottleneck matrix of the Perron branch B_t at v_{i+1} containing the vertex v_i and $\widehat{M}_t$ is the bottleneck matrix of the Perron branch $\widehat{B}_t$ at v_i containing the vertex v_{i+1}. From Theorem 2.10, it follows

LEMMA 3.2. *If $C_t \in \mathcal{C}$ is a Type I tree and v_i is its characteristic vertex then*

$$\frac{1}{a(C_t)} = \rho(M_t) = \rho\left(\widehat{M}_t\right) \tag{3.2}$$

where M_t and $\widehat{M}_t$ are the bottleneck matrices of B_t and $\widehat{B}_t$, respectively, in which B_t is the branch containing the vertex v_1 and $\widehat{B}_t$ is the branch containing the vertex v_{d-1}.

3.2. Maximum algebraic connectivity. We first consider the case of d even. This case is already studied in [1, Theorem 9]. Here, we give a different proof.

THEOREM 3.3. *If $d = 2s+2$ then among all trees in $\mathcal{C}$ the maximum algebraic connectivity is attained by the caterpillar $C(\mathbf{q})$ where $q_i = 1$ for all $i \neq s+1$ and $q_{s+1} = n - 2d + 3$.*

Proof. Observe that $C(\mathbf{q}) \in \mathcal{C}$. Let $\mathbf{e}$ be the all ones vector of order $d-1$. Let $C_1 = C(\mathbf{p}) \in \mathcal{C}$, $C_2 = C(\mathbf{e})$ and $C_3 = C(\mathbf{q})$. Clearly C_2 is a subtree of C_1. Applying Corollary 2.2, $a(C_1) \leq a(C_2)$. Moreover, it is clear that C_2 and C_3 are both trees of Type I at the same characteristic vertex v_s having isomorphic Perron branches. Hence, applying Theorem 2.10, $a(C_2) = a(C_3)$. Therefore $a(C_1) \leq a(C_3)$. □

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

For the case of odd d, to get the caterpillar in $\mathcal{C}$ with the maximum algebraic connectivity is not easy as in Theorem 3.3.

At this point, it is convenient to recall that if A is an irreducible nonnegative (in particular, positive) matrix then $\rho(A)$ strictly increases when any entry of A strictly increases [16, Theorem 2.1.].

LEMMA 3.4. *Let* $d = 2s+1$. *Let*

$$C = C(p_1, ..., p_s, p_{s+1}, ..., p_{2s}) \in \mathcal{C}$$
$$\widetilde{p}_s = \sum_{l=1}^{s} p_l - s + 1$$
$$\widetilde{p}_{s+1} = \sum_{l=s+1}^{2s} p_l - s + 1.$$

Then

$$a(C) \leq a\left(\widetilde{C}\right) \tag{3.3}$$

where

$$\widetilde{C} = C(1, ..., 1, \widetilde{p}_s, \widetilde{p}_{s+1}, 1, ..., 1).$$

The equality holds if and only if $C = \widetilde{C}$.

Proof. Let $C_1 = C$, $C_3 = \widetilde{C}$ and

$$C_2 = C(1, ..., 1, \widetilde{p}_s, p_{s+1}, ..., p_{2s}).$$

Clearly $C_2, C_3 \in \mathcal{C}$ and $s = \frac{1}{2}(d-1)$. Let $v_1, ..., v_s, v_{s+1}, ..., v_{2s}$ be the vertices of the path P_{2s}. We first prove that

$$a(C_1) \leq a(C_2) \tag{3.4}$$

with equality if and only if $C_1 = C_2$.

If $C_1 = C_2$ then $a(C_1) = a(C_2)$. Suppose $C_1 \neq C_2$. Then $p_t > 1$ for some $1 \leq t \leq s-1$. Four cases can occur. In each case, the vertices of C_1 and C_2 are labeled appropriately to compare the entries of the corresponding bottleneck matrices in (3.1) or (3.2).

Case C_1 *and* C_2 *are both of Type I tree*. Let w_1 and w_2 be the characteristic vertices of C_1 and C_2, respectively. Then

$$\frac{1}{a(C_1)} = \rho(M_1) = \rho\left(\widehat{M_1}\right) \text{ and } \frac{1}{a(C_2)} = \rho(M_2) = \rho\left(\widehat{M_2}\right).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

ELA

We have $w_1 = v_l$ for some l and $w_2 = v_k$ for some $k \geq s+1$. If $l < k$, $\widehat{B}_2$ is isomorphic to a proper subtree of $\widehat{B}_1$ and then $\widehat{M}_1(p,q) - \widehat{M}_2(p,q) > 1$ for each (p,q). Hence

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1\right) > \rho\left(\widehat{M}_2\right) = \frac{1}{a(C_2)}.$$

