Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

RECOVERING THE CHARACTERISTIC POLYNOMIAL OF A GRAPH FROM ENTRIES OF THE ADJUGATE MATRIX*

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Abstract. The adjugate matrix of G, denoted by $\operatorname{adj}(G)$, is the adjugate of the matrix $x\mathbf{I} - \mathbf{A}$, where $\mathbf{A}$ is the adjacency matrix of G. The polynomial reconstruction problem (PRP) asks if the characteristic polynomial of a graph G can always be recovered from the multiset $\mathcal{PD}(G)$ containing the n characteristic polynomials of the vertex-deleted subgraphs of G. Noting that the n diagonal entries of $\operatorname{adj}(G)$ are precisely the elements of $\mathcal{PD}(G)$, we investigate variants of the PRP in which multisets containing entries from $\operatorname{adj}(G)$ successfully reconstruct the characteristic polynomial of G. Furthermore, we interpret the entries off the diagonal of $\operatorname{adj}(G)$ in terms of characteristic polynomials of graphs, allowing us to solve versions of the PRP that utilize alternative multisets to $\mathcal{PD}(G)$ containing polynomials related to characteristic polynomials of graphs, rather than entries from $\operatorname{adj}(G)$.

Key words. Adjugate matrix, Polynomial reconstruction problem, Characteristic polynomial, Matrix trace.

AMS subject classifications. 05C50, 05C31, 05C83.

1. Introduction. Let G be a simple graph on n vertices, labeled $1, 2, \dots, n$, and let G have m edges. We write $u \overset{G}{\sim} v$ if u and v are two vertices in G that are joined by an edge. If G is implied from the context, then we may write $u \sim v$ instead of $u \overset{G}{\sim} v$ to express the same assertion. If u and v are not joined by an edge in G, then we write $u \overset{G}{\nsim} v$ to state this fact, or simply $u \nsim v$ if G is clear from the context. The entry in the i^{th} row and j^{th} column of the adjacency matrix $\mathbf{A}$ is 1 if $i \overset{G}{\sim} j$ and is 0 if $i \overset{G}{\nsim} j$. We sometimes write $\mathbf{A}_{ij}$ to refer to this entry. Clearly, $\mathbf{A}$ is a symmetric matrix containing zero entries on its diagonal.

The degree of vertex v, denoted by d_v, is the number of vertices adjacent to v. We write $i \overset{G}{\sim} j \overset{G}{\sim} k$ (or $i \sim j \sim k$) if there is an edge joining vertices i and j and an edge joining vertices j and k in G; this notation may be extended as needed. A walk of length ℓ in G between the two vertices u and v is a sequence $u, v_1, v_2, \dots, v_{\ell-1}, v$ of vertices of G such that $u \sim v_1 \sim v_2 \sim \dots \sim v_{\ell-1} \sim v$; the vertices in this sequence do not need to be distinct. The walk is closed if $u = v$. It is well known that the entry $(\mathbf{A}^\ell)_{ij}$ is equal to the number of walks of length ℓ between vertices i and j [6, Proposition 1.3.4]. We denote the number of walks of length two between the distinct vertices i and j by $w_{ij}^{(2)}$; thus, $w_{ij}^{(2)} = (\mathbf{A}^2)_{ij}$. Trivially, there is one walk of length zero between any vertex and itself. In this context, we observe that $\mathbf{A}^0 = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix with columns $\mathbf{e}_1, \dots, \mathbf{e}_n$.

The characteristic polynomial of G, denoted by $\phi(G, x)$, is the determinant of $x\mathbf{I}-\mathbf{A}$, written as $|x\mathbf{I} - \mathbf{A}|$. The vertex-deleted subgraph $G - v$ is the graph obtained after deleting the vertex v and all edges incident to v from G. The two-vertex-deleted subgraph $G - u - v$ is the graph $(G - u) - v$, which is isomorphic to the graph $(G - v) - u$.

The graph $\underset{u \sim v}{G}$ is the graph obtained after introducing an edge between u and v to G, or the graph G itself if $u \overset{G}{\sim} v$. Likewise, the graph $\underset{u \nsim v}{G}$ is the graph obtained after removing the edge joining u and v from G,

*Received by the editors on June 27, 2022. Accepted for publication on October 11, 2022. Handling Editor: Froilán Dopico. Corresponding Author: Alexander Farrugia.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

