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EIGENVALUES OF SUMS OF PSEUDO-HERMITIAN MATRICES[*]

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Abstract. We study analogues of classical inequalities for the eigenvalues of sums of Hermitian matrices for the cone of admissible elements in the pseudo-Hermitian case. In particular, we obtain analogues of the Lidskii-Wielandt inequalities.

Key words. Eigenvalue, Pseudo-Hermitian, Admissible, Convexity.

AMS subject classifications. 15A42, 15B57.

1. Introduction. The classical triangle inequality says that for a triangle with side lengths a, b and c, one has $|a - b| \leq c \leq |a + b|$. If one considers the space $\mathbb{R}^3$ with the Minkowski norm $|(x, y, z)|^2 = z^2 - x^2 - y^2$, then in the future timelike cone, defined by $z^2 - x^2 - y^2 > 0$, $z > 0$, the triangle inequality gets reversed, and the sides of a triangle $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$ satisfy $|\overrightarrow{c}| \geq |\overrightarrow{a}| + |\overrightarrow{b}|$. This can be interpreted in terms of 2×2 traceless pseudo-Hermitian matrices, if one puts into correspondence to a vector with coordinates (x, y, z) the matrix

$$\begin{pmatrix} z & x + \sqrt{-1} \cdot y \\ -x + \sqrt{-1} \cdot y & -z \end{pmatrix} .$$

The eigenvalues of this matrix are $\pm\sqrt{z^2 - x^2 - y^2}$ and therefore the Minkowski triangle inequality answers the following question: given two traceless pseudo-Hermitian matrices with real spectra $(a, -a)$ and $(b, -b)$ and non-negative upper-left entries, what are the possible eigenvalues of their sum? Explorations of this and related questions for Hermitian symmetric matrices (and more generally for triangles in dual vector spaces of compact Lie algebras) led to many exciting developments bridging across algebra, Lie theory, representation theory, symplectic geometry, geometric invariant theory, vector bundles, and combinatorics, see for example, [5], [8] and references therein. A brief answer to this question can be formulated as follows: given two Hermitian symmetric matrices A and B, the set of eigenvalues for their sum $A + B$ necessarily belongs to a convex polytope defined by certain linear inequalities on the sets of eigenvalues of A and B.

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In the present paper, we begin answering a similar question in the non-compact setting. Let p and q be non-negative integers, $n = p + q$, and let $G = \mathrm{U}(p,q)$ be the pseudounitary group of $n \times n$ matrices M, satisfying $M J_{pq} M^* = J_{pq}$, where J_{pq} is the diagonal matrix

$$J_{pq} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \, .$$

Here 1_p and 1_q are the identity $p \times p$ and $q \times q$ matrices respectively. Let $\mathfrak{g} = \mathfrak{u}(p,q)$ be its Lie algebra of matrices B, satisfying $B J_{pq} + J_{pq} B^* = 0$ and let $\mathfrak{g}^*$ be its dual vector space, which is identified with the space $\sqrt{-1} \cdot \mathfrak{g}$ of pseudo-Hermitian matrices A, satisfying $A J_{pq} = J_{pq} A^*$. In the block form,

$$A = \begin{pmatrix} H_p & B \\ -\bar{B}^T & H_q \end{pmatrix} \, ,$$

where H_p and H_q are $p \times p$ and $q \times q$ Hermitian symmetric matrices respectively and B is a complex $p \times q$ matrix. In general, eigenvalues of pseudo-Hermitian matrices are not necessarily real, unless A is elliptic. And moreover, as computations with 4×4 matrices can already show, the eigenvalues of the sum of even two elliptic elements can be pretty much arbitrary complex numbers. However, if one restricts to the convex cone of *admissible* elements [10], then the question about possible eigenvalues of the sum becomes more meaningful, because the sum of two admissible elements is again admissible. In our situation, the convex cone of admissible elements $\mathfrak{g}^*_{\mathrm{adm}}$ will consist of matrices, which are G-conjugate to diagonal (and thus real) matrices $\mathrm{diag}(\lambda_1, ..., \lambda_p, \mu_1, ..., \mu_q)$ such that $\lambda_i > \mu_j$ for all pairs i, j. We can certainly assume that λ's are arranged in the non-increasing order $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_p$ and μ's are in the non-decreasing order $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_q$ (this is done for convenience), and thus the condition of admissibility becomes rather simple: $\lambda_1 > \mu_1$.