If $l \geq k$, since $p_t > 1$ for some $1 \leq t \leq s-1$, we have $M_1(p,q) - M_2(p,q) \geq 0$ for each (p,q) with strict inequality in at least one entry. Hence

$$\frac{1}{a(C_1)} = \rho(M_1) > \rho(M_2) = \frac{1}{a(C_2)}.$$

Case C_1 is a Type I tree and C_2 is a Type II tree. Let v_l be the characteristic vertex of C_1. Let v_i and v_{i+1} be the characteristic vertices of C_2. There is $0 < \gamma_2 < 1$ such that

$$\frac{1}{a(C_2)} = \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right).$$

Clearly, $i \geq s$. If $i = l$ then

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1\right) = \rho\left(\widehat{M}_2\right) > \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

If $i > l$ then $\widehat{B}_2$ is isomorphic to a proper subtree of $\widehat{B}_1$, thus $\widehat{M}_1(p,q) - \widehat{M}_2(p,q) > 1$ for each (p,q) and hence

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1\right) > \rho\left(\widehat{M}_2\right) > \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

If $i < l$, since $p_t > 1$ for some $1 \leq t \leq s-1$, we have $M_1(p,q) - M_2(p,q) \geq 0$ for each (p,q) with strict inequality in at least one entry. Hence

$$\frac{1}{a(C_1)} = \rho(M_1) > \rho(M_2) > \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

Case C_1 is a Type II tree and C_2 is a Type I tree. Let v_k be the characteristic vertex of C_2. Then $k \geq s+1$. Let v_i and v_{i+1} be the characteristic vertices of C_1. There is $0 < \gamma_1 < 1$ such that

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right).$$

If $i < k$ then the branch $\widehat{B}_1$ contains the vertex v_k and an isomorphic copy of $\widehat{B}_2$, thus $\widehat{M}_1(p,q) - \widehat{M}_2(p,q) \geq 1$ for each (p,q). Hence

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right) > \rho\left(\widehat{M}_2\right) = \frac{1}{a(C_2)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

If $i \geq k$ then $M_1(p,q) - M_2(p,q) \geq 1$ for each (p,q). Hence

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) > \rho(M_2) = \frac{1}{a(C_2)}.$$

Case C_1 and C_2 are both of Type II. Let v_j and v_{j+1} the characteristic vertices of C_1. Let v_i and v_{i+1} be the characteristic vertices of C_2. There are $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$ such that

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right)$$
$$\frac{1}{a(C_2)} = \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right).$$

As before, $i \geq s$. If $j < i$, $\widehat{M}_1(p,q) - \widehat{M}_2(p,p) \geq 1$ for each (p,q). Then

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right) > \rho\left(\widehat{M}_2\right)$$
$$> \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

If $j > i$, $M_1(p,q) - M_2(p,p) \geq 1$ for each (p,q). Then

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) > \rho(M_2) > \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

If $j = i$, then $\widehat{M}_1 = \widehat{M}_2$ and $M_1 \geq M_2$. Moreover, since $p_t > 1$ for some $1 \leq t \leq s-1$, we have $M_1(p,q) > M_2(p,q)$ for some (p,q). If $\gamma_1 = \gamma_2$ then

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}. \tag{3.5}$$

Moreover

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) > \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}$$

which contradicts (3.5). Therefore $\gamma_1 \neq \gamma_2$. If $\gamma_1 < \gamma_2$ then

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right) < \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}. \tag{3.6}$$

Moreover

$$\frac{1}{a(C_1)} = \rho\left(M_1 - \gamma_1 \mathbf{e}\mathbf{e}^T\right) > \rho\left(M_2 - \gamma_2 \mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}$$

which contradicts (3.6). Consequently, $\gamma_1 > \gamma_2$ and hence

$$\frac{1}{a(C_1)} = \rho\left(\widehat{M}_1 - (1-\gamma_1)\mathbf{e}\mathbf{e}^T\right) > \rho\left(\widehat{M}_2 - (1-\gamma_2)\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_2)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

ELA

We have proved that $a(C_1) \leq a(C_2)$ with equality if and only if $C_1 = C_2$.

Observe that C_2 is isomorphic to

$$\widetilde{C}_1 = C(p_{2s}, ..., p_{s+1}, \widetilde{p}_s, 1, ..., 1)$$

and that C_3 is isomorphic to

$$\widetilde{C}_2 = C(1, ..., 1, \widetilde{p}_{s+1}, \widetilde{p}_s, 1, ..., 1).$$

We apply (3.4) to conclude $a\left(\widetilde{C}_1\right) \leq a\left(\widetilde{C}_2\right)$ with equality if and only if $\widetilde{C}_1 = \widetilde{C}_2$. Therefore $a(C_2) \leq a(C_3)$ with equality if and only if $C_2 = C_3$. By transitivity, $a(C) = a(C_1) \leq a(C_3) = a\left(\widetilde{C}\right)$ with equality if and only if $C = \widetilde{C}$. □

For $d = 2s + 1$, let

$$\mathcal{M} = \{C(1, ..., 1, p_s, p_{s+1}, 1, ..., 1) \in \mathcal{C} : p_s \leq p_{s+1}\}$$

Let $p_s = a$ and $p_{s+1} = b$. Thus a caterpillar in $\mathcal{M}$ is of the form $C(1, ..., 1, a, b, 1, ..., 1)$ in which $a + b = n - 2d + 4$ and $a \leq b$. For brevity, we write $C(a, b)$ instead of $C(1, ..., 1, a, b, 1, ..., 1)$.

