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THE MAXIMUM SPECTRAL RADIUS OF GRAPHS WITH A LARGE CORE*

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Abstract. The $(k+1)$-core of a graph G, denoted by $C_{k+1}(G)$, is the subgraph obtained by repeatedly removing any vertex of degree less than or equal to k. $C_{k+1}(G)$ is the unique induced subgraph of minimum degree larger than k with a maximum number of vertices. For $1 \leq k \leq m \leq n$, we denote $R_{n,k,m} = K_k \vee (K_{m-k} \cup I_{n-m})$. In this paper, we prove that $R_{n,k,m}$ obtains the maximum spectral radius and signless Laplacian spectral radius among all n-vertex graphs whose $(k+1)$-core has at most m vertices. Our result extends a recent theorem proved by Nikiforov [*Electron. J. Linear Algebra*, 27:250–257, 2014]. Moreover, we also present the bipartite version of our result.

Key words. Adjacency matrices, Core, Extremal graph theory, Bipartite graph.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $|V(G)|$ be the *order* of G and $|E(G)|$ be the *size* of G. Two vertices of a graph G are said to be *independent* if they are not adjacent. A subset I of $V(G)$ is called an *independent set* if any two vertices of I are independent in G. Let I_r be an independent set of size r. The *neighborhood* of a vertex v, written by $N_G(v)$, is the set of vertices adjacent to v in G. The *degree* of v is defined as the number $d_G(v) = |N_G(v)|$. The minimum degree of G is denoted by $\delta(G)$. If $v \in V(G)$, then $G - v$ denotes the graph obtained from G by deleting the vertex v and all its incident edges. If $uv \in E(G)$, then $G - uv$ is a graph obtained from G by removing the edge uv. The *null graph* is the graph whose vertex set and edge set are empty. We adopt the notation and terminologies in [3] except as stated otherwise.

The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix with the (i,j)-entry equals to 1 if vertices v_i and v_j are adjacent and 0 otherwise. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G and denoted by $\rho(G)$. Let $D(G) = \mathrm{diag}\big(d(v_1), d(v_2), \ldots, d(v_n)\big)$ be the diagonal matrix of vertex degrees. The *signless Laplacian matrix* is $Q(G) = D(G) + A(G)$, and we call its largest eigenvalue, denoted by $q(G)$, the Q-index of G. It is well known that $A(G)$ $(Q(G))$ is irreducible if G is connected. From the Perron–Frobenius Theorem, if G is connected, then there is a unique positive eigenvector corresponding to $\rho(G)$ $(q(G))$ whose entries sum to 1. We call this eigenvector *principle eigenvector*. Spectral graph theory is an important branch of algebraic graph theory. In particular, eigenvalues of graphs are important structural invariants which have numerous applications in quantum chemistry and theoretical chemistry. Many upper bounds on $\rho(G)$ and $q(G)$ have been obtained (see [4, 5, 6, 7, 8, 15, 16, 17, 18, 19] for example).

The *(k+1)-core* of a graph G, denoted by $C_{k+1}(G)$, is the subgraph obtained by repeatedly removing any vertex of degree less than or equal to k. It is easy to see that $C_{k+1}(G)$ is the unique induced subgraph of minimum degree larger than k with a maximum number of vertices. Cores were introduced by S.B. Seidman

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[13] and have been studied extensively in [2]. Note that we allow $C_{k+1}(G)$ to be a null graph; in this case, the graph has no subgraph of minimum degree larger than k. Graphs whose $(k+1)$-core is a null graph are referred to as *k-degenerate*; see [1] and [10] for recent breakthrough in extremal graph theory. For two vertex-disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H; $G \vee H$ denotes the join of G and H, which is obtained from $G \cup H$ by adding all possible edges between G and H. For instance, $K_{r,s} = I_r \vee I_s$. We can observe that $K_k \vee I_{n-k}$ is a k-degenerate graph on n vertices.

In 2014, Nikiforov [12] proved the following results for k-degenerate graphs.

Theorem 1 (Nikiforov [12]). *If G is a k-degenerate graph of order $n \geq k$, then*

$$\rho(G) \leq \rho(K_k \vee I_{n-k}),$$

equality holds if and only if $G = K_k \vee I_{n-k}$.

Theorem 2 (Nikiforov [12]). *If G is a k-degenerate graph of order $n \geq k$, then*

$$q(G) \leq q(K_k \vee I_{n-k}),$$

equality holds if and only if $G = K_k \vee I_{n-k}$.

Let $1 \leq k \leq m \leq n$ be positive integers. We denote (see Fig. 1)

$$R_{n,k,m} := K_k \vee (K_{m-k} \cup I_{n-m}).$$

Clearly, the $(k+1)$-core of $R_{n,k,m}$ has at most m vertices. In particular, when $m = k$ or $k+1$, $C_{k+1}(R_{n,k,m})$ is a null graph; when $m \geq k+2$, we can see that $C_{k+1}(R_{n,k,m})$ is the complete graph K_m.

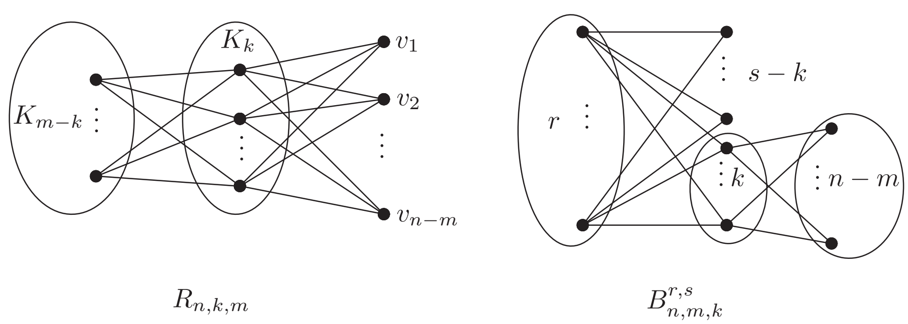

Fig. 1. *Graphs $R_{n,k,m}$ and $B^{r,s}_{n,m,k}$.*

In what follows, we generalize Nikiforov's results on both Theorems 1 and 2.

Theorem 3. *Let G be an n-vertex graph with $|C_{k+1}(G)| \leq m$. Then,*

$$\rho(G) \leq \rho(R_{n,k,m}),$$

equality holds if and only if $G = R_{n,k,m}$.

Theorem 4. *Let G be an n-vertex graph with $|C_{k+1}(G)| \leq m$. Then,*

$$q(G) \leq q(R_{n,k,m}),$$

equality holds if and only if $G = R_{n,k,m}$.