For two admissible matrices $A, B \in \mathfrak{g}^*_{\mathrm{adm}}$ with given spectra, the question of finding possible eigenvalues of their sum can be formulated in terms of the non-abelian convexity theorem in symplectic geometry. The coadjoint orbits $\mathcal{O}_A$ and $\mathcal{O}_B$ of A and B carry natural invariant symplectic structures and so does their product $\mathcal{O}_A \times \mathcal{O}_B$. A generalization due to Weinstein [12] of the original Kirwan's theorem to the case of non-compact semisimple groups implies that the possible spectrum of $A + B$ forms a convex polyhedral set in the positive Weyl chamber $\mathfrak{t}^*_+$ of the dual space to the diagonal torus.

The primary purpose of this note is to reveal some of the defining conditions on this set, in particular obtaining an analogue of classical Lidskii-Wielandt inequalities [13]. Let us formulate our result and explain its geometric meaning. For $A, B \in \mathfrak{g}^*_{\mathrm{adm}}$ and $C = A + B$, let $\lambda_i(A), \mu_j(A), \lambda_i(B), \mu_j(B), \lambda_i(C), \mu_j(C)$ be their eigenvalues

in the order as above. Then for each m integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$ and ℓ integers $1 \leq j_1 < j_2 < \cdots < j_\ell \leq q$ we have

$$\sum_{k=1}^{m} \lambda_{i_k}(C) \geq \sum_{k=1}^{m} \lambda_{i_k}(A) + \sum_{k=1}^{m} \lambda_k(B)$$

and

$$\sum_{k=1}^{\ell} \mu_{j_k}(C) \leq \sum_{k=1}^{\ell} \mu_{j_k}(A) + \sum_{k=1}^{\ell} \mu_k(B) .$$

Of course, in addition, we have the trace condition:

$$\sum_{i=1}^{p} \lambda_i(C) + \sum_{j=1}^{q} \mu_j(C) = \sum_{i=1}^{p} \lambda_i(A) + \sum_{j=1}^{q} \mu_j(A) + \sum_{i=1}^{p} \lambda_i(B) + \sum_{j=1}^{q} \mu_j(B) .$$

A particular case of these inequalities for consecutive sets of integers $1, 2, ..., m$ and $1, 2, ..., \ell$ has appeared in Bebiano *et al.*, [2]. We also state a more general analogue of Thompson-Freede inequalities [11]. Recall from [10, Theorem VIII.1.19] that the set of possible diagonal entries of an admissible matrix A with eigenvalues

$$(\overrightarrow{\lambda}, \overrightarrow{\mu}) = (\lambda_1, ..., \lambda_p, \mu_1, ..., \mu_q)$$

as above, form a convex polyhedral set $\mathcal{S}_A$, which can be described as the sum $\Pi + \mathcal{C}$ of a polytope Π and a cone $\mathcal{C}$. The polytope Π is the convex hull of $S_p \times S_q.(\overrightarrow{\lambda}, \overrightarrow{\mu})$, where S_p and S_q are the symmetric groups on p and q elements, - so its vertices are obtained by the action of the Weyl group for the maximal compact subgroup (the product of two symmetric groups in our case). The cone $\mathcal{C}$ is given by the non-compact roots, which in our case means that it is the $\mathbb{R}_+$-span of the diagonal differences $a_{ii} - a_{jj}$ for $1 \leq i \leq p$ and $p+1 \leq j \leq n$. The above inequalities have then the following geometric interpretation: possible eigenvalues of $A + B$ belong to the convex polyhedral region $(\overrightarrow{\lambda}(A), \overrightarrow{\mu}(A)) + \mathcal{S}_B$ (of course, due to symmetry, we can interchange A and B and get another set of conditions).

In this note we only deal with analogues of classical eigenvalue inequalities, leaving out natural questions of relationship with tensor products of representations of G and combinatorics.