In order to obtain the caterpillar having the maximum algebraic connectivity in

$$\mathcal{C} = \{C(p_1, ..., p_s, p_{s+1}, ..., p_{2s}) : p_i \geq 1 \text{ for all } i\},$$

we recall the following theorem.

THEOREM 3.5. [14, Theorem 5] *Let $d = 2s + 1$. The algebraic connectivity of $C(a, b) = C(1, ..., 1, a, b, 1, ..., 1) \in \mathcal{M}$ is a strictly decreasing function for $1 \leq a \leq \frac{1}{2}(n - 2d + 4)$.*

We are ready to give the caterpillar with the maximum algebraic connectivity among the caterpillars in $\mathcal{C}$ whenever d is odd.

THEOREM 3.6. *If $d = 2s + 1$ then among all trees in $\mathcal{C}$ the maximum algebraic connectivity is attained by the caterpillar $C(1, ..., 1, 1, p_{s+1}, 1, ..., 1)$ where $p_{s+1} = n - 2d + 3$.*

Proof. Let

$$C_1 = C(p_1, ..., p_s, p_{s+1}, ..., p_{2s}) \in \mathcal{C}.$$

We define

$$C_3 = C(1, ..., 1, \widetilde{p}_s, \widetilde{p}_{s+1}, 1, ..., 1)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

where $\widetilde{p}_s = \sum_{l=1}^{s} p_l - s + 1$, $\widetilde{p}_{s+1} = \sum_{l=s+1}^{2s} p_l - s + 1$, and

$$C_4 = C\left(1, 1, ..., 1, n - 2d + 3, 1, ..., 1\right).$$

Clearly, $C_3 \in \mathcal{M}$ and $C_4 \in \mathcal{M}$. From (3.3), $a\left(C_1\right) \leq a\left(C_3\right)$. If $\widetilde{p}_s = 1$ then $\widetilde{p}_{s+1} = n - 2d + 3$ and thus $a\left(C_1\right) \leq a\left(C_3\right) = a\left(C_4\right)$. If $\widetilde{p}_s > 1$, from Theorem 3.5, we get $a\left(C_3\right) < a\left(C_4\right)$ and thus $a\left(C_1\right) < a\left(C_4\right)$. □

3.3. Minimum algebraic connectivity. Let

$$\mathcal{E} = \left\{C\left(a, 1, ..., 1, b\right) \in \mathcal{C} : a \leq b,\ a + b = n - 2d + 4\right\}.$$

We recall the following theorem.

THEOREM 3.7. [14, Theorem 4] *The algebraic connectivity of* $C\left(a, 1, ..., 1, b\right) \in \mathcal{E}$ *is a strictly decreasing function for* $1 \leq a \leq \frac{1}{2}\left(n - 2d + 4\right)$.

THEOREM 3.8. *Among all trees in* $\mathcal{C}$ *the minimum algebraic connectivity is attained by the caterpillar* $C\left(x, 1, 1, \ldots, 1, 1, y\right)$ *where* $x = \left\lfloor \frac{n-2d+4}{2} \right\rfloor$ *and* $y = \left\lceil \frac{n-2d+4}{2} \right\rceil$.

Proof. Let $C_1 = C\left(\mathbf{p}\right) \in \mathcal{C}$. As in the proof of Lemma 3.4, four cases occur and, in each case, the vertices of the caterpillars are labeled appropriately to compare the entries of the corresponding bottleneck matrices in (3.1) or (3.2).

Case C_1 *is a Type I tree.* Let v_l be the characteristic vertex of C_1. We have

$$\frac{1}{a\left(C_1\right)} = \rho\left(M_1\right) = \rho\left(\widehat{M}_1\right).$$

We define $C_0 = C\left(\widetilde{p}_1, 1, 1, \ldots, 1, 1, \widetilde{p}_{d-1}\right) \in \mathcal{C}$ where

$$\widetilde{p}_1 = \sum_{j=1}^{l} p_j - (l - 1) \text{ and } \widetilde{p}_{d-1} = \sum_{j=l+1}^{d-1} p_j - (d - l - 2).$$

We want to prove that $a\left(C_0\right) \leq a\left(C_1\right)$. We may assume that $C_0 \neq C_1$. Then $p_t > 1$ for some $2 \leq t \leq d - 2$. Suppose that C_0 *is a Type I tree with characteristic vertex* v_k. We have

$$\frac{1}{a\left(C_0\right)} = \rho\left(M_0\right) = \rho\left(\widehat{M}_0\right).$$