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Recall that a graph is k-degenerate if and only if its $(k+1)$-core is a null graph. Hence, G is k-degenerate if and only if $|C_{k+1}(G)| \leq k$. Our results extend Theorems 1 and 2. Indeed, setting $m = k$ in Theorem 3, we get Theorem 1. Similarly, Theorem 4 can imply Theorem 2.

We continue our investigation on bipartite graphs. Let $k \leq s \leq r$ be integers with $r + s = m$. We define $B^{r,s}_{n,m,k}$ as the bipartite graph obtained from a bipartite graph $K_{r,s}$ and an independent set I_{n-m} by joining $n - m$ vertices of I_{n-m} to the same k vertices of $K_{r,s}$ in the color class of size s (see Fig. 1).

In particular, if $s = k$, then $B^{r,s}_{n,m,k} = K_{k,n-k}$. Let $\mathfrak{B} = \{B^{r,s}_{n,m,k} \mid r + s = m\}$.

Now, we present another main result, which is a bipartite version of Theorem 3.

Theorem 5. *Let G be a connected bipartite graph whose spectral radius $\rho(G)$ is maximum among all n-vertex connected bipartite graphs whose $(k+1)$-core has at most m vertices.*

(1) *If $m \leq 2k+1$, then $\rho(G) \leq \rho(K_{k,n-k})$ with equality if and only if $G = K_{k,n-k}$.*
(2) *If $m \geq 2k+2$, then $G \in \mathfrak{B}$.*

Corollary 1. *If G is a k-degenerate connected bipartite graph of order $n \geq k$, then,*

$$\rho(G) \leq \rho(K_{k,n-k}),$$

equality holds if and only if $G = K_{k,n-k}$.

2. Technical lemmas. In this section, we introduce four specific graph operations, and our technique is to employ these specific operations to make the transformed graph with larger spectral radius.

Lemma 6 ([11]). *Let M and N be two nonnegative irreducible matrices with same order. If $(N)_{ij} \leq (M)_{ij}$ for each i, j, then $\mu(N) \leq \mu(M)$ with equality if and only if $N = M$, where $\mu(N)$ and $\mu(M)$ denote the spectral radius of N and M.*

Lemma 7 ([14]). *Let u and v be two vertices of a connected graph G. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u)$ $(1 \leq s \leq d(v))$, $v_1, v_2, \ldots, v_s$ are different from u and $\mathbf{x}$ is the Perron vector. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $\mathbf{x}_u \geq \mathbf{x}_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 8 ([9]). *Let G be a connected graph and $q(G)$ be the spectral radius of $Q(G)$. Let u and v be two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u)$ $(1 \leq s \leq d(v))$, $v_1, v_2, \ldots, v_s$ are different from u and $\mathbf{x}$ is the Perron vector of $Q(G)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $\mathbf{x}_u \geq \mathbf{x}_v$, then $q(G) < q(G^*)$.*

Lemma 9. *Let G be a connected graph. Let v be a vertex of G and U, W be two vertex-disjoint sets of $V(G)$. Suppose $v \notin W$, $N_G(v) \cap U = U, N_G(v) \cap W = \emptyset$ and $\mathbf{x}$ is the Perron vector of $A(G)$. Let G^* be the graph obtained from G by deleting the edge set $\cup_{u\in U}\{uv\}$ and adding the edge set $\cup_{w\in W}\{wv\}$. If $\sum_{u\in U} \mathbf{x}_u \leq \sum_{w\in W} \mathbf{x}_w$, then $\rho(G) < \rho(G^*)$.*

Proof.

$$\begin{aligned}\rho(G^*) - \rho(G) &\geq \frac{\mathbf{x}^\top A(G^*)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} - \frac{\mathbf{x}^\top A(G)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \\ &= \frac{2\mathbf{x}_v(\sum_{w\in W} \mathbf{x}_w - \sum_{u\in U} \mathbf{x}_u)}{\mathbf{x}^\top \mathbf{x}} \\ &\geq 0.\end{aligned}$$

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If $\rho(G^*) = \rho(G)$, then $\mathbf{x}$ is a Perron vector of $A(G^*)$. Hence, we have

$$\begin{aligned} 0 &= \rho(G^*) \sum_{w \in W} \mathbf{x}_w - \rho(G) \sum_{w \in W} \mathbf{x}_w \\ &= |W|\mathbf{x}_v \\ &> 0, \end{aligned}$$

a contradiction. Hence, $\rho(G) < \rho(G^*)$. ∎

3. Proof of Theorems 3 and 4. Let $\mathcal{G}_{n,m}^k$ be the class of all n-vertex graphs whose $(k+1)$-core has at most m vertices.

Choose $G \in \mathcal{G}_{n,m}^k$ such that $\rho(G)$ is as large as possible. Order the vertices of G as follows: choose a vertex $v \in V(G)$ with $d_G(v) = \delta(G)$ and set $v_1 = v$; further, having chosen $v_1, v_2, \ldots, v_j$, letting H be the graph induced by $V(G) \backslash \{v_1, v_2, \ldots, v_j\}$, choose $v \in V(H)$ with $d_H(v) = \delta(H) \leq k$ and set $v_{j+1} = v$. Continuing the process until we obtain $C_{k+1}(G)$. Since $|C_{k+1}(G)| \leq m$, in the ordering $v_1, v_2, \ldots, v_{n-m}$, every vertex has at most k neighbors in the remaining vertices of $V(G)$. Let $G_0 = G - \{v_1, v_2, \ldots, v_{n-m}\}$. In what follows, we shall first show that G_0 is a complete graph on m vertices, and then we shall apply a series of claims to prove that $v_1, \ldots, v_{n-m-1}, v_{n-m}$ form an independent set of G and each of these vertices has the same neighborhood in G_0.

First of all, we have

$$|E(G)| \leq k(n-m) + |E(G_0)| \leq k(n-m) + \tbinom{m}{2}.$$

By the maximality of $\rho(G)$, in the ordering $v_1, v_2, \ldots, v_{n-m}$, every vertex has exactly k neighbors in the remaining vertices of $V(G)$ and $G_0 = K_m$. Thus, we have $|E(G)| = k(n-m) + \binom{m}{2}$ and G is a connected graph. Let $V(G_0) = \{u_1, u_2, \ldots, u_m\}$. Without loss of generality, let $N_{G_0}(v_{n-m}) = \{u_1, u_2, \ldots, u_k\}$ and $\mathbf{x}$ be the Perron vector of $A(G)$.

Claim 1. $v_{n-m-1}v_{n-m} \notin E(G)$.

Proof. Suppose to the contrary that $v_{n-m-1}v_{n-m} \in E(G)$. Since v_{n-m-1} has exactly k neighbors in $\{v_{n-m}\} \cup V(G_0)$, there is at least one vertex in $\{u_1, u_2, \ldots, u_k\}$ nonadjacent to v_{n-m-1}. Without loss of generality, we may assume that $v_{n-m-1}u_1 \notin E(G)$.