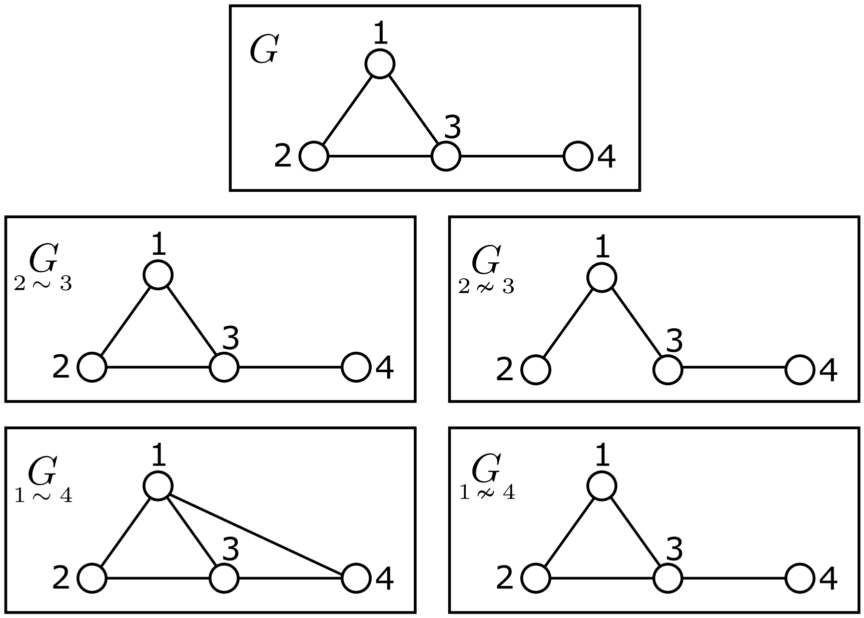

Figure 1. *Illustrating* $\underset{u\sim v}{G}$ *and* $\underset{u\nsim v}{G}$ *vis-a-vis the graph* G.

or the graph G itself if $u \overset{G}{\nsim} v$. The adjacency matrices of $\underset{u\sim v}{G}$ and $\underset{u\nsim v}{G}$ are denoted by $\underset{u\sim v}{\mathbf{A}}$ and $\underset{u\nsim v}{\mathbf{A}}$, respectively. Note that, for any graph G, exactly one of these two adjacency matrices is necessarily equal to $\mathbf{A}$. Figure 1 illustrates the graphs introduced in this paragraph.

The polynomial reconstruction problem (PRP), originally posed in 1973 by Cvetković [4] and independently by Schwenk [17], asks if $\phi(G,x)$ can be recovered from the multiset $\mathcal{PD}(G) = \{\phi(G-1,x),\ldots,\phi(G-n,x)\}$ whenever $n > 2$. (If G_1 is the graph on two vertices joined by a single edge and G_2 is the graph on two vertices and no edges, then $\mathcal{PD}(G_1) = \mathcal{PD}(G_2)$, so the case $n = 2$ is not included in the PRP.) While this problem has been solved in the affirmative for a variety of classes of graphs [3,5,8,18,20–23], it is still an open problem in general. Schwenk [17] expresses his doubt that $\phi(G,x)$ can be recovered from $\mathcal{PD}(G)$ for all graphs, arguing that a large graph will eventually be uncovered which will turn out to be a counterexample to the PRP. He also adds, however, that such a counterexample will be hard to discover. No such graph counterexample has been discovered yet; a computer search revealed that none of the graphs having up to 10 vertices is a counterexample to the PRP [5].

In this paper, we shall be extensively using the adjugate of the matrix $x\mathbf{I}-\mathbf{A}$, written as $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$. Section 2 is devoted to this matrix and its main properties, so we postpone its definition until that section. We shall refer to $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ as the adjugate matrix of G, often denoting it by $\operatorname{adj}(G)$ instead. The adjugate matrix of G, whose entries are all polynomials in x of degree at most $n-1$, contains an extremely rich amount of information about the graph G. It is not surprising, then, that the knowledge of the matrix $\operatorname{adj}(G)$ is equivalent to the knowledge of G itself, as shall be proved in the next section.

As shall be also elaborated upon in the next section, the polynomials contained in $\mathcal{PD}(G)$ are the n diagonal entries of $\operatorname{adj}(G)$. So the PRP can be equivalently restated as asking the question: does the knowledge of the diagonal entries of $\operatorname{adj}(G)$ allow us to deduce $\phi(G,x)$?

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This paper thus takes a different approach to the majority of papers in the literature that tackle the PRP. Rather than attempting to prove or disprove the truth or falsity of the PRP for some class of graphs, we try to recover $\phi(G,x)$ from different minimal sets of entries of the adjugate matrix other than those on its diagonal. We reveal two such sets in Section 4: one is a superset of $\mathcal{PD}(G)$, while the other is completely disjoint from $\mathcal{PD}(G)$. Moreover, in Section 5, we shall provide a meaning to the polynomial entries that are off the diagonal of $\operatorname{adj}(G)$ by writing them in terms of characteristic polynomials of graphs related to G, specifically those of the graphs $G-u-v$, $\underset{u\nsim v}{G}$ and/or $\underset{u\sim v}{G}$ for certain vertices u and v. This allows us to recover $\phi(G,x)$ from multisets containing characteristic polynomials of the aforementioned graphs. To help us in the discovery of these results, Section 2 presents the main properties of the adjugate matrix that is then used in the subsequent sections. Section 3 provides our principal means of determining $\phi(G,x)$ from entries of $\operatorname{adj}(G)$: the matrix traces of the product of $\mathbf{A}^k$ and $\operatorname{adj}(G)$ for nonnegative values of k. Indeed, this paper's main results on the PRP hinge upon Theorem 3.3. The paper finishes with some concluding remarks in Section 6.