2. Courant-Fischer theorem for pseudo-Hermitian matrices. Let $G = \mathrm{U}(p, q)$ be the pseudounitary Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual vector space identified with the space of pseudo-Hermitian matrices A, defined by the condition $A = J_{pq} A^* J_{pq}$, where $J_{pq} = \mathrm{diag}(\underbrace{1, ..., 1}_{p}, \underbrace{-1, ..., -1}_{q})$ and A^* is the conjugate transpose. Let $\mathfrak{g}^*_{\mathrm{adm}}$ denote a convex component of the open cone of admissible elements,

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in the terminology of [10]. In general, an element $A \in \mathfrak{g}^*$ is said to be admissible if the co-adjoint orbit $\mathcal{O}_A$ is closed and its convex hull contains no lines. In the pseudounitary case, this translates to the requirement that the coadjoint orbit of A contains a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_p, ..., \lambda_1, \mu_1, .., \mu_q)$, where $\lambda_p \geq \lambda_{p-1} \geq \cdots \geq \lambda_1$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_q$, and either $\lambda_1 > \mu_1$, or $\mu_q > \lambda_p$. There are two open cone components, and without loss of generality we choose $\mathfrak{g}^*_{\mathrm{adm}}$ to be the component in which $\lambda_1 > \mu_1$.

Let us consider the complex vector space $\mathbb{C}^n$ with the pseudo-Hermitian scalar product of signature (p, q):

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{i=1}^{p} z_i \bar{w}_i - \sum_{j=p+1}^{n} z_j \bar{w}_j \ .$$

If we introduce the notation

$$\mathbf{x}^\dagger = (J_{pq} \bar{\mathbf{x}})^T \ ,$$

then we can rewrite the above pairing in terms of the usual product:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^\dagger \cdot \mathbf{z} \ .$$

Let us also denote by $\mathbb{C}^n_+$ the open cone of positive vectors, satisfying $\langle \mathbf{z}, \mathbf{z} \rangle > 0$, and similarly by $\mathbb{C}^n_-$ the cone of negative vectors. Our condition that A is admissible is equivalent to saying that it has real eigenvalues, and the p eigenvalues corresponding to the eigenvectors in $\mathbb{C}^n_+$ are larger than the q eigenvalues corresponding to the eigenvectors in $\mathbb{C}^n_-$.

Now we shall examine an appropriate analogue of the Rayleigh-Ritz ratio, defined as

$$\mathcal{R}_A(\mathbf{x}) = \frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} \ .$$

This ratio and its properties in the pseudo-Hermitian context were also considered in [1], [2], and [4]. For example, it was established that if $A \in \mathfrak{g}^*_{\mathrm{adm}}$ has the eigenvalues

$$(2.1) \qquad \lambda_p \geq \lambda_{p-1} \geq \cdots \geq \lambda_1 > \mu_1 \geq \mu_2 \geq \cdots \geq \mu_q \ ,$$

then one has

$$\lambda_1 = \min_{\mathbf{x} \in \mathbb{C}^n_+} \mathcal{R}_A(\mathbf{x}) \quad \text{and} \quad \mu_1 = \max_{\mathbf{x} \in \mathbb{C}^n_-} \mathcal{R}_A(\mathbf{x}) \ .$$

Next, let $\mathbf{v}_1, .., \mathbf{v}_p, \mathbf{w}_1, ..., \mathbf{w}_q$ be a basis of eigenvectors of A in $\mathbb{C}^n$, corresponding to the eigenvalues $\lambda_1, ..., \lambda_p, \mu_1, ..., \mu_q$ respectively and orthonormal with respect

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 115-125, February 2010

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to $\langle \cdot, \cdot \rangle$. In particular, we have that $||\mathbf{v}_i||^2 = 1$, $||\mathbf{w}_j||^2 = -1$ and the pairing of any two different vectors from this basis equals zero. Let also, for convenience, denote $V = \mathrm{Span}\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ and $W = \mathrm{Span}\{\mathbf{w}_1, ..., \mathbf{w}_q\}$. Note that for