If $\mathbf{x}_{u_1} \geq \mathbf{x}_{v_{n-m}}$, then let $G_1 = G - v_{n-m-1}v_{n-m} + v_{n-m-1}u_1$. It is easy to see that $G_1 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_1) > \rho(G)$, a contradiction. If $\mathbf{x}_{u_1} < \mathbf{x}_{v_{n-m}}$, we complete the proof according to the following two possible cases.

Case 1. $m = k$.

Swap the labels of two vertices u_1 and v_{n-m}, that is, letting $u_1 = v_{n-m}$ and $v_{n-m} = u_1$. Thus, $v_{n-m-1}v_{n-m}$
$\notin E(G)$. In the new vertex sequence $v_1, v_2, \ldots, v_{n-m}$, every vertex still has exactly k neighbors in the remaining vertices of $V(G)$.

Case 2. $m > k$.

Let $G_2 = G - \cup_{i=k+1}^{m}\{u_1u_i\} + \cup_{i=k+1}^{m}\{v_{n-m}u_i\}$. Remove the vertices $v_1, v_2, \ldots,\ v_{n-m-1}, u_1$ in turn, and it is easy to see that any vertex in this sequence has at most k neighbors in the remaining vertices of G_2. Then, we have $G_2 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_2) > \rho(G)$, a contradiction. ∎

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Recall that $N_{G_0}(v_{n-m}) = \{u_1, u_2, \ldots, u_k\}$. We have the following claim.

Claim 2. $N_{G_0}(v_{n-m-1}) = \{u_1, u_2, \ldots, u_k\}$.

Proof. Suppose to the contrary that $N_{G_0}(v_{n-m-1}) \neq \{u_1, u_2, \ldots, u_k\}$. Since v_{n-m-1} has exactly k neighbors in $\{v_{n-m}\} \cup V(G_0)$, by Claim 1, there is at least one u_i $(i > k)$ adjacent to v_{n-m-1}. Without loss of generality, assume that $v_{n-m-1}u_1 \notin E(G)$ and $v_{n-m-1}u_{k+1} \in E(G)$.

If $\mathbf{x}_{u_1} \geq \mathbf{x}_{u_{k+1}}$, then let $G_3 = G - v_{n-m-1}u_{k+1} + v_{n-m-1}u_1$. It is easy to see that $G_3 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_3) > \rho(G)$, a contradiction.

If $\mathbf{x}_{u_1} < \mathbf{x}_{u_{k+1}}$, then let $G_4 = G - v_{n-m}u_1 + v_{n-m}u_{k+1}$. It is easy to see that $G_4 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_4) > \rho(G)$, a contradiction. ∎

Claim 3. $\{v_{n-m}, v_{n-m-1}, v_{n-m-2}\}$ *is an independent set.*

Proof. Suppose $v_{n-m-2}v_{n-m} \in E(G)$. Since v_{n-m-2} has exactly k neighbors in $\{v_{n-m-1}, v_{n-m}\} \cup V(G_0)$, without loss of generality, we may assume that $v_{n-m-2}u_1 \notin E(G)$.

If $\mathbf{x}_{u_1} \geq \mathbf{x}_{v_{n-m}}$, then let $G_5 = G - v_{n-m-2}v_{n-m} + v_{n-m-2}u_1$. It is easy to see that $G_5 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_5) > \rho(G)$, a contradiction.

If $\mathbf{x}_{u_1} < \mathbf{x}_{v_{n-m}}$, let $G_6 = G - u_1v_{n-m-1} + v_{n-m}v_{n-m-1}$. We have $G_6 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_6) > \rho(G)$, a contradiction. Similarly, we have $v_{n-m-2}v_{n-m-1} \notin E(G)$. ∎

Claim 4. $N_{G_0}(v_{n-m-2}) = \{u_1, u_2, \ldots, u_k\}$.

Proof. Suppose to the contrary that $N_{G_0}(v_{n-m-2}) \neq \{u_1, u_2, \ldots, u_k\}$. Since v_{n-m-2} has exactly k neighbors in $\{v_{n-m-1}, v_{n-m}\} \cup V(G_0)$, by Claim 3, there is at least one u_i $(i > k)$ adjacent to v_{n-m-2}. Without loss of generality, assume that $v_{n-m-2}u_1 \notin E(G)$ and $v_{n-m-2}u_{k+1} \in E(G)$.

If $\mathbf{x}_{u_1} \geq \mathbf{x}_{u_{k+1}}$, then let $G_7 = G - v_{n-m-2}u_{k+1} + v_{n-m-2}u_1$. We have $G_7 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_7) > \rho(G)$, a contradiction.

If $\mathbf{x}_{u_1} < \mathbf{x}_{u_{k+1}}$, then let $G_8 = G - v_{n-m}u_1 + v_{n-m}u_{k+1}$. We have $G_8 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_8) > \rho(G)$, a contradiction. ∎

By Claims 1–4, we have $\{v_{n-m}, v_{n-m-1}, v_{n-m-2}\}$ is an independent set and

$$N_{G_0}(v_{n-m}) = N_{G_0}(v_{n-m-1}) = N_{G_0}(v_{n-m-2}) = \{u_1, u_2, \ldots, u_k\}.$$

Next, assume that $N_i = \{v_{n-m}, v_{n-m-1}, \cdots, v_{n-m-(i-1)}\}$ is an independent set for some $3 \leq i < n-m$, and $N_{G_0}(v_{n-m-j}) = \{u_1, u_2, \cdots, u_k\}$ for $0 \leq j \leq i-1$. In the following, we prove $N_{i+1} = N_i \cup \{v_{n-m-i}\}$ is an independent set and $N_{G_0}(v_{n-m-i}) = \{u_1, u_2, \cdots, u_k\}$.

Assume $N_{i+1} = N_i \cup \{v_{n-m-i}\}$ is not an independent set. Without loss of generality, we may assume $v_{n-m-i}v_{n-m} \in E(G)$. Since v_{n-m-i} has exactly k neighbors in $\{v_{n-m-(i-1)}, \cdots, v_{n-m-1}, v_{n-m}\} \cup V(G_0)$, v_{n-m-i} is not adjacent to all vertices of $\{u_1, u_2, \ldots, u_k\}$. Without loss of generality, assume $v_{n-m-i}u_1 \notin E(G)$.

If $\mathbf{x}_{u_1} \geq \mathbf{x}_{v_{n-m}}$, then let $G_9 = G - v_{n-m-i}v_{n-m} + v_{n-m-i}u_1$. We have $G_9 \in \mathcal{G}_{n,m}^k$. By Lemma 7, we obtain $\rho(G_9) > \rho(G)$, a contradiction.

