

## RANGES OF SYLVESTER MAPS AND A MINIMAL RANK PROBLEM\*

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**Abstract.** It is proved that the range of a Sylvester map defined by two matrices of sizes  $p \times p$  and  $q \times q$ , respectively, plus matrices whose ranks are bounded above, cover all  $p \times q$  matrices. The best possible upper bound on the ranks is found in many cases. An application is made to a minimal rank problem that is motivated by the theory of minimal factorizations of rational matrix functions.

**Key words.** Sylvester maps, Invariant subspaces, Rank.

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**1. Introduction.** Let  $\mathbb{F}$  be a (commutative) field. We let  $\mathbb{F}^{p \times q}$  stand for the set of  $p \times q$  matrices with entries in  $\mathbb{F}$ ;  $\mathbb{F}^{p \times 1}$  will be abbreviated to  $\mathbb{F}^p$ . The following minimal rank problem was stated in [8, Section 6] for the case when  $\mathbb{F}$  is the complex field  $\mathbb{C}$ :

**PROBLEM 1.1.** *Given  $A \in F^{n \times n}$ , and given an  $A$ -invariant subspace  $\mathcal{M} \subseteq \mathbb{F}^n$ , find the smallest possible rank, call it  $\mu(A, \mathcal{M})$ , for the difference  $A - Z$ , where  $Z$  runs over the set of all  $n \times n$  matrices with entries in  $\mathbb{F}$  for which there is a  $Z$ -invariant subspace  $\mathcal{N} \subseteq \mathbb{F}^n$  complementary to  $\mathcal{M}$ . Also, find structural properties, or description, of such matrices  $Z$ .*

The problem (for  $\mathbb{F} = \mathbb{C}$ ) is intimately connected with minimal factorizations of rational matrix functions, in particular, if certain additional symmetry properties of  $A$ ,  $\mathcal{M}$ , and  $Z$  are assumed; see [8] for more details. Pairs of matrices  $(A, Z)$  that have a pair of complementary subspaces  $\mathcal{M}, \mathcal{N}$ , of which the first is  $A$ -invariant and the second is  $Z$ -invariant, but without explicit rank conditions on  $A - Z$ , are studied in [1, 2], for example, in connection with complete minimal factorization of rational matrix functions.

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In the general formulation, Problem 1.1 appears to be difficult, even intractable, especially the part concerning properties or description of all matrices  $Z$ . To illustrate, assume  $\mathbb{F}$  is algebraically closed, and let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M} = \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then  $\mu(A, \mathcal{M}) = 1$ , and a matrix  $Z = \begin{pmatrix} x & w \\ y & z \end{pmatrix}$ ,  $x, y, z, w \in \mathbb{F}$ , has the properties that  $\text{rank}(A - Z) = 1$  and  $Z$  has an invariant subspace complemented to  $\mathcal{M}$  if and only if  $Z$  is not a nonzero scalar multiple of  $A$  and the equality  $xz + y(1 - w) = 0$  holds.

If  $A$  and  $\mathcal{M}$  are as in Problem 1.1, by applying a similarity transformation we can assume without loss of generality that  $\mathcal{M}$  is spanned by first  $p$  unit coordinate vectors in  $\mathbb{F}^n$ ; thus  $A$  has the block form

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where  $A_1 \in \mathbb{F}^{p \times p}$ ,  $A_2 \in \mathbb{F}^{(n-p) \times (n-p)}$ . If the minimal polynomials of  $A_1$  and  $A_2$  are coprime, then it is easy to see that  $\mu(A, \mathcal{M}) = 0$ , i.e.,  $A$  has an invariant subspace  $\mathcal{N}$  complementary to  $\mathcal{M}$ . Indeed, such  $\mathcal{N}$  is spanned by the columns of  $\begin{pmatrix} Q \\ I \end{pmatrix}$ , where the matrix  $Q$  satisfies the equation

$$\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} = \begin{pmatrix} Q \\ I \end{pmatrix} A_2,$$

or

$$QA_2 - A_1Q = A_{12}. \quad (1.1)$$

It is well known that the *Sylvester map*  $Q \mapsto QA_2 - A_1Q$  is invertible if and only if the minimal polynomials of  $A_1$  and  $A_2$  are coprime. See, e.g., [5, 7] for this fact; although this was established in [5, 7] only for the complex field, the extension to any algebraically closed field is immediate, and to prove this fact for the general field  $\mathbb{F}$  one considers the algebraic closure of  $\mathbb{F}$ . Hence (1.1) can be solved for  $Q$  for any given  $A_{12}$ , and  $\mu(A, \mathcal{M}) = 0$  is established. This example shows close connections of Problem 1.1 with properties of Sylvester maps. There is a large literature (in mathematical and engineering journals) on numerical analysis involving Sylvester maps; see, e.g., [3, 4] and the references cited there.