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p + \beta_1 \mathbf{w}_1 + \cdots \beta_q \mathbf{w}_q$$

the quotient $\mathcal{R}_A(\mathbf{x})$ can be written as

$$\mathcal{R}_A(\mathbf{x}) = \frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} = \frac{\sum_{i=1}^p |\alpha_i|^2 \lambda_i - \sum_{j=1}^q |\beta_j|^2 \mu_j}{\sum_{i=1}^p |\alpha_i|^2 - \sum_{j=1}^q |\beta_j|^2} \ .$$

From the fact that $\langle \cdot, \cdot \rangle$ restricts to a positive definite Hermitian pairing on the subspace V, which is orthogonal to W with respect to $\langle \cdot, \cdot \rangle$, one can easily deduce:

$$\lambda_k = \min_{\mathbf{x} \in \mathbb{C}_+^n, \ \mathbf{x} \perp \mathbf{v}_1, ..., \mathbf{v}_{k-1}} \mathcal{R}_A(\mathbf{x}) \quad \text{and} \quad \lambda_k = \max_{\mathbf{x} \in V \setminus \{0\}, \ \mathbf{x} \perp \mathbf{v}_{k+1}, ..., \mathbf{v}_p} \mathcal{R}_A(\mathbf{x}) \ .$$

A similar statement is, of course, valid for μ_k's:

$$\mu_k = \max_{\mathbf{x} \in \mathbb{C}_-^n, \ \mathbf{x} \perp \mathbf{w}_1, ..., \mathbf{w}_{k-1}} \mathcal{R}_A(\mathbf{x}) \quad \text{and} \quad \mu_k = \min_{\mathbf{x} \in W \setminus \{0\}, \ \mathbf{x} \perp \mathbf{w}_{k+1}, ..., \mathbf{w}_q} \mathcal{R}_A(\mathbf{x}) \ .$$

Now we will state a result similar to the classical Courant-Fischer theorem. A similar result was independently obtained in [2, Theorem 2.3 (I)]. Note that in [1] and [2] a larger class of elements is considered, what they call J-Hermitian matrices with non-interlacing eigenvalues. Our admissible elements fall into their category (I). The proof follows a standard argument and thus is not given. It can also be derived using Hermitian pencils and applying the results of Najman and Ye [9].

THEOREM 2.1. *Let* $A \in \mathfrak{g}_{adm}^*$ *be an admissible pseudo-Hermitian matrix with eigenvalues as in (2.1). Let* k *be an integer,* $1 \le k \le p$. *Then*

$$(2.2) \qquad \lambda_k = \min_{\mathbf{u}_1, ..., \mathbf{u}_{n-k} \in \mathbb{C}^n} \ \max_{\mathbf{x} \in \mathbb{C}_+^n, \ \mathbf{x} \perp \mathbf{u}_1, ..., \mathbf{u}_{n-k}} \mathcal{R}_A(\mathbf{x})$$

$$(2.3) \qquad \lambda_k = \max_{\mathbf{u}_1, ..., \mathbf{u}_{k-1} \in \mathbb{C}^n} \ \min_{\mathbf{x} \in \mathbb{C}_+^n, \ \mathbf{x} \perp \mathbf{u}_1, ..., \mathbf{u}_{k-1}} \mathcal{R}_A(\mathbf{x})$$

Note that, in general, the ratio $\mathcal{R}_A(\mathbf{x})$ is not bounded from above on $\mathbb{C}_+^n$. Therefore in the right hand side of the formula (2.2), the maximum should be taken over the $(n-k)$-tuples of vectors for which it is actually achieved, and otherwise one might want to use sup instead of max.

The above theorem obviously has a natural counterpart, consisting of two series of minimax and maximin identities, for μ_k's. We omit stating and proving those, since it can easily be done if one replaces A by its negative.

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It is also worth noticing that one can rewrite the equality (2.2) in the following form:

$$(2.4) \qquad \lambda_k = \min_{W_k} \ \max_{\mathbf{x} \in \mathbb{C}^n_+, \ \mathbf{x} \in W_k} \ \mathcal{R}_A(\mathbf{x}) \ ,$$

where W_k is a subspace of dimension k, which in fact can be taken entirely lying in $\mathbb{C}^n_+$ (with the exception of the origin, of course).