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If $\mathbf{x}_{u_1} < \mathbf{x}_{v_{n-m}}$, let $G_{10} = G - u_1v_{n-m-1} + v_{n-m}v_{n-m-1}$. We have $G_{10} \in \mathcal{G}^k_{n,m}$. By Lemma 7, we obtain $\rho(G_{10}) > \rho(G)$, a contradiction. Similarly, we have $v_{n-m-i}v_{n-m-j} \notin E(G)$, for $1 \leq j \leq i-1$. Thus, $N_{i+1} = N_i \cup \{v_{n-m-i}\}$ is an independent set in G.

Assume $N_{G_0}(v_{n-m-i}) \neq \{u_1, u_2, \cdots, u_k\}$. Without loss of generality, assume $v_{n-m-i}u_1 \notin E(G)$ and $v_{n-m-i}u_{k+1} \in E(G)$. If $\mathbf{x}_{u_1} \geq \mathbf{x}_{u_{k+1}}$, then let $G_{11} = G - v_{n-m-i}u_{k+1} + v_{n-m-i}u_1$. We have $G_{11} \in \mathcal{G}^k_{n,m}$. By Lemma 7, we obtain $\rho(G_{11}) > \rho(G)$, a contradiction. If $\mathbf{x}_{u_1} < \mathbf{x}_{u_{k+1}}$, then let $G_{12} = G - v_{n-m}u_1 + v_{n-m}u_{k+1}$. We have $G_{12} \in \mathcal{G}^k_{n,m}$. By Lemma 7, we obtain $\rho(G_{12}) > \rho(G)$, a contradiction.

Performing the above process, since $V(G)$ is a finite set, we will obtain that $\{v_{n-m}, v_{n-m-1}, \cdots, v_1\}$ is an independent set and

$$N_{G_0}(v_{n-m}) = N_{G_0}(v_{n-m-1}) = \cdots = N_{G_0}(v_1) = \{u_1, u_2, \ldots, u_k\}.$$

Thus, $G = R_{n,k,m}$. Therefore, our result holds. □

Proof of Theorem 1 Suppose G is a graph with maximum spectral radius among the set of k-*degenerate* graphs of order $n \geq k$. Note that G is k-degenerate if and only if $|C_{k+1}(G)| \leq k$. Letting $m = k$, by Theorem 3, we obtain $G = R_{n,k,k}$. □

By Lemmas 6 and 8, and performing the same graph transformations as in the proof of Theorem 3, we may obtain Theorems 4 and 2.

4. Proof of Theorem 5. Let $\mathcal{B}^k_{n,m}$ be the class of all n-vertex connected bipartite graphs whose $(k+1)$-core has at most m vertices.

Choose $G \in \mathcal{B}^k_{n,m}$ such that $\rho(G)$ is as large as possible. Similarly, as in the proof of Theorem 3, we have a vertex sequence $v_1, v_2, \ldots, v_{n-m}$ such that, in the ordering, every vertex has at most k neighbors in the remaining vertices of $V(G)$. Let $G_0 = G - \{v_1, v_2, \ldots, v_{n-m}\}$. Then, by the maximality of $\rho(G)$, G_0 is a complete bipartite graph. Assume $G = (X, Y)$ and $G_0 = (X_0, Y_0)$ with $X_0 \subseteq X$ and $Y_0 \subseteq Y$. Let $\mathbf{x}$ be the Perron vector of $A(G)$.

Remark 1. *If $m \leq 2k+1$, then G is k-degenerate. Hence, $V(C_{k+1}(G)) = \emptyset$ and we have $\mathcal{B}^k_{n,0} = \mathcal{B}^k_{n,1} = \mathcal{B}^k_{n,2} = \cdots = \mathcal{B}^k_{n,2k+1}$.*

By Remark 1, in the following, assume $m \geq 2k+1$.

Claim 5. $|X_0|, |Y_0| > 0$.

Proof. If $|X_0| = 0$, then $|Y_0| = m \geq 2k+1$ and $G_0 = Y_0$ is an independent set. We claim that for any $1 \leq i \neq j \leq n-m$, either $N_{Y_0}(v_i) \subseteq N_{Y_0}(v_j)$ or $N_{Y_0}(v_j) \subseteq N_{Y_0}(v_i)$. Otherwise, there exist $u \in N_{Y_0}(v_i)$ and $w \in N_{Y_0}(v_j)$ such that $uv_j \notin E(G)$ and $wv_i \notin E(G)$. Without loss of generality, let $\mathbf{x}_u \geq \mathbf{x}_w$. Clearly, $G - wv_j + uv_j \in \mathcal{B}^k_{n,m}$. By Lemma 7, $\rho(G - wv_j + uv_j) > \rho(G)$, a contradiction.

Since $|N_{Y_0}(v_i)| \leq k$ $(1 \leq i \leq n-m)$, we can conclude that $\left|\bigcup_{1\leq i\leq n-m} N_{Y_0}(v_i)\right| \leq k$. Since $|Y_0| = m \geq 2k+1$, there are at least $k+1$ isolated vertices of G in Y_0. Hence, G is disconnected, a contradiction. Similarly, by the symmetry of X_0 and Y_0, we also have $|Y_0| > 0$. □

Claim 6. *If $m \geq 2k+1$ and at least one of X_0 and Y_0 is less than or equal to k, then $G = K_{k,n-k}$.*

Proof. Note that $K_{k,n-k} \in \mathcal{B}^k_{n,m}$ and if we delete $n-m$ vertices of $K_{k,n-k}$ with degree k, then the remaining subgraph is $K_{k,m-k}$ with one color class of size k. By the maximality of $\rho(G)$, we have $\rho(G) \geq \rho(K_{k,n-k})$. Without loss of generality, assume $|X_0| \leq k$.

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Case 1. If $|X_0| < k$ and $v_{n-m} \in X$, by the maximality of $\rho(G)$, then $|N_{Y_0}(v_{n-m})| = k$. Since $|Y_0| \geq k+2$, there is a vertex $u \in Y_0$ nonadjacent to v_{n-m}. Let $G^* = G + uv_{n-m}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m-1}, u$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$ and $X_0 \cup \{v_{n-m}\}$ is a color class of $G^* - \{v_1, v_2, \ldots, v_{n-m-1}, u\}$ with size at most k. By Lemma 6, we have $\rho(G^*) > \rho(G)$, a contradiction.