If $k = l$ then $M_0\left(p, q\right) - M_1\left(p, q\right) \geq 0$ for each (p, q) with strict inequality in at least one entry whenever $2 \leq t \leq l$ or $\widehat{M}_0\left(p, q\right) - \widehat{M}_1\left(p, q\right) \geq 0$ for each (p, q) with strict inequality in at least one entry whenever $l + 1 \leq t \leq d - 2$. Hence

$$\frac{1}{a\left(C_0\right)} = \rho\left(M_0\right) > \rho\left(M_1\right) = \frac{1}{a\left(C_1\right)}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 136-157, February 2010

ELA

or

$$\frac{1}{a\left(C_0\right)}=\rho\left(\widehat{M_0}\right)>\rho\left(\widehat{M_1}\right)=\frac{1}{a\left(C_1\right)}.$$

If $k>l$ then $M_0\left(p,q\right)-M_1\left(p,q\right)>0$ for each $\left(p,q\right)$. Hence

$$\frac{1}{a\left(C_0\right)}=\rho\left(M_0\right)>\rho\left(M_1\right)=\frac{1}{a\left(C_1\right)}.$$

If $k<l$ then $\widehat{M_0}\left(p,q\right)-\widehat{M_1}\left(p,q\right)>0$ for each $\left(p,q\right)$. Hence

$$\frac{1}{a\left(C_0\right)}=\rho\left(\widehat{M_0}\right)>\rho\left(\widehat{M_1}\right)=\frac{1}{a\left(C_1\right)}.$$

Suppose now that C_0 is a Type II tree with characteristic vertices v_r and v_{r+1}. There is $0<\gamma_0<1$ such that

$$\frac{1}{a\left(C_0\right)}=\rho\left(M_0-\gamma_0\mathbf{e}\mathbf{e}^T\right)=\rho\left(\widehat{M_0}-\left(1-\gamma_0\right)\mathbf{e}\mathbf{e}^T\right).$$

If $r\geq l$ then $M_0\left(p,q\right)-M_1\left(p,q\right)\geq 1$ for each $\left(p,q\right)$ and thus

$$\frac{1}{a\left(C_0\right)}=\rho\left(M_0-\gamma_0\mathbf{e}\mathbf{e}^T\right)>\rho\left(M_1\right)=\frac{1}{a\left(C_1\right)}.$$

If $r<l$ then $\widehat{M_0}\left(p,q\right)-\widehat{M_1}\left(p,q\right)\geq 1$ for each $\left(p,q\right)$ and thus

$$\frac{1}{a\left(C_0\right)}=\rho\left(\widehat{M_0}-\left(1-\gamma_0\right)\mathbf{e}\mathbf{e}^T\right)>\rho\left(\widehat{M_1}\right)=\frac{1}{a\left(C_1\right)}.$$

We have proved that if C_1 is a Type I tree and $C_1\neq C_0$ then $a\left(C_0\right)<a\left(C_1\right)$.

Case C_1 is a Type II tree. Let v_i and v_{i+1} be the characteristic vertices of C_1. There is $0<\gamma_1<1$ such that

$$\frac{1}{a\left(C_1\right)}=\rho\left(M_1-\gamma_1\mathbf{e}\mathbf{e}^T\right)=\rho\left(\widehat{M_1}-\left(1-\gamma_1\right)\mathbf{e}\mathbf{e}^T\right).$$

In this case, we define $C_0=C\left(\widetilde{p}_1,1,1,\ldots,1,1,\widetilde{p}_{d-1}\right)$ where

$$\widetilde{p}_1=\sum_{j=1}^{i}p_j-\left(i-1\right)\ \text{ and }\widetilde{p}_{d-1}=\sum_{j=i+1}^{d-1}p_j-\left(d-i-2\right).$$

We want to prove $a\left(C_0\right)\leq a\left(C_1\right)$. Suppose $C_0\neq C_1$. Then $p_t>1$ for some $2\leq t\leq d-2$. Suppose that *C_0 is a Type I tree with characteristic vertex v_k.* We have

$$\frac{1}{a\left(C_0\right)}=\rho\left(M_0\right)=\rho\left(\widehat{M_0}\right).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 136-157, February 2010

If $k \leq i$ then $\widehat{M}_0(p,q) - \widehat{M}_1(p,q) \geq 0$ for each (p,q). Hence

$$\frac{1}{a(C_0)} = \rho\left(\widehat{M}_0\right) \geq \rho\left(\widehat{M}_1\right) > \rho\left(\widehat{M}_1 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.$$

If $k > i$ then $M_0(p,q) - M_1(p,q) \geq 0$ for each (p,q). Hence

$$\frac{1}{a(C_0)} = \rho(M_0) \geq \rho(M_1) > \rho\left(M_1 - \gamma_1\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.$$