One can also obtain a result similar to Ky Fan's [3], that for an admissible pseudo-Hermitian matrix A as above, and a positive integer $k \leq p$, one has

$$(2.5) \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_k = \min_{\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}} \ \sum_{i=1}^k \mathcal{R}_A(\mathbf{x}_i) \ .$$

Note that the condition $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}$ automatically implies that all of the $\mathbf{x}_i$'s belong to $\mathbb{C}^n_+$.

As another easy corollary to Theorem 2.1, we have the following analogue of classical Weyl inequalities:

PROPOSITION 2.2. *Let $A, B \in \mathfrak{g}^*_{\mathrm{adm}}$ and let $\lambda_i(A)$, $\mu_j(A)$, $\lambda_i(B)$, $\mu_j(B)$, $\lambda_i(A + B)$, $\mu_j(A + B)$ be the eigenvalues of A, B, and $A + B$ arranged in the order as in (2.1). Then for each $1 \leq k \leq p$ and $1 \leq \ell \leq q$ we have:*

$$\lambda_k(A + B) \geq \lambda_k(A) + \lambda_1(B) \quad \text{and} \quad \mu_\ell(A + B) \leq \mu_\ell(A) + \mu_1(B) \ .$$

3. Lidskii-Wielandt and Thompson-Freede type inequalities. In this section we will establish stronger inequalities for the eigenvalues of the sum of two admissible pseudo-Hermitian matrices. The first goal of this section is to prove the following

THEOREM 3.1. *Let $A, B \in \mathfrak{g}^*_{\mathrm{adm}}$ and let $\lambda_i(A)$, $\mu_j(A)$, $\lambda_i(B)$, $\mu_j(B)$, $\lambda_i(C)$, $\mu_j(C)$ be the eigenvalues of A, B, and $C = A + B$ arranged in the order as in (2.1). Then for each m integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$ and ℓ integers $1 \leq j_1 < j_2 < \cdots < j_\ell \leq q$ we have*

$$(3.1) \qquad \sum_{k=1}^m \lambda_{i_k}(C) \geq \sum_{k=1}^m \lambda_{i_k}(A) + \sum_{k=1}^m \lambda_k(B)$$

and

$$(3.2) \qquad \sum_{k=1}^\ell \mu_{j_k}(C) \leq \sum_{k=1}^\ell \mu_{j_k}(A) + \sum_{k=1}^\ell \mu_k(B) \ .$$

In what follows, we will only work on proving (3.1), as (3.2) is similar.

For $m \leq p$, let us have a fixed m-tuple of integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$. Consider a flag of subspaces $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$, where $V_{i_j} \setminus \{0\} \subset \mathbb{C}_+^n$ and the subscript indicates the dimension of the corresponding subspace. We say that a set of vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$, orthogonal with respect to $\langle \cdot, \cdot \rangle$, is *subordinate* to this flag, if $\mathbf{x}_{i_j} \in V_{i_j}$ and $\langle \mathbf{x}_{i_j}, \mathbf{x}_{i_k} \rangle = \delta_{jk}$.

Denote by P_m the projection operator onto $Y = \mathrm{Span}\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$. Here the projection is taken with respect to $\langle \cdot, \cdot \rangle$, and is therefore given by the matrix $\mathbf{X}\mathbf{X}^\dagger$, where the j-th column of $\mathbf{X}$ is $\mathbf{x}_{i_j}$. For any $A \in \mathfrak{g}^*$, the operator $P_m A P_m$ is also pseudo-Hermitian, but its restriction to Y is actually Hermitian, and we let $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_m$ be its eigenvalues. We have the following analogue of a classical result of Wielandt [13]:

LEMMA 3.2. *For $A \in \mathfrak{g}^*_{\mathrm{adm}}$ with eigenvalues as in (2.1), and η_i's as above, we have*

$$\sum_{j=1}^m \lambda_{i_j} = \min_{V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}} \max_{\mathbf{x}_{i_j} \in V_{i_j}} \sum_{j=1}^m \eta_j \ .$$

We postpone proving this rather technical lemma till the next section, and now state an easy corollary:

PROPOSITION 3.3. *For $A \in \mathfrak{g}^*_{\mathrm{adm}}$ with eigenvalues as in (2.1) and an m-tuple of integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$, one has*