Case 2. If $|X_0| < k$ and $v_{n-m} \in Y$, by the maximality of $\rho(G)$, then $N_{X_0}(v_{n-m}) = X_0$. We will prove $v_j \in Y$ for any $1 \leq j \leq n-m$. Suppose to the contrary that there is a vertex $v_{n-m-i} \in X$. We choose $v_{n-m-i} \in X$ such that $\{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\} \subseteq Y$. Since $|Y_0| \geq k+2$, there is a vertex $u \in Y_0$ nonadjacent to v_{n-m-i}. Let $G^* = G + uv_{n-m-i}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m-(i+1)}, v_{n-m-(i-1)}, \ldots, v_{n-m}, u$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$ and $X_0 \cup \{v_{n-m-i}\}$ is a color class of $G^* - \{v_1, \ldots, v_{n-m-(i+1)}, v_{n-m-(i-1)}, \ldots, v_{n-m}, u\}$ with size at most k. By Lemma 6, we have $\rho(G^*) > \rho(G)$, a contradiction. Thus, $G = K_{|X_0|, n-|X_0|}$ and $\rho(G) < \rho(K_{k,n-k})$, a contradiction.

Case 3. If $|X_0| = k$ and $v_{n-m} \in X$, we claim that $v_j \in X$ for any $1 \leq j \leq n-m$. Suppose to the contrary that there is a $v_{n-m-i} \in Y$. We choose $v_{n-m-i} \in X$ such that $\{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\} \subseteq X$. By the maximality of $\rho(G)$, we have $|N_{Y_0}(v_{n-m})| = k$, $N_{Y_0}(v_{n-m}) = N_{Y_0}(v_{n-m-1}) = \cdots = N_{Y_0}(v_{n-m-(i-1)})$ and v_{n-m-i} has exactly k neighbors in $X_0 \cup \{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\}$. We claim that v_{n-m-i} is adjacent to all k vertices of X_0. Otherwise, there is a vertex $v_{n-m-i'}$ adjacent to v_{n-m-i}, where $0 \leq i' \leq i-1$. Let z be a vertex of X_0 nonadjacent to v_{n-m-i}.

- If $\mathbf{x}_z \leq \mathbf{x}_{v_{n-m-i'}}$, let

$$G^* = G - \bigcup_{y \in Y_0, yv_{n-m-i'} \notin E(G)} \{zy\} + \bigcup_{y \in Y_0, yv_{n-m-i'} \notin E(G)} \{yv_{n-m-i'}\}.$$

  Since any vertex in the sequence $v_1, \ldots, v_{n-m-(i'+1)}, v_{n-m-(i'-1)}, \ldots, v_{n-m}$, z has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction.
- If $\mathbf{x}_z > \mathbf{x}_{v_{n-m-i'}}$, let $G^* = G - v_{n-m-i'}v_{n-m-i} + zv_{n-m-i}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, $\rho(G^*) > \rho(G)$, a contradiction.

Hence, v_{n-m-i} is adjacent to all k vertices of X_0. Similarly, if $v_{n-m-(i+1)} \in Y$, we also have $v_{n-m-(i+1)}$ is adjacent to all k vertices of X_0. If $v_{n-m-(i+1)} \in X$, we claim that $v_{n-m-(i+1)}v_{n-m-i} \notin E(G)$. Otherwise, there is a vertex $w \in N_{Y_0}(v_{n-m})$ nonadjacent to $v_{n-m-(i+1)}$. If $\mathbf{x}_w \leq \mathbf{x}_{v_{n-m-i}}$, let $G^* = G - wv_{n-m} + v_{n-m}v_{n-m-i}$. Since any vertex in the sequence $v_1, \ldots, v_{n-m-(i+1)}, v_{n-m-(i-1)}, \ldots, v_{n-m}, v_{n-m-i}$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction. If $\mathbf{x}_w > \mathbf{x}_{v_{n-m-i}}$, let $G^* = G - v_{n-m-(i+1)}v_{n-m-i} + wv_{n-m-(i+1)}$. Clearly, $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence, $v_{n-m-(i+1)}v_{n-m-i} \notin E(G)$ and $v_{n-m-(i+1)}$ is adjacent to exactly k vertices of Y_0. By the maximality of $\rho(G)$, we have $N_{Y_0}(v_{n-m-(i+1)}) = N_{Y_0}(v_{n-m})$. Recursively, we obtain G is isomorphic to G' (see Fig. 2).

If $\mathbf{x}_x \leq \mathbf{x}_y$ for any $x \in X_0$ and $y \in N_{Y_0}(v_{n-m})$, let

$$G^* = G - \bigcup_{x \in X_0} \{xv_{n-m-i}\} + \bigcup_{y \in N_{Y_0}(v_{n-m})} \{yv_{n-m-i}\}.$$

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Clearly, $G^* \in \mathcal{B}^k_{n,m}$. By Lemma 9, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence, $\mathbf{x}_x > \mathbf{x}_y$ for any $x \in X_0$ and $y \in N_{Y_0}(v_{n-m})$. Let

$$G^* = G - \bigcup_{y \in N_{Y_0}(v_{n-m})} \{yv_{n-m}\} + \bigcup_{x \in X_0} \{xv_{n-m}\}.$$

Clearly, $G^* \in \mathcal{B}^k_{n,m}$. By Lemma 9, we have $\rho(G^*) > \rho(G)$, a contradiction.

Hence, $v_j \in X$ for any $1 \le j \le n-m$. By a direct calculation, we have

$$\rho(G) = \frac{\sqrt{2}}{2}\sqrt{(kn-k^2) + \sqrt{(kn-k^2)^2 - 4k^2(n-m)(m-2k)}}.$$

It is easy to see that $\rho(G) < \sqrt{k(n-k)} = \rho(K_{k,n-k})$, a contradiction.

Case 4. If $|X_0| = k$ and $v_{n-m} \in Y$, by the maximality of $\rho(G)$, v_{n-m} is adjacent to all vertices of X_0. We claim that $v_j \in Y$ for any $1 \le j \le n-m$. Suppose to the contrary that there is a $v_{n-m-i} \in X$. We choose $v_{n-m-i} \in X$ such that $\{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\} \subseteq Y$. Then, $v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}$ are adjacent to the k vertices of X_0. Assume v_{n-m-i} has exactly s neighbors in Y_0, $k-s$ neighbors in $\{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\}$, where $0 \le s \le k$. Let $u_1, u_2, \ldots, u_{i-(k-s)}$ be the $i-k+s$ vertices in $\{v_{n-m}, v_{n-m-1}, \ldots, v_{n-m-(i-1)}\}$ nonadjacent to v_{n-m-i}. Let $y_1, y_2, \ldots, y_{k-s}$ be $k-s$ vertices in Y_0 nonadjacent to v_{n-m-i}. Then, $v_1, \ldots, v_{n-m-(i+1)}, u_1, \ldots, u_{i-(k-s)}, y_1, \ldots, y_{k-s}, v_{n-m-i}$ is a vertex sequence in which any vertex has at most k neighbors in the remaining vertices of G. Note that v_{n-m-i} is the finally deleted vertex in the above vertex sequence. Since $v_{n-m-i} \in X$, similarly as in Case 3, we can prove $\rho(G) < \rho(K_{k,n-k})$. Hence, $v_j \in Y$ for any $1 \le j \le n-m$. Thus, $G = K_{k,n-k}$. ∎