Connected to the Sylvester map the following problem appears in the theory of control for coordination (see [6]). Given is a linear system  $\dot{x}(t) = Ax(t)$ , where the

matrix  $A$  has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix}$$

with respect to a fixed decomposition  $X = X_1 \dot{+} X_2 \dot{+} X_c$  of the state space. One allows transformations  $A \mapsto S^{-1}AS$ , where  $S$  is of the form

$$S = \begin{pmatrix} I & 0 & S_1 \\ 0 & I & S_2 \\ 0 & 0 & I \end{pmatrix}.$$

Then

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 & A_{11}S_1 - S_1A_{cc} + A_{1c} \\ 0 & A_{22} & A_{22}S_2 - S_2A_{cc} + A_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix}.$$

From the point of view of communicating as little as possible between the coordinator acting in  $X_c$  and the subsystems acting in  $X_1$  and  $X_2$ , it is of interest to study when the ranks of  $A_{ii}S_i - S_iA_{cc} + A_{ic}$  are as small as possible for  $i = 1, 2$ . It is precisely this problem we shall discuss in the next section.

**2. Ranges of Sylvester maps.** We recall the definition of invariant polynomials. For  $A \in \mathbb{F}^{p \times p}$ , we let

$$\lambda I - A = E(\lambda) \text{diag} (\phi_{A,1}(\lambda), \dots, \phi_{A,p}(\lambda)) F(\lambda),$$

where  $E(\lambda)$ ,  $F(\lambda)$  are everywhere invertible matrix polynomials, and  $\phi_{A,j}$  are scalar monic polynomials such that  $\phi_{A,j}$  is divisible by  $\phi_{A,j+1}$ , for  $j = 1, \dots, p-1$ . The polynomials  $\phi_{A,j}$  are called the *invariant polynomials* of  $A$ ;  $\phi_{A,1}$  is in fact the minimal polynomial of  $A$ .

For two matrices  $A_1$  and  $A_2$  over the field  $\mathbb{F}$  of sizes  $p \times p$  and  $q \times q$ , respectively, define the nonnegative integer  $s(A_1, A_2)$  as

$$\max \{j \mid 1 \leq j \leq \min\{p, q\}, \phi_{A_1,j}(\lambda) \text{ and } \phi_{A_2,j}(\lambda) \text{ are not coprime}\}.$$

The maximum of the empty set in this formula is assumed to be zero. Clearly,  $s(A_1, A_2) = 0$  if and only if the minimal polynomials of  $A_1$  and  $A_2$  are coprime.

If all eigenvalues of  $A_1$  and  $A_2$  are in  $\mathbb{F}$  (in particular if  $\mathbb{F}$  is algebraically closed), then

$$s(A_1, A_2) = \max_{\lambda \in \mathbb{F}} \min \{\dim \text{Ker} (A_1 - \lambda I), \dim \text{Ker} (A_2 - \lambda I)\}.$$

Consider the linear Sylvester map (for fixed  $A_1$  and  $A_2$ )  $T : \mathbb{F}^{p \times q} \longrightarrow \mathbb{F}^{p \times q}$  defined by

$$T(S) = SA_2 - A_1S, \quad S \in \mathbb{F}^{p \times q},$$

THEOREM 2.1.

(a) Every matrix  $X \in \mathbb{F}^{p \times q}$  can be written in the form

$$X = T(S) + Y,$$

for some  $S \in \mathbb{F}^{p \times q}$  and some  $Y \in \mathbb{F}^{p \times q}$  with  $\text{rank } Y \leq s(A_1, A_2)$ .