Suppose now that C_0 is a Type II tree with characteristic vertices v_r and v_{r+1}. There is $0 < \gamma_0 < 1$ such that

$$\frac{1}{a(C_0)} = \rho\left(M_0 - \gamma_0\mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_0 - (1-\gamma_0)\,\mathbf{e}\mathbf{e}^T\right).$$

If $r < i$ then $\widehat{M}_0(p,q) - \widehat{M}_1(p,q) \geq 1$ for each (p,q) and thus

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(\widehat{M}_0 - (1-\gamma_0)\,\mathbf{e}\mathbf{e}^T\right) > \rho\left(\widehat{M}_1\right)\\ &> \rho\left(\widehat{M}_1 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

If $r > i$ then $M_0(p,q) - M_1(p,q) \geq 1$ for each (p,q) and thus

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(M_0 - \gamma_0\mathbf{e}\mathbf{e}^T\right) > \rho(M_1)\\ &> \rho\left(M_1 - \gamma_1\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

If $r = i$ and $\gamma_0 = \gamma_1$ then $M_0(p,q) - M_1(p,q) \geq 0$ for each (p,q) with strict inequality in at least one entry whenever $2 \leq t \leq i$ or $\widehat{M}_0(p,q) - \widehat{M}_1(p,q) \geq 0$ for each (p,q) with strict inequality in at least one entry whenever $i+1 \leq t \leq d-2$. Hence

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(M_0 - \gamma_0\mathbf{e}\mathbf{e}^T\right) = \rho\left(M_0 - \gamma_1\mathbf{e}\mathbf{e}^T\right)\\ &> \rho\left(M_1 - \gamma_1\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

or

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(\widehat{M}_0 - (1-\gamma_0)\,\mathbf{e}\mathbf{e}^T\right) = \rho\left(\widehat{M}_0 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right)\\ &> \rho\left(\widehat{M}_1 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

If $r = i$ and $\gamma_0 < \gamma_1$ then $M_0(p,q) - M_1(p,q) \geq 0$ for each (p,q) and thus

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(M_0 - \gamma_0\mathbf{e}\mathbf{e}^T\right) > \rho\left(M_0 - \gamma_1\mathbf{e}\mathbf{e}^T\right)\\ &\geq \rho\left(M_1 - \gamma_1\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

ELA

If $r = i$ and $\gamma_0 > \gamma_1$ then $\widehat{M}_0(p,q) - \widehat{M}_1(p,q) \geq 0$ for each (p,q) and thus

$$\begin{aligned}\frac{1}{a(C_0)} &= \rho\left(\widehat{M}_0 - (1-\gamma_0)\,\mathbf{e}\mathbf{e}^T\right) > \rho\left(\widehat{M}_0 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right)\\ &\geq \rho\left(\widehat{M}_1 - (1-\gamma_1)\,\mathbf{e}\mathbf{e}^T\right) = \frac{1}{a(C_1)}.\end{aligned}$$

We have proved that if C_1 is a Type II tree and $C_1 \neq C_0$ then $a(C_0) < a(C_1)$.

Finally, let μ be the algebraic connectivity of the caterpillar $C(x,1,\ldots,1,y)$ in which $x = \left\lfloor \frac{n-2d+4}{2} \right\rfloor$ and $y = \left\lceil \frac{n-2d+4}{2} \right\rceil$. Observe that this caterpillar lies in $\mathcal{C}$. We apply Theorem 3.7, to obtain $\mu \leq a(C_0)$. This completes the proof. □

3.4. Extremal algebraic connectivities of symmetric caterpillars. We have

$$\mathcal{S} = \left\{ C(\mathbf{p}) \in \mathcal{C} : p_j = p_{d-j}, j = 1,2,...,\left\lfloor \frac{d-1}{2} \right\rfloor \right\}.$$

To find the caterpillars in $\mathcal{S}$ having the maximum and the minimum algebraic connectivity is immediate. In fact, if $d = 2s+2$ then, from Theorem 3.3,

$$\begin{aligned}\max\{a(C) : C \in \mathcal{S}\} = a(C_0) &= \max\{a(C) : C \in \mathcal{C}\}\\ C_0 &= C(1,...,1,p_{s+1},1,...,1)\\ p_{s+1} &= n-2d+3\end{aligned}$$

and if $d = 2s+1$ then, from Lemma 3.4,

$$\begin{aligned}\max\{a(C) : C \in \mathcal{S}\} &= a(C_0)\\ C_0 &= C(1,...,1,p_s,p_{s+1},1,...,1)\\ p_s = p_{s+1} &= \frac{1}{2}(n-2d+4).\end{aligned}$$