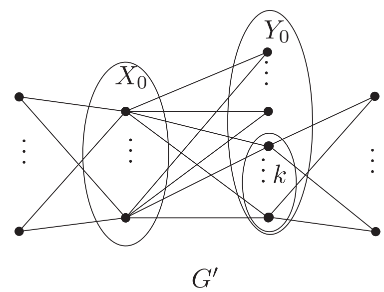

FIG. 2. *Graph G' in the proof of Claim 6.*

CLAIM 7. *If $|X_0|, |Y_0| > k$, then $G = B^{r,s}_{n,m,k}$ for some integers r, s with $s > k$.*

Proof. Without loss of generality, assume $v_{n-m} \in X$. We proceed with the following two cases.

Case 1. $v_{n-m-1} \in Y$.

- If $v_{n-m-2} \in Y$, we claim that v_{n-m-2} and v_{n-m-1} have exactly the same neighbors in $\{v_{n-m}\} \cup X_0$. Otherwise, since both v_{n-m-2} and v_{n-m-1} have exactly k neighbors in $\{v_{n-m}\} \cup X_0$, there are vertices $z_1, z_2 \in \{v_{n-m}\} \cup X_0$ such that $z_1v_{n-m-1}, z_2v_{n-m-2} \in E(G)$ and $z_1v_{n-m-2}, z_2v_{n-m-1} \notin E(G)$. If $\mathbf{x}_{z_1} \ge \mathbf{x}_{z_2}$, let $G^* = G - z_2v_{n-m-2} + z_1v_{n-m-2}$. If $\mathbf{x}_{z_1} < \mathbf{x}_{z_2}$, let $G^* = G - z_1v_{n-m-1} + z_2v_{n-m-1}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}^k_{n,m}$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction.
- If $v_{n-m-2} \in X$, we claim that $v_{n-m-2}v_{n-m-1} \notin E(G)$ and $N_{Y_0}(v_{n-m-2}) = N_{Y_0}(v_{n-m})$. We first prove $v_{n-m-2}v_{n-m-1} \notin E(G)$. Otherwise, v_{n-m-2} has exactly $k-1$ neighbors in Y_0. Recall that

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v_{n-m} has exactly k neighbors in Y_0. Then, there is a vertex $w \in N_{Y_0}(v_{n-m})$ satisfying $wv_{n-m-2} \notin E(G)$. If $\mathbf{x}_w \geq \mathbf{x}_{v_{n-m-1}}$, let $G^* = G - v_{n-m-1}v_{n-m-2} + wv_{n-m-2}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction. If $\mathbf{x}_w < \mathbf{x}_{v_{n-m-1}}$, let

$$G^* = G - \bigcup_{x \in X_0, xv_{n-m-1} \notin E(G)} \{xw\} + \bigcup_{x \in X_0, xv_{n-m-1} \notin E(G)} \{xv_{n-m-1}\}.$$

Note that $N_{G^*}(w) \cap X_0 = N_G(v_{n-m-1}) \cap X_0$. We have $|N_{G^*}(w) \cap X_0| = k-1$ if $v_{n-m-1}v_{n-m} \in E(G)$, and $|N_{G^*}(w) \cap X_0| = k$ if $v_{n-m-1}v_{n-m} \notin E(G)$. For $v_{n-m-1}v_{n-m} \in E(G)$, any vertex in the sequence $v_1, v_2, \ldots, v_{n-m-2}, w, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*. For $v_{n-m-1}v_{n-m} \notin E(G)$, any vertex in the sequence $v_1, v_2, \ldots, v_{n-m-2}, v_{n-m}, w$ has at most k neighbors in the remaining vertices of G^*. Hence, $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction.

Now, we prove $N_{Y_0}(v_{n-m-2}) = N_{Y_0}(v_{n-m})$. Otherwise, there are vertices $y_1 \in N_{Y_0}(v_{n-m})$ and $y_2 \in N_{Y_0}(v_{n-m-2})$ such that $y_1v_{n-m-2}, y_2v_{n-m} \notin E(G)$. If $\mathbf{x}_{y_1} \geq \mathbf{x}_{y_2}$, let $G^* = G - y_2v_{n-m-2} + y_1v_{n-m-2}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction. Similarly, if $\mathbf{x}_{y_1} < \mathbf{x}_{y_2}$, we also have a contradiction.

Recursively, we have G is isomorphic to graphs G_1 or G_2 (see Fig. 3). In the following, we prove G is not isomorphic to G_1. Suppose to the contrary that G is isomorphic to G_1.

For graph G_1, since $|X_0| > k$, there is a vertex $x \in X_0 \backslash N_{X_0}(v_{n-m-1})$. If $\mathbf{x}_x \leq \mathbf{x}_{v_{n-m}}$, let

$$G^* = G_1 - \bigcup_{y \in Y_0 \backslash N_{Y_0}(v_{n-m})} \{xy\} + \bigcup_{y \in Y_0 \backslash N_{Y_0}(v_{n-m})} \{v_{n-m}y\}.$$

Remove the vertices $v_1, v_2, \ldots, v_{n-m-2}, v_{n-m-1}, x$ in turn, and it is easy to see that any vertex in this sequence has at most k neighbors in the remaining vertices of G^*. Then, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G_1)$, a contradiction. If $\mathbf{x}_x > \mathbf{x}_{v_{n-m}}$. Let

$$G^* = G_1 - v_{n-m}v_{n-m-1} + xv_{n-m-1}.$$

Remove the vertices $v_1, v_2, \ldots, v_{n-m}$ in turn, and it is easy to see that any vertex in this sequence has at most k neighbors in the remaining vertices of G^*. Then, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G_1)$, a contradiction. Hence, G is isomorphic to G_2.

Case 2. $v_{n-m-1} \in X$.