(b) Assume that  $s(A_1, A_2) \neq 0$ , and that the greatest common divisor of the minimal polynomials of  $A_1$  and  $A_2$  have all their roots in  $\mathbb{F}$  (in particular, this condition is always satisfied if  $\mathbb{F}$  is algebraically closed). Then for fixed  $A_1$  and  $A_2$ , there is a Zariski open nonempty set  $\Omega$  of  $\mathbb{F}^{p \times q}$  such that for every  $X \in \Omega$ , there is no representation of  $X$  in the form

$$X = T(S) + Y,$$

where  $S, Y \in \mathbb{F}^{p \times q}$  are such that  $\text{rank } Y < s(A_1, A_2)$ .

Theorem 2.1 can be thought of as a generalization of the well known fact that the Sylvester map is a bijection if and only if the minimal polynomials of  $A_1$  and  $A_2$  are coprime.

Consider the following example to illustrate Theorem 2.1. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The invariant polynomials are

$$\phi_{A_1,1}(\lambda) = \phi_{A_2,1}(\lambda) = \lambda(\lambda - 1),$$

$$\phi_{A_1,2}(\lambda) = \lambda, \quad \phi_{A_1,2}(\lambda) = \lambda - 1, \quad \phi_{A_1,3}(\lambda) = \phi_{A_2,3}(\lambda) = 1.$$

We have  $s(A_1, A_2) = 1$ . The range of the Sylvester map is easy to find:

$$\text{Range } T = \left\{ \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix} \right\},$$

where by  $*$  we denote arbitrary entries which are independent free variables. For every  $X = [x_{i,j}] \in \mathbb{F}^{3 \times 3}$ , we have

$$X = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & x_{3,2} & x_{3,3} \end{bmatrix} + \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix}.$$

**3. Proof of Theorem 2.1. Part (a).** We use the rational canonical forms for  $A_1$  and  $A_2$  (see, e.g., [5]), together with the invariance of the statement of the theorem and of its conclusions under similarity transformations

$$A_1 \longrightarrow G_1^{-1} A_1 G_1, \quad A_2 \longrightarrow G_2^{-1} A_2 G_2, \quad G_1 \text{ and } G_2 \text{ invertible.}$$

We may assume therefore without loss of generality that

$$A_1 = \text{diag}(A_1^{(1)}, \dots, A_1^{(u)}) \quad \text{and} \quad A_2 = \text{diag}(A_2^{(1)}, \dots, A_2^{(v)}), \quad (3.1)$$

where

$$A_1^{(j)} = \text{diag}(A_1^{(j,1)}, \dots, A_1^{(j,\gamma_{1,j})}), \quad j = 1, \dots, u,$$

and where the characteristic polynomials of  $A_1^{(j,k)}$ ,  $k = 1, \dots, \gamma_{1,j}$ , are all powers of the same monic irreducible polynomial  $f_{1,j}$ . The matrices  $A_1^{(j,k)}$  are nonderogatory, i.e., the minimal and the characteristic polynomials of  $A_1^{(j,k)}$  coincide. In addition, we require that the irreducible polynomials  $f_{1,1}, \dots, f_{1,u}$  are all distinct.

Similarly,

$$A_2^{(j)} = \text{diag}(A_2^{(j,1)}, \dots, A_2^{(j,\gamma_{2,j})}), \quad j = 1, \dots, v,$$

where the characteristic polynomials of  $A_2^{(j,k)}$ ,  $k = 1, \dots, \gamma_{2,j}$ , are all powers of the same monic irreducible polynomial  $f_{2,j}$ , and the polynomials  $f_{2,1}, \dots, f_{2,v}$  are all distinct. Again, the matrices  $A_2^{(j,k)}$  are nonderogatory.

Moreover, we arrange the blocks  $A_1^{(1)}, \dots, A_1^{(u)}$  and  $A_2^{(1)}, \dots, A_2^{(v)}$  so that

$$f_{1,1} = f_{2,1}, \dots, f_{1,\ell} = f_{2,\ell},$$

but the irreducible polynomials

$$f_{1,1}, \dots, f_{1,u}, f_{2,\ell+1}, \dots, f_{2,v}$$

are all distinct. (The case when  $\ell = 0$ , i.e., the polynomials  $f_{1,j}$  and  $f_{2,j}$  are all distinct, is not excluded; in this case the subsequent arguments should be modified in

obvious ways.) Note that since powers of distinct irreducible polynomials are coprime, it follows that the characteristic polynomials of the matrices

$$A_1^{(1)}, \dots, A_1^{(u)}, A_2^{(\ell+1)}, \dots, A_2^{(v)} \quad (3.2)$$

are pairwise coprime.