(3.3) $$\sum_{j=1}^m \lambda_{i_j} = \min_{V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}} \max_{\mathbf{x}_{i_j} \in V_{i_j}} \sum_{j=1}^m \mathcal{R}_A(\mathbf{x}_{i_j}) \ .$$

Proof. One can easily see that $\sum_{j=1}^m \mathcal{R}_A(\mathbf{x}_{i_j})$ is exactly the trace of the Hermitian operator $P_m A P_m$ acting on the space Y, because

$$\langle P_m A P_m \mathbf{x}_{i_j}, \mathbf{x}_{i_k} \rangle = \langle A\mathbf{x}_{i_j}, \mathbf{x}_{i_k} \rangle \ ,$$

and as such, equals $\sum_{j=1}^m \eta_j$. □

Now we can establish an analogue of Lidskii-Wieland inequalities and prove Theorem 3.1.

Proof. For a given m-tuple of integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$, let us choose a flag of subspaces $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$ in $\mathbb{C}_+^n$ so that for any orthogonal set of

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vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$ subordinate to this flag, one has

$$\sum_{j=1}^{m} \lambda_{i_j}(C) \geq \sum_{j=1}^{m} \mathcal{R}_C(\mathbf{x}_{i_j}) \ .$$

As Proposition 3.3 shows, this is always possible. Now note that

$$\sum_{j=1}^{m} \mathcal{R}_C(\mathbf{x}_{i_j}) = \sum_{j=1}^{m} \mathcal{R}_A(\mathbf{x}_{i_j}) + \sum_{j=1}^{m} \mathcal{R}_B(\mathbf{x}_{i_j}) \ ,$$

and use Proposition 3.3 once again to choose an orthogonal set of vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2},$ ..., $\mathbf{x}_{i_m}\}$ subordinate to the flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$ such that

$$\sum_{j=1}^{m} \mathcal{R}_A(\mathbf{x}_{i_j}) \geq \sum_{j=1}^{m} \lambda_{i_j}(A).$$

Next, note that (2.5) implies that

$$\sum_{j=1}^{m} \mathcal{R}_B(\mathbf{x}_{i_j}) \geq \sum_{j=1}^{m} \lambda_j(B) \ ,$$

and the result follows. □

A particular case of the above theorem for consecutive sets of integers when $i_k = k$ and $j_k = k$ has independently appeared in [2].

We now state an analogue of Thompson-Freede inequalities [11] (without proof). Let us have two m-tuples of integers $1 \leq i_1 < i_2 < \cdots < i_m \leq p$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq p$ such that $i_m + j_m \leq m + p$. Then

$$\sum_{h=1}^{m} \lambda_{i_h + j_h - h}(C) \geq \sum_{h=1}^{m} \lambda_{i_h}(A) + \sum_{h=1}^{m} \lambda_{j_h}(B) \ .$$

A similar inequality can be stated for μ's as well.

4. Proof of Lemma 3.2. Following the standard path of proving such results as outlined, for example, in the Appendix by B.V. Lidskii to [6], the lemma will follow if we prove the following two statements:

I. For any flag of subspaces $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$ in $\mathbb{C}^n_+$, there exist a subordinate set of vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$, such that

$$\sum_{j=1}^{m} \eta_j \geq \sum_{j=1}^{m} \lambda_{i_j} \ .$$

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II. There exists a flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$ such that for any subordinate set of vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$, one has

$$\sum_{j=1}^{m} \lambda_{i_j} \geq \sum_{j=1}^{m} \eta_j \ .$$

We will first prove II. Set

$$V_{i_j} = \mathrm{Span}\{\mathbf{v}_1, ..., \mathbf{v}_{i_j}\} \ ,$$

where $\mathbf{v}_1, .., \mathbf{v}_p \in \mathbb{C}_+^n$ are eigenvectors of A, corresponding to the eigenvalues $\lambda_1, ...,$ λ_p respectively. Note that $V_{i_j} \setminus \{0\} \subset \mathbb{C}_+^n$. Let $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$ be a set of vectors subordinate to the chosen flag, and let W_ℓ be an ℓ-dimensional subspace in their span. We know from the classical minimax identities that

$$\eta_\ell \leq \max_{\mathbf{x} \in W_\ell} \mathcal{R}_{P_m A P_m}(\mathbf{x}) \ .$$