If $d = 2s+2$, from Theorem 3.8, for even n,

$$\begin{aligned}\min\{a(C) : C \in \mathcal{S}\} &= a(C_0)\\ C_0 &= C(x,1,...,1,1,1,...,1,x)\\ x &= \frac{1}{2}(n-2d+4).\end{aligned}$$

and, for odd n,

$$\begin{aligned}\min\{a(C) : C \in \mathcal{S}\} &= a(C_0)\\ C_0 &= C(x,1,...,1,2,1,...,1,x)\\ x &= \frac{1}{2}(n-2d+3).\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 136-157, February 2010

If $d = 2s + 1$, from Theorem 3.8,

$$\begin{aligned} \min\{a(C) : C \in \mathcal{S}\} &= a(C_0) \\ C_0 &= C(x, 1, ..., 1, 1, ..., 1, x) \\ x &= \frac{1}{2}(n - 2d + 3). \end{aligned}$$

3.5. A summary of the extremal algebraic connectivities. Table 1 summarizes the results on the extremal algebraic connectivities on $\mathcal{C}$ and on $\mathcal{S}$ that we have found in this paper. The table includes also the corresponding results for the subclasses $\mathcal{E}$ and $\mathcal{M}$. We observe that these results are immediate consequences of Theorem 3.7 and Theorem 3.5. The columns 2 and 3 give the caterpillars that attain the maximum and the minimum algebraic connectivity, respectively, in the class of caterpillars indicated in column 1.

Class	$\max a(C(\mathbf{p}))$	$\min a(C(\mathbf{p}))$
$\mathcal{C}$	$d = 2s+1$ or $d = 2s+2$ $C(1, ..., 1, p_{s+1}, 1, ..., 1)$ $p_{s+1} = n - 2d + 3$	$d = 2s+1$ or $d = 2s+2$ $C(x, 1, ..., 1, 1, ..., 1, y)$ $x = \lfloor \frac{n-2d+4}{2} \rfloor, y = \lceil \frac{n-2d+4}{2} \rceil$
$\mathcal{E}$	d odd or d even $C(1, 1, ..., 1, ..., 1, y)$ $y = n - 2d + 3$	d odd or d even $C(x, 1, ..., 1, ..., 1, y)$ $x = \lfloor \frac{n-2d+4}{2} \rfloor, y = \lceil \frac{n-2d+4}{2} \rceil$
$\mathcal{M}$	$d = 2s+1$ $C(1, ..., 1, 1, p_{s+1}, 1, ..., 1)$ $p_{s+1} = n - 2d + 3$	$d = 2s+1$ $C(1, ..., 1, p_s, p_{s+1}, 1, ..., 1)$ $p_s = \lfloor \frac{n-2d+4}{2} \rfloor, p_{s+1} = \lceil \frac{n-2d+4}{2} \rceil$
$\mathcal{S}$	$d = 2s+2$ $C(1, ..., 1, p_{s+1}, 1, ..., 1)$ $p_{s+1} = n - 2d + 3$	$d = 2s+2$ and n even $C(x, 1, ..., 1, ..., 1, x)$ $x = \frac{1}{2}(n - 2d + 4)$
$\mathcal{S}$		$d = 2s+2$ and n odd $C(x, 1, ..., 2, ..., 1, x)$ $x = \frac{1}{2}(n - 2d + 3)$
$\mathcal{S}$	$d = 2s+1$ $C(1, ..., 1, p_s, p_{s+1}, 1, ..., 1)$ $p_s = p_{s+1} = \frac{1}{2}(n - 2d + 4)$	$d = 2s+1$ $C(x, 1, ..., 1, ..., 1, x)$ $x = \frac{1}{2}(n - 2d + 4)$

Table 1: Extremal algebraic connectivities

4. Characterizing the algebraic connectivity of symmetric caterpillars. A generalized Bethe tree is a rooted tree in which vertices at the same distance from the root have the same degree. In [13], we characterize completely the Laplacian eigenvalues of the tree $P_m\{B_i\}$ obtained from the path P_m and m generalized Bethe

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

trees $B_1, B_2, ..., B_m$ by identifying the root vertex of B_i with the i-th vertex of P_m. This is the case for the caterpillar $C(\mathbf{p})$ in which the path is P_{d-1} and the $(d-1)$ generalized Bethe trees are the stars S_{p_i} $(1 \leq i \leq d-1)$. Thus we may apply the results in [13] to characterize the algebraic connectivity of $C(\mathbf{p})$. Previously we introduce some notation. Let

$$A(x) = \begin{bmatrix} 1 & \sqrt{x} \\ \sqrt{x} & x+1 \end{bmatrix}$$

$$B(x) = \begin{bmatrix} 1 & \sqrt{x} \\ \sqrt{x} & x+2 \end{bmatrix}$$

and

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying Theorem 4 of [13] to $C(\mathbf{p})$, one can obtain

THEOREM 4.1. *The algebraic connectivity of $C(\mathbf{p})$ is the smallest positive eigenvalue of the $(2d-2) \times (2d-2)$ positive semidefinite matrix*

$$M(\mathbf{p}) = \begin{bmatrix} A(p_1) & E & & & \\ E & B(p_2) & E & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & B(p_{d-2}) & E \\ & & & E & A(p_{d-1}) \end{bmatrix}. \tag{4.1}$$

Next we characterize the algebraic connectivity of a caterpillar in $\mathcal{S}$ as the smallest eigenvalue of a 2×2 - block tridiagonal matrix of order $2s\times 2s$ if $d = 2s+1$ or $d = 2s+2$.