We assume in addition that  $A_1^{(j,k)}$  are companion matrices. To set up notation, we let  $e_j$  be a row with 1 in the  $j$ th position and zeros in all other positions (the number of components in  $e_j$  will be evident from context), and analogously let  $e_j^T$  (the transpose of  $e_j$ ) be the column with 1 in the  $j$ th position and zeros in all other positions. Let  $\xi_{1,j,k}$  (resp.,  $\xi_{2,j,k}$ ) be the size of the matrix  $A_1^{(j,k)}$  (resp.,  $A_2^{(j,k)}$ ). Thus, we let

$$A_1^{(j,k)} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{1,j,k}} \\ \alpha_{1,j,k} \end{bmatrix} \quad (3.3)$$

or

$$A_1^{(j,k)} = \begin{bmatrix} e_2^T & e_3^T & \dots & e_{\xi_{1,j,k}}^T & \alpha_{1,j,k}^T \end{bmatrix} \quad (3.4)$$

for some row  $\alpha_{1,j,k}$  (with entries in  $\mathbb{F}$ ), and analogously,

$$A_2^{(j,k)} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{2,j,k}} \\ \alpha_{2,j,k} \end{bmatrix} \quad (3.5)$$

or

$$A_2^{(j,k)} = \begin{bmatrix} e_2^T & e_3^T & \dots & e_{\xi_{2,j,k}}^T & \alpha_{2,j,k}^T \end{bmatrix} \quad (3.6)$$

for some row  $\alpha_{2,j,k}$ .

The forms (3.3) and (3.5) will be used if  $\gamma_{1,j} \leq \gamma_{2,j}$ , and the forms (3.4) and (3.6) will be used if  $\gamma_{1,j} > \gamma_{2,j}$ .

We return to the Sylvester map  $T$ . Conformably with (3.1), we partition

$$S = [S_{j_1, j_2}]_{j_1=1, \dots, u; j_2=1, \dots, v}.$$

Thus,

$$T(S) = SA_2 - A_1S = [S_{j_1, j_2} A_2^{(j_2)} - A_1^{(j_1)} S_{j_1, j_2}]_{j_1=1, \dots, u; j_2=1, \dots, v}.$$

Also, if  $X \in \mathbb{F}_{p \times q}$  is an arbitrary matrix, then we partition again conformably with (3.1):

$$X = [X_{j_1, j_2}]_{j_1=1, \dots, u; j_2=1, \dots, v}.$$

We will show that for any given  $X \in \mathbb{F}^{p \times q}$ , there exist

$$S_{j,j} \in \mathbb{F}^{(\xi_{1,j,1} + \dots + \xi_{1,j,\gamma_{1,j}}) \times (\xi_{2,j,1} + \dots + \xi_{2,j,\gamma_{2,j}})}, \quad j = 1, \dots, \ell,$$

with the property that

$$X_{j,j} = Y_{j,j} + (S_{j,j} A_2^{(j)} - A_1^{(j)} S_{j,j}), \quad (3.7)$$

for some matrix  $Y_{j,j}$  such that

$$\text{rank } Y_{j,j} \leq \min\{\gamma_{1,j}, \gamma_{2,j}\}, \quad j = 1, \dots, \ell. \quad (3.8)$$

Assuming that we have already shown the existence of  $S_{j,j}$  satisfying (3.8) and (3.7), we can easily complete the proof of Part (a).

Indeed, let

$$\mu = \max_{j=1, \dots, \ell} (\min\{\gamma_{1,j}, \gamma_{2,j}\}),$$

and notice that  $\mu = s(A_1, A_2)$ . Now let

$$Y_{j,j} = W_{j,j} Z_{j,j}, \quad j = 1, \dots, \ell$$

be a rank decomposition, where the matrix  $W_{j,j}$  (resp.,  $Z_{j,j}$ ) has  $\mu$  columns (resp.,  $\mu$  rows). We also put formally  $W_{j,j} = 0$  for  $j = \ell + 1, \dots, u$ , and  $Z_{j,j} = 0$  for  $j = \ell + 1, \dots, v$ . Using the property that the characteristic polynomials of matrices (3.2) are pairwise coprime, and that the Sylvester map  $S \mapsto SB_2 - B_1S$  is onto if the characteristic polynomials of  $B_1$  and  $B_2$  are coprime, we find