Note that for $\mathbf{x} \in W_\ell$, we have $\mathcal{R}_{P_m A P_m}(\mathbf{x}) = \mathcal{R}_A(\mathbf{x})$. Thus if we let

$$W_\ell = \mathrm{Span}\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_\ell}\} \ ,$$

then the fact that $W_\ell \subset V_{i_\ell}$ will imply

$$\max_{\mathbf{x} \in W_\ell} \mathcal{R}_A(\mathbf{x}) \leq \max_{\mathbf{x} \in V_\ell} \mathcal{R}_A(\mathbf{x}) \ .$$

But the maximum in the right-hand side is achieved on the eigenvector $\mathbf{v}_{i_\ell}$ and equals λ_{i_ℓ}. (We recall that the operator A is trivially Hermitian on the span of its eigenvectors from $\mathbb{C}_+^n$.) Thus

$$\eta_\ell \leq \max_{\mathbf{x} \in W_\ell} \mathcal{R}_{P_m A P_m}(\mathbf{x}) = \max_{\mathbf{x} \in W_\ell} \mathcal{R}_A(\mathbf{x}) \leq \max_{\mathbf{x} \in V_\ell} \mathcal{R}_A(\mathbf{x}) = \lambda_{i_\ell} \ ,$$

proving II.

Now we turn to proving I, by induction on p. Note that for $p = 1$, the statement amounts to showing that

$$\lambda_1 = \min_{V_1} \ \eta_1 \ ,$$

where V_1 is a one-dimensional subspace in $\mathbb{C}_+^n$. This is not hard to establish directly, and in any case, is an easy consequence of [4, Proposition 4.1].

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Now we can take $m < p$, since in the case when $m = p$, the statement is again a consequence of [4]. We consider two subcases:

1). When $i_m < p$, there exists a $(p-1)$-dimensional subspace R_{p-1} of $\mathbb{C}_+^n$, containing the whole flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$. Let P_{p-1} be the operator of projection onto R_{p-1}, taken with respect to the pseudo-Hermitian scalar product $\langle \cdot, \cdot \rangle$. Consider the pseudo-Hermitian operator $A_{p-1} = P_{p-1}AP_{p-1}$, which is actually Hermitian, being restricted to R_{p-1}. Clearly for all $\mathbf{x} \in R_{p-1}$, one has $\mathcal{R}_{A_{p-1}}(\mathbf{x}) = \mathcal{R}_A(\mathbf{x})$. If we denote by $\xi_1, .., \xi_{p-1}$ the eigenvalues of A_{p-1}, in the non-decreasing order, then according to [4], one has

(4.1) $\xi_i \geq \lambda_i \quad \text{for} \quad 1 \leq i \leq p-1 \ .$

By the induction hypothesis, for any flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m}$ in R_{n-1}, there exists a subordinate system of vectors $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_m}\}$ such that

$$\sum_{j=1}^{m} \eta_j \geq \sum_{j=1}^{m} \xi_{i_j} \ ,$$

and we are done in this case.

2). Now consider the case $i_m = p$. Assume $i_m = p$, $i_{m-1} = p-1$, ..., $i_{m-s} = p-s$ and that the number $(p - s - 1)$ is not a part of the m-tuple $1 \leq i_1 < i_2 < \cdots < i_m \leq p$. Let i_t be the largest remaining element of this m-tuple (the case when there is no such left requires only a minor and trivial modification of our discussion). The corresponding flag of subspaces now takes the form

$$V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_t} \subset V_{i_t+1} \subset \cdots \subset V_p \ .$$

Let $\mathbf{v}_{p-s}, \mathbf{v}_{p-s+1}, ..., \mathbf{v}_p$ be the eigenvectors of A corresponding to the $s+1$ largest eigenvalues. Let R_{n-1} be the subspace of $\mathbb{C}^n$ spanned by these vectors and containing V_{i_t} and all the $\mathbf{w}_j$'s. Such a subspace exists since $i_t \leq p-s-2$ and thus $s+1+i_t \leq p-1$.