- If $v_{n-m-2} \in Y$, we claim that v_{n-m-2} is neither adjacent to v_{n-m-1} nor adjacent to v_{n-m}. Suppose to the contrary that v_{n-m-2} is adjacent to $u \in \{v_{n-m-1}, v_{n-m}\}$. By Lemma 7, v_{n-m-1} and v_{n-m} have exactly the same neighbors in Y_0. Since $|X_0| > k$, there is a vertex $x \in X_0$ nonadjacent to v_{n-m-2}. If $\mathbf{x}_x \geq \mathbf{x}_u$, let $G^* = G - uv_{n-m-2} + xv_{n-m-2}$. Since any vertex in the sequence $v_1, v_2, \ldots, v_{n-m}$ has at most k neighbors in the remaining vertices of G^*. Then, we have $G^* \in \mathcal{B}_{n,m}^k$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction. If $\mathbf{x}_x < \mathbf{x}_u$, let

$$G^* = G - \bigcup_{y \in Y_0 \backslash N_{Y_0}(u)} \{xy\} + \bigcup_{y \in Y_0 \backslash N_{Y_0}(u)} \{uy\}.$$

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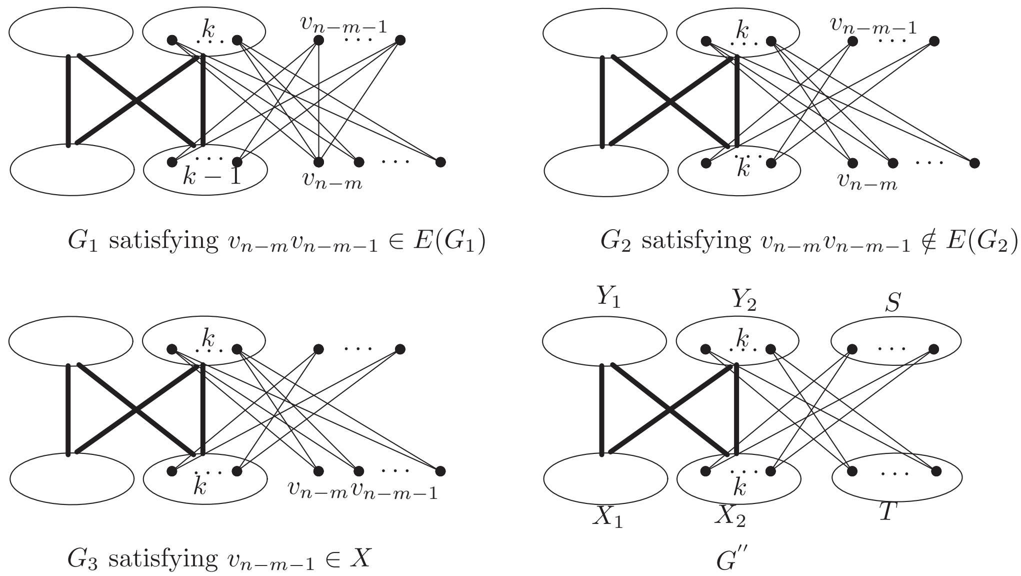

Fig. 3. *Graphs* G_1, G_2, G_3 *and* $G^{''}$.

Let w be the vertex different from u in $\{v_{n-m-1}, v_{n-m}\}$. Since any vertex in the sequence $v_1, v_2, \ldots,$ v_{n-m-2}, w, x has at most k neighbors in the remaining vertices of G^*. Then, we have $G^* \in \mathcal{B}^k_{n,m}$. By Lemma 7, we have $\rho(G^*) > \rho(G)$, a contradiction.

- If $v_{n-m-2} \in X$, by Lemma 7, v_{n-m-2}, v_{n-m-1} and v_{n-m} have exactly the same neighbors in Y_0.

Recursively, we have G is isomorphic to graph G_3 (see Fig. 3).

Note that both G_2 and G_3 have the form $G^{''}$, where $X_1 \cup X_2 = X_0$, $Y_1 \cup Y_2 = Y_0$ and $|S| + |T| = n - m$ (see Fig. 3).

Let $\mathbf{x}$ be a principle eigenvector of $G^{''}$ corresponding to $\rho(G^{''})$. By symmetry, we have $\mathbf{x}_{x_1} = \mathbf{x}_{x_2}$ for any $x_1, x_2 \in X_2$ (resp. $\mathbf{x}_{y_1} = \mathbf{x}_{y_2}$ for any $y_1, y_2 \in Y_2$). Without loss of generality, assume $\mathbf{x}_x \leq \mathbf{x}_y$ for $x \in X_2$ and $y \in Y_2$. We claim $|S| = 0$. Otherwise, let $u \in S$ and

$$G^* = G^{''} - \bigcup_{x \in X_2} \{ux\} + \bigcup_{y \in Y_2} \{uy\}.$$

It is easy to see that $G^* \in \mathcal{B}^k_{n,m}$. By Lemma 9, we have $\rho(G^*) > \rho(G^{''})$, a contradiction.

We can see that $A(G)$ has the following equitable quotient matrix (with respect to $V(G) = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup T$):

$$B = \begin{pmatrix} 0 & 0 & |Y_1| & k & 0 \\ 0 & 0 & |Y_1| & k & 0 \\ |X_1| & k & 0 & 0 & 0 \\ |X_1| & k & 0 & 0 & n-m \\ 0 & 0 & 0 & k & 0 \end{pmatrix}.$$

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By a simple calculation, the characteristic polynomial of B is equal to

$$\begin{aligned}f_{|X_1|,|Y_1|}(\lambda)=&\lambda^5-(k^2-km+kn+k|X_1|+k|Y_1|+|X_1||Y_1|)\lambda^3\\&-(k^2m|Y_1|-k^2n|Y_1|+km|X_1||Y_1|-kn|X_1||Y_1|)\lambda.\end{aligned}$$

Fact 1. $|Y_1|\leq|X_1|$.

Proof. By a direct calculation, we have $f_{|X_1|+1,|Y_1|-1}(\lambda)=f_{|X_1|,|Y_1|}(\lambda)+(|X_1|-|Y_1|+1)\lambda^3+k(n-m)(|Y_1|-|X_1|-k-1)\lambda$. If $|Y_1|=|X_1|+1$, then $f_{|X_1|+1,|Y_1|-1}(\lambda)=f_{|X_1|,|Y_1|}(\lambda)-k^2(n-m)\lambda$. Note that $-k^2(n-m)\lambda<0$ in $[\rho(G),\infty)$. Hence, G is not the graph in $\mathcal{B}^k_{n,m}$ with maximum spectral radius, a contradiction. If $|Y_1|>|X_1|+1$, then $f_{|X_1|+1,|Y_1|-1}(\lambda)-f_{|X_1|,|Y_1|}(\lambda)=(|X_1|-|Y_1|+1)\lambda^3+k(n-m)(|Y_1|-|X_1|-k-1)\lambda$ is a decreasing function of λ in $(0,\infty)$. Since $\rho(G)>\rho(K_{k,n-m})=\sqrt{k(n-m)}$, when $\lambda\geq\rho(G)$, we have

$$\begin{aligned}&f_{|X_1|+1,|Y_1|-1}(\lambda)-f_{|X_1|,|Y_1|}(\lambda)\\<&f_{|X_1|+1,|Y_1|-1}(\sqrt{k(n-m)})-f_{|X_1|,|Y_1|}(\sqrt{k(n-m)})\\=&\sqrt{k(n-m)}\big[(|X_1|-|Y_1|+1)k(n-m)+k(n-m)(|Y_1|-|X_1|-k-1)\big]\\=&-\sqrt{k(n-m)}\cdot k^2(n-m)\\<&0.\end{aligned}$$