$$S_{j_1, j_2} \in \mathbb{F}^{(\xi_{1,j_1,1} + \dots + \xi_{1,j_1,\gamma_{1,j_1}}) \times (\xi_{2,j_2,1} + \dots + \xi_{2,j_2,\gamma_{2,j_2}})}$$

for

$$j_1 = 1, \dots, u, \quad j_2 = 1, \dots, v, \quad (j_1, j_2) \notin \{(1, 1), \dots, (\ell, \ell)\},$$

such that

$$X_{j_1, j_2} = W_{j_1, j_1} Z_{j_2, j_2} + (S_{j_1, j_2} A_2^{(j_2)} - A_1^{(j_1)} S_{j_1, j_2}).$$

Now letting

$$Y = \begin{bmatrix} W_{1,1} \\ W_{2,2} \\ \vdots \\ W_{u,u} \end{bmatrix} \begin{bmatrix} Z_{1,1} & Z_{2,2} & \cdots & Z_{v,v} \end{bmatrix},$$

we have  $X = T(S) + Y$ , and obviously,  $\text{rank } Y \leq \mu$ .

Thus, it remains to show the existence of  $S_{j,j}$  satisfying (3.7) and (3.8). We fix  $j$ ,  $j = 1, \dots, \ell$ . We assume that  $\gamma_{1,j} \leq \gamma_{2,j}$ , thus (3.3) and (3.5) will be used; if  $\gamma_{1,j} > \gamma_{2,j}$  the proof is completely analogous using (3.4) and (3.6). Choose rows  $\alpha'_{1,j,k}$  ( $k = 1, \dots, \gamma_{1,j}$ ) so that the characteristic polynomials of the matrices

$$B_1^{(j,k)} := \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{1,j,k}} \\ \alpha'_{1,j,k} \end{bmatrix}, \quad k = 1, \dots, \gamma_{1,j},$$

are coprime to  $f_{2,j} = f_{1,j}$ , and let

$$B_1^{(j)} = \text{diag} \left( B_1^{(j,1)}, \dots, B_1^{(j,\gamma_{1,j})} \right).$$

Therefore, we can find  $S_{j,j}$  so that

$$X_{j,j} = S_{j,j} A_2^{(j)} - B_1^{(j)} S_{j,j}.$$

Now (3.7) holds with

$$Y_{j,j} = \left( B_1^{(j)} - A_1^{(j)} \right) S_{j,j},$$

and since the matrix  $Y_{j,j}$  has at most  $\gamma_{1,j}$  nonzero rows, we have  $\text{rank } Y_{j,j} \leq \gamma_{1,j}$ , as required.

**Part (b).** We assume that  $A_1$  and  $A_2$  have the form (3.1), and use the notation introduced in the proof of Part (a). We have  $\ell \geq 1$ . Let  $j_0$  be such that  $\mu = \min\{\gamma_{1,j_0}, \gamma_{2,j_0}\}$ . Without loss of generality we may assume  $j_0 = 1$ . Let  $p_1 \times p_1$  and  $q_1 \times q_1$ , be the size of

$$A_1^{(1)} = \text{diag} (A_1^{(1,1)}, \dots, A_1^{(1,\gamma_{1,1})}) \quad \text{and} \quad A_2^{(1)} = \text{diag} (A_2^{(1,1)}, \dots, A_2^{(1,\gamma_{2,1})}),$$

respectively. It is easy to see that it suffices to find a Zariski open nonempty set  $\Omega_1$  of  $\mathbb{F}^{p_1 \times q_1}$  such that for every  $X_1 \in \Omega_1$ , there is no representation of  $X_1$  in the form  $X_1 =$