4.1. Case $d = 2s+1$. Let $d = 2s+1$. Let $C(\mathbf{p}) \in \mathcal{S}$. Then

$$\mathbf{p} = [p_1, ..., p_s, p_s, ..., p_1],$$

and the matrix $M(\mathbf{p})$ in (4.1) becomes

$$M(\mathbf{p}) = \begin{bmatrix} A(p_1) & E & & & \\ E & B(p_2) & E & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & B(p_2) & E \\ & & & E & A(p_1) \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

of order $4s \times 4s$. Let J be the permutation matrix

$$J = \begin{bmatrix} & & & I \\ & & I & \\ & & & \\ & I & & \\ I & & & \end{bmatrix} \tag{4.2}$$

of order $2s \times 2s$ in which the 2×2 blocks in the secondary diagonal are the 2×2 identity matrix. We may write

$$M(\mathbf{p}) = \begin{bmatrix} N(p_1, ..., p_s) & F \\ F & JN(p_1, ..., p_s) J \end{bmatrix} \tag{4.3}$$

where

$$N(p_1, ..., p_s) = \begin{bmatrix} A(p_1) & E & & & & \\ E & B(p_2) & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & & & E \\ & & & E & B(p_s) \end{bmatrix} \tag{4.4}$$

and

$$F = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 0 & 0 \\ E & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We claim that $M(\mathbf{p})$ in (4.3) is orthogonally similar to a block diagonal matrix. In fact, let us consider the orthogonal matrix Q of order $4s \times 4s$,

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & J \\ I & -J \end{bmatrix}.$$

A computation shows

$$QM(\mathbf{p}) Q^T = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 136-157, February 2010

ELA

where

$$R = \begin{bmatrix} A(p_1) & E & & & \\ E & B(p_2) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & E \\ & & & E & B(p_s)+E \end{bmatrix}$$

and

$$S = \begin{bmatrix} A(p_1) & E & & & \\ E & B(p_2) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & E \\ & & & E & B(p_s)-E \end{bmatrix}.$$

Therefore

$$\sigma(M(\mathbf{p})) = \sigma(R) \cup \sigma(S). \tag{4.5}$$

Since $\det R = 2$ and $\det S = 0$, R is a positive definite matrix and S is positive semidefinite matrix.

We recall the Monotonicity Theorem for eigenvalues of Hermitian matrices [9]:

THEOREM 4.2. *Let $A, B \in M_n$ be Hermitian matrices. Let $C = A + B$. Let*

$$\alpha_1 \le \alpha_2 \le ..... \le \alpha_n$$
$$\beta_1 \le \beta_2 \le ..... \le \beta_n$$
$$\gamma_1 \le \gamma_2 \le ..... \le \gamma_n$$

be the eigenvalues of A, B and C respectively. Then

$$\begin{array}{ccc} \alpha_j + \beta_{i-j+1} \le \gamma_i & \text{if} & i \ge j \\ \gamma_i \le \alpha_j + \beta_{i-j+n} & \text{if} & i \le j \\ \alpha_i + \beta_1 \le \gamma_i \le \alpha_i + \beta_n & & i = 1, 2, ..., n. \end{array}$$

THEOREM 4.3. *If $d = 2s + 1$ and $\mathbf{p} = [p_1, ..., p_s, p_s, ..., p_1]$ then the algebraic connectivity of the caterpillar $C(\mathbf{p})$ is the smallest eigenvalue of*

$$R = \begin{bmatrix} A(p_1) & E & & \\ E & B(p_2) & \ddots & \\ & \ddots & \ddots & E \\ & & E & B(p_s)+E \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 136-157, February 2010

ELA

Proof. We have

$$R = S + T$$

where all the entries of T are zeros except for $T(2s, 2s) = 2$. Let

$$\begin{aligned} \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_{2s} \\ 0 = \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_{2s} \\ 0 = \beta_1 = ... = \beta_{2s-1} < \beta_2 = 2 \end{aligned}$$

be the eigenvalues of R, S and T respectively. From (4.5)

$$\sigma(M(\mathbf{p})) = \{\gamma_1, \gamma_2, ..., \gamma_{2s}\} \cup \{0, \alpha_2, ..., \alpha_{2s}\}.$$

From Theorem 4.2

$$\gamma_i \leq \alpha_j + \beta_{i-j+2s} \text{ for } i \leq j.$$

In particular, for $i = 1$ and $j = 2$, we have $\gamma_1 \leq a_2 + \beta_{2s-1} = \alpha_2 + 0 = \alpha_2$. This shows that the smallest positive eigenvalue of R is the smallest positive eigenvalue of $M(\mathbf{p})$. This fact and Theorem 4.1 complete the proof. □