Hence, G is not the graph in $\mathcal{B}^k_{n,m}$ with maximum spectral radius, a contradiction. ◻

Combining Cases 1 and 2, we obtain $G=B^{r,s}_{n,m,k}$ for some integers r,s with $s\geq k$. ◻

Proof of Theorem 5. If $m\leq 2k+1$, by Remark 1, Claims 5 and 6, we have $\rho(G)\leq\rho(K_{k,n-k})$ with equality if and only if $G=K_{k,n-k}$. Note that if $s=k$, then $B^{r,s}_{n,m,k}=K_{k,n-k}$. If $m\geq 2k+2$, by Remark 1 and Claims 5–7, we have $G=B^{r,s}_{n,m,k}$ for some integers r,s with $s\geq k$. ☐

Proof of Corollary 1. G is k-degenerate if and only if $|C_{k+1}(G)|\leq 2k+1$. Letting $m=2k+1$, by Theorem 5, we have $\rho(G)\leq\rho(K_{k,n-k})$ with equality if and only if $G=K_{k,n-k}$. ☐

Remark 2. *In case (ii) of Theorem 5, the extremal graphs are not unique and determining them does not seem easy. We take a few examples to illustrate.*

Example 1. $m=2k+2$. *In this case, if $k=2$ and $n=7$, then $B^{3,3}_{7,6,2}$ is the unique graph having the largest spectral radius among $\mathcal{B}^2_{7,6}$. If $k=2$ and $n=8$, then $K_{2,6}$ and $B^{3,3}_{8,6,2}$ are the only two graphs having the largest spectral radius among $\mathcal{B}^2_{8,6}$. If $k=2$ and $n\geq 9$, then $K_{2,n-2}$ is the unique graph having the largest spectral radius among $\mathcal{B}^2_{n,6}$. If $k\geq 3$, then $K_{k,n-k}$ is the unique graph having the largest spectral radius among $\mathcal{B}^k_{n,2k+2}$.*

Example 2. $m=2k+3$ *and* $k\geq 2$. *In this case, $B^{k+2,k+1}_{n,2k+3,k}$ is the unique graph having the largest spectral radius among $\mathcal{B}^k_{n,2k+3}$.*

Example 3. $m=2k+4$ *and* $k\geq 2$. *In this case, if $n\leq\lfloor\frac{(k+2)(2k^2+3k+3)}{k(k+1)}\rfloor$, then $\rho(G)\leq\rho(B^{k+2,k+2}_{n,2k+3,k})$ with equality if and only if $G=B^{k+2,k+2}_{n,2k+3,k}$. Otherwise, $\rho(G)\leq\rho(B^{k+3,k+1}_{n,2k+4,k})$ with equality if and only if $G=B^{k+3,k+1}_{n,2k+3,k}$.*

Example 4. $n=91,m=41,$ *and* $k=10$. *By a direct calculation, $B^{k+19,k+2}_{n,4k+1,k}=B^{29,12}_{91,41,10}$ is the unique graph having the largest spectral radius among $\mathcal{B}^{10}_{91,41}$.*

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REFERENCES

[1] N. Alon, M. Krivelevich, and B. Sudakov. Turán numbers of bipartite graphs and related Ramsey-type questions. *Combin. Probab. Comput.,* 12:477–494, 2003.
[2] A. Bickle. The k-Cores of a graph. Ph.D. Dissertation. Western Michigan University, 2010.
[3] J.A. Bondy and U.S.R. Murty. Graph theory. In: *Graduate Texts in Mathematics*, Vol. 244, Springer, 2008.
[4] A. Berman and X.D. Zhang. On the spectral radius of graphs with cut vertices. *J. Combin. Theory Ser. B,* 83:233–240, 2001.
[5] S. Cioabă, L.H. Feng, M. Tait, and X.D. Zhang. The maximum spectral radius of graphs without friendship subgraphs. *Electron. J. Combin.,* 27(4), Paper No. 4.22: 19 pp., 2020.
[6] L.H. Feng, L. Lu, T. Réti, and D. Stevanović. A bound on the spectral radius of graphs in terms of their Zagreb indices. *Linear Algebra Appl.,* 597:33–45, 2020.
[7] L.H. Feng, J.X. Cao, W.J. Liu, S.F. Ding, and H. Liu. The spectral radius of edge chromatic critical graphs. *Linear Algebra Appl.,* 492:78–88, 2016.
[8] Y. Hong, J.L. Shu, and K. Fang. A sharp upper bound of the spectral radius of graphs. *J. Combin. Theory Ser. B,* 81:177–183, 2001.
[9] Y. Hong and X.-D. Zhang. Sharp upper and lower bounds for largest eigenvalue of the Laplacian matrix of trees. *Discrete Math.,* 296:187–197, 2005.
[10] C. Lee. Ramsey numbers of degenerate graphs. *Ann. Math.* (2), 185(3):791–829, 2017.
[11] H. Minc, Nonnegative Matrices. New York, 1998.
[12] V. Nikiforov. Maxima of the Q-index: Degenerate graphs. *Electron. J. Linear Algebra*, 27:250–257, 2014.
[13] S.B. Seidman. Network structure and minimum degree. *Social Netw.,* 5:269–287, 1983.
[14] B. Wu, E. Xiao, and Y. Hong. The spectral radius of the trees on k pendent vertices. *Linear Algebra Appl.,* 395:343–349, 2005.
[15] G. Yu, L.H. Feng, A. Ilić, and D. Stevanović. The Signless Laplacian spectral radius of bounded degree graphs on surfaces. *Appl. Anal. Discrete Math.,* 9:332–346, 2015.
[16] G.L. Yu, S.G. Guo and Y.R. Wu, Maxima of the Q-index for outer-planar graphs, Linear Multilinear Algebra. 63 (2015) 1837-1848.
[17] M.Q. Zhai, H.Q. Lin, and J.L. Shu. Spectral extrema of graphs with fixed size: Cycles and complete bipartite graphs. *European J. Combin.,* 95, Paper No. 103322: 18 pp., 2021.
[18] M.Q. Zhai and H.Q. Lin. Spectral extrema of graphs: Forbidden hexagon. *Discrete Math.,* 343(10):112028, 6 pp., 2020.
[19] M.Q. Zhai, J. Xue, and Z.Z. Lou, The signless Laplacian spectral radius of graphs with a prescribed number of edges. *Linear Algebra Appl.,* 603:154–165, 2020.