$(SA_2^{(1)} - A_1^{(1)}S) + Y$ , where  $S, Y \in F^{p_1 \times q_1}$  and  $\text{rank } Y < \min\{\gamma_{1,1}, \gamma_{2,1}\}$ . Because of the hypotheses of Part (b), we may assume that every matrix  $A_k^{1,w}$ ,  $w = 1, 2, \dots, \gamma_{k,1}$ ,  $k = 1, 2$ , is an (upper triangular) Jordan block with the same eigenvalue  $\lambda$ ; let the size of this block be  $p_{k,w} \times p_{k,w}$ . Consider a matrix of the form  $SA_2^{(1)} - A_1^{(1)}S$ ,  $S \in \mathbb{F}^{p_1 \times q_1}$ , which is partitioned

$$SA_2^{(1)} - A_1^{(1)}S = [Q_{\alpha,\beta}]_{\alpha=1, \beta=1}^{\gamma_{1,1}, \gamma_{2,1}}, \quad (3.9)$$

where the block  $Q_{\alpha,\beta}$  has the size

$$\left(\text{size of } A_1^{1,\alpha}\right) \times \left(\text{size of } A_2^{1,\beta}\right).$$

Since the  $A_k^{1,w}$ 's are Jordan blocks, the bottom left corners of the blocks  $Q_{\alpha,\beta}$  are all zeros. Now partition

$$X_1 = [X_{\alpha,\beta}]_{\alpha=1, \beta=1}^{\gamma_{1,1}, \gamma_{2,1}} \in \mathbb{F}^{p_1 \times q_1}$$

conformably with the right hand side of (3.9). The Zariski open set  $\Omega_1$  consists of exactly those matriceds  $X_1$  for which the  $\gamma_{1,1} \times \gamma_{2,1}$  matrix formed by the bottom left corners of the  $X_{\alpha,\beta}$ 's has the full rank, equal to  $\min\{\gamma_{1,1}, \gamma_{2,1}\}$ .  $\square$

**4. A special case of the minimal rank problem.** Given a subspace  $\mathcal{M} \subseteq \mathbb{F}^n$  and a matrix  $Z \in \mathbb{F}^{n \times n}$ , we say that  $\mathcal{M}$  is a *complementary  $Z$ -invariant subspace* if  $\mathcal{M}$  is  $Z$ -invariant and some direct complement to  $\mathcal{M}$  in  $\mathbb{F}^n$  is also  $Z$ -invariant. Denote by  $\mathcal{CI}(\mathcal{M})$  the set of all matrices  $Z$  for which  $\mathcal{M}$  is a complementary invariant subspace.

The following problem is closely related to Problem 1.1.

**PROBLEM 4.1.** *Given a matrix  $A \in \mathbb{F}^{n \times n}$  and its invariant subspace  $\mathcal{M} \subseteq \mathbb{F}^n$ , find the smallest possible rank of the differences  $A - Y$ , where  $Y$  is an arbitrary matrix in  $\mathcal{CI}(\mathcal{M})$ , and find a matrix  $Z \in \mathcal{CI}(\mathcal{M})$  such that the difference  $A - Z$  has this smallest possible rank.*

In fact, Problem 4.1 requires an extra condition for  $Z$  in comparison with Problem 1.1, namely, that  $\mathcal{M}$  is an invariant subspace for  $Z$ .

Using similarity, we assume without loss of generality that

$$\mathcal{M} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in F^p \right\}, \quad A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad A_1 \in \mathbb{F}^{p \times p}, \quad A_2 \in \mathbb{F}^{q \times q}.$$

Theorem 2.1 sheds some light on Problem 4.1, as follows.

**THEOREM 4.2.** *Let  $\kappa := s(A_1, A_2)$ . There exists a matrix  $Z \in \mathcal{CI}(\mathcal{M})$  such that*

$$\text{rank}(A - Z) \leq \kappa. \quad (4.1)$$

Moreover,  $Z$  can be taken in the form

$$Z = \begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix} \quad (4.2)$$

for some  $W$ .

**Proof.** We use Theorem 2.1. Indeed, if  $Z$  is in the form (4.2), then  $Z \in \mathcal{CI}(\mathcal{M})$  if and only if for some matrix  $Q \in \mathbb{F}^{p \times q}$  the subspace  $\text{Span} \begin{pmatrix} Q \\ I \end{pmatrix}$  is  $Z$ -invariant, i.e., the equation

$$\begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} = \begin{pmatrix} Q \\ I \end{pmatrix} A_2$$

holds, or equivalently,

$$QA_2 - A_1Q - (W - A_{12}) = A_{12}.$$

By Theorem 2.1, such  $Q$  exists for some  $W$  with the property that  $\text{rank}(A_{12} - W) \leq \kappa$ . Since obviously

$$\text{rank}(A - Z) = \text{rank}(A_{12} - W),$$

the result follows.  $\square$

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