Consider yet another flag of subspaces:

(4.2) $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_t} \subset R_{p-s-1} \subset R_{p-s} \subset \cdots \subset R_{p-1} \ ,$

where $R_j = V_{j+1} \cap R_{n-1}$. (In the degenerate case when the dimension of the intersection does not drop by 1, we can artificially remove one extra dimension.)

Again, let us introduce the operator $A_{p-1} = P_{p-1}AP_{p-1}$ on the space R_{p-1} as before. Using our inductive assumption, we can find a subordinate system of vectors

$$\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_t}, \mathbf{x}_{p-s-1}, ..., \mathbf{x}_{p-1}\}$$

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such that

$$\sum_{j=1}^{m} \eta_j \geq \sum_{j=1}^{t} \xi_{i_j} + \sum_{j=p-s-1}^{p-1} \xi_j \ ,$$

where ξ's are the eigenvalues of A_{p-1} arranged in the non-decreasing order. According to (4.1), we have

$$\xi_{i_1} \geq \lambda_{i_1}, \ \xi_{i_2} \geq \lambda_{i_2}, \ ..., \ \xi_{i_t} \geq \lambda_{i_t} \ .$$

The vectors $\mathbf{v}_{p-s}, \mathbf{v}_{p-s+1}, ..., \mathbf{v}_p$ belong to the subspace R_{p-1} and are eigenvectors for A_{p-1}. Thus the corresponding eigenvalues $\lambda_{p-s}, ..., \lambda_p$ are dominated by ξ_{p-s-1}, ..., ξ_{p-1}, which are the largest $(s+1)$ eigenvalues of A_{p-1}. Thus we conclude that

$$\xi_{i_1} + \xi_{i_2} + \cdots + \xi_{i_t} + \xi_{p-s} + \cdots + \xi_p \geq \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_t} + \lambda_{p-s} + \cdots + \lambda_p$$

Since the system $\{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, ..., \mathbf{x}_{i_t}, \mathbf{x}_{p-s-1}, ..., \mathbf{x}_{p-1}\}$ is subordinate not only to the original flag, but also to (4.2), we have completed the proof.

REFERENCES

[1] N. Bebiano, H. Nakazato, J. da Providência, R. Lemos, and G. Soares. Inequalities for *J*-Hermitian matrices. *Linear Algebra Appl.*, 407:125–139, 2005.

[2] N. Bebiano, H. Nakazato, and J. da Providência. On the Courant-Fischer theory for Krein spaces. *Linear Algebra Appl.*, 430:1867–1876, 2009.

[3] Ky Fan. On a theorem of Weyl concerning eigenvalues of linear transformations. *Proc. Nat. Acad. Sci. USA*, 35:652–655, 1949.

[4] P. Foth. Polygons in Minkowski space and the Gelfand-Tsetlin method for pseudo-unitary groups. *J. Geom. Phys.*, 58:825–832, 2008.

[5] W. Fulton. Eigenvalues, invariant factors, heighest weights, and Schubert calculus. *Bull. Amer. Math. Soc.*, 37:209–249, 2000.

[6] F.R. Gantmacher. *Theory of Matrices* [in Russian]. With an Appendix by V.B. Lidskii. Nauka, Moscow, 1967.

[7] R. Horn and C. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[8] M. Kapovich, B. Leeb, and J. Millson. The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra. *Memoirs Amer. Math. Soc.*, 192, 2008.

[9] B. Najman and Q. Ye. A minimax characterization for eigenvalues of hermitian pencils. *Linear Algebra Appl.*, 144:217–230, 1991.

[10] K.-H. Neeb. *Holomorphy and convexity in Lie Theory.* De Gruyter expositions in Mathematics, 28, Walter de Gruyter & Co. Berlin, 2000.

[11] R. Thompson and L. Freede. On the eigenvalues of sums of Hermitian matrices. *Linear Algebra Appl.*, 4:369–376, 1971.

[12] A. Weinsten. Poisson geometry of discrete series orbits, and momentum convexity for noncompact group actions. *Lett. Math. Phys.*, 56:17–30, 2001.

[13] H. Wielandt. An extremum property of sums of eigenvalues. *Proc. Amer. Math. Soc.*, 6:106–110, 1955.