EXAMPLE 4.4. *Consider the symmetric caterpillar*

We have $\mathbf{p} = \begin{bmatrix} 4 & 3 & 2 & 2 & 3 & 4 \end{bmatrix}$ *and its algebraic connectivity is the smallest eigenvalue of*

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} & 0 & 0 \\ 0 & 1 & \sqrt{3} & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 & 1 & \sqrt{2} & 5 \end{bmatrix}.$$

4.2. Case $d = 2s + 2$**.** Let $d = 2s + 2$. Let $C(\mathbf{p}) \in \mathcal{S}$. Then

$$\mathbf{p} = [p_1, p_2, ..., p_s, p_{s+1}, p_s, ..., p_2, p_1],$$

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and the matrix $M(\mathbf{p})$ in (4.1) becomes

$$M(\mathbf{p}) = \begin{bmatrix} N(p_1,...,p_s) & \begin{bmatrix} 0 \\ E \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & E \end{bmatrix} & \begin{bmatrix} 1 & \sqrt{p_{s+1}} \\ \sqrt{p_{s+1}} & p_{s+1}+2 \end{bmatrix} & \begin{bmatrix} E & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} E \\ 0 \end{bmatrix} & JN(p_1,...,p_s)\,J \end{bmatrix}$$

where $N(p_1,...,p_s)$ and J are as in (4.4) and (4.2) respectively. Let us consider the orthogonal matrix Q of order $(4s+2)\times(4s+2)$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & J \\ 0 & \sqrt{2}I & 0 \\ I & 0 & -J \end{bmatrix}.$$

A computation shows

$$\begin{aligned} &QM(\mathbf{p})\,Q^T \\ &= \begin{bmatrix} N(p_1,...,p_s) & \begin{bmatrix} 0 \\ \sqrt{2}E \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & \sqrt{2}E \end{bmatrix} & \begin{bmatrix} 1 & \sqrt{p_{s+1}} \\ \sqrt{p_{s+1}} & p_{s+1}+2 \end{bmatrix} & 0 \\ 0 & 0 & N(p_1,...,p_s) \end{bmatrix}. \end{aligned}$$

Hence

$$\sigma(M(\mathbf{p})) = \sigma(U) \cup \sigma(N) \tag{4.6}$$

where

$$U = \begin{bmatrix} N(p_1,...,p_s) & \begin{bmatrix} 0 \\ \sqrt{2}E \end{bmatrix} \\ \begin{bmatrix} 0 & \sqrt{2}E \end{bmatrix} & \begin{bmatrix} 1 & \sqrt{p_{s+1}} \\ \sqrt{p_{s+1}} & p_{s+1}+2 \end{bmatrix} \end{bmatrix}.$$

Since $\det U = 0$ and $\det N = 1$, U is a positive semidefinite matrix and N is a positive definite matrix.

The proof of the following theorem as well as the following example are given in [1, Theorem 8 and Example 3]. However, for completeness, they are also given here.

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THEOREM 4.5. *If* $d = 2s+2$ *and* $\mathbf{p} = [p_1, ..., p_s, p_{s+1}, p_s, ..., p_1]$ *then the algebraic connectivity of* $C(\mathbf{p})$ *is the smallest eigenvalue of*

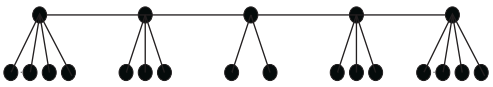
$$N = \begin{bmatrix} A(p_1) & E & & \\ E & B(p_2) & \ddots & \\ & \ddots & \ddots & E \\ & & E & B(p_s) \end{bmatrix}.$$

Proof. We know that the smallest positive eigenvalue of $M(\mathbf{p})$ is the algebraic connectivity of $C(\mathbf{p})$. From (4.6) the eigenvalues of $M(\mathbf{p})$ are the eigenvalues of U and N. Let U_1 the principal submatrix of U obtained by deleting its last row and its last column. That is

$$U_1 = \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $\det N = 1$. Then at least one eigenvalue of N is strictly less than 1. By the eigenvalue interlacing property for Hermitian matrices, the eigenvalues of N and the eigenvalue 1 interlace the eigenvalues of U. These facts all together imply that the smallest eigenvalue of N is the smallest positive eigenvalue of $M(\mathbf{p})$. This completes the proof. □

EXAMPLE 4.6. *The algebraic connectivity of the caterpillar*



is the smallest eigenvalue of

$$N = \begin{bmatrix} 1 & \sqrt{4} & 0 & 0 \\ \sqrt{4} & 5 & 0 & 1 \\ 0 & 0 & 1 & \sqrt{3} \\ 0 & 1 & \sqrt{3} & 5 \end{bmatrix}.$$

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