2. The adjugate of a graph. The adjugate matrix $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ is the transpose of the matrix of cofactors of the matrix $x\mathbf{I}-\mathbf{A}$ [14]. Since $|x\mathbf{I}-\mathbf{A}|=\phi(G,x)$, it is not the zero polynomial. Thus, $(x\mathbf{I}-\mathbf{A})^{-1}$ exists. This means that the adjugate matrix of G may be alternatively defined as the matrix $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ satisfying

$$(x\mathbf{I}-\mathbf{A})\operatorname{adj}(x\mathbf{I}-\mathbf{A})=(\operatorname{adj}(x\mathbf{I}-\mathbf{A}))(x\mathbf{I}-\mathbf{A})=\phi(G,x)\,\mathbf{I}.$$

By definition of the matrix of cofactors and the fact that the diagonal entries of $(x\mathbf{I}-\mathbf{A})$ are polynomials of degree 1, the entries of $\operatorname{adj}(G)$ are polynomials in x of degree at most $n-1$. Moreover, $\operatorname{adj}(G)$ is a symmetric matrix, since it is the adjugate of a symmetric matrix.

We may also write down $\operatorname{adj}(G)$ as a polynomial in x with appropriate matrix coefficients $\mathbf{M}_0,\dots,\mathbf{M}_{n-1}$, each of which containing solely integers [7]:

$$\operatorname{adj}(x\mathbf{I}-\mathbf{A})=\mathbf{M}_{n-1}x^{n-1}+\mathbf{M}_{n-2}x^{n-2}+\cdots+\mathbf{M}_0.$$

Let $\phi(G,x)=\sum_{j=0}^{n}a_jx^j$ for appropriate coefficients $a_0,\dots,a_n$. We remark that $a_n=1$, $a_{n-1}=0$, $a_{n-2}=-m$, and $a_{n-3}=-2t$, where t is the number of triangles (circuits of size three) in G [13, 16]. By expanding the brackets on the left-hand side of

$$(x\mathbf{I}-\mathbf{A})(\mathbf{M}_{n-1}x^{n-1}+\mathbf{M}_{n-2}x^{n-2}+\cdots+\mathbf{M}_0)=\sum_{j=0}^{n}c_j\mathbf{I}x^j,$$

and by comparing coefficients of $x^n,x^{n-1},x^{n-2},x^{n-3},\dots$, we get

$$\mathbf{M}_{n-1}=\mathbf{I},\quad \mathbf{M}_{n-2}=\mathbf{A},\quad \mathbf{M}_{n-3}=\mathbf{A}^2-m\mathbf{I},\quad \mathbf{M}_{n-4}=\mathbf{A}^3-m\mathbf{A}-2t\mathbf{I},\ \dots\ . \tag{2.1}$$

By using an inductive argument, $\mathbf{M}_k$ is a polynomial in $\mathbf{A}$ of degree $n-1-k$ for all k between 0 and $n-1$. Remembering that the ij^{th} entry of $\mathbf{A}^k$ is equal to the number of walks of length k from vertex i to vertex j, we deduce the following useful result.

THEOREM 2.1. *If there is a walk between vertices i and j in G, then the ij^{th} entry of $\operatorname{adj}(G)$ is a polynomial of degree $n-1-k$, where k is the length of the* **shortest** *walk from vertex i to vertex j. Moreover, the leading coefficient of this polynomial is the number of possible shortest walks between i and j. If there are no walks between i and j, then the ij^{th} entry of $\operatorname{adj}(G)$ is zero.*

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Theorem 2.1 is saying that for any two vertices i and j, the leading term of the ij^{th} entry of $\operatorname{adj}(G)$ contains the number of shortest paths from i to j in its coefficient, as well as the distance between i and j in its index (after subtracting it from $n-1$). Thus, the distance matrix of G is easily deduced from $\operatorname{adj}(G)$ as being the matrix whose ij^{th} entry is $n-1-d$, where d is the degree of the ij^{th} entry of $\operatorname{adj}(G)$.

In view of this, the following corollary is immediate.

Corollary 2.2. *The polynomials on the diagonal of the adjugate matrix of G have degree $n-1$. Moreover, if vertices i and j are connected by an edge in G, then the ij^{th} entry of* $\operatorname{adj}(G)$ *is a polynomial of degree $n-2$; otherwise, it is a polynomial of degree at most $n-3$.*

Thus, the adjacency matrix $\mathbf{A}$ (and hence, the graph G) may be readily deduced from $\operatorname{adj}(G)$ by replacing each polynomial entry of degree $n-2$ in $\operatorname{adj}(G)$ with 1 and replacing all the other entries with 0.

We shall also focus on the coefficient of x^{n-3} of entries of $\operatorname{adj}(G)$ quite often. Here is an important theorem.

Theorem 2.3. *The coefficient of x^{n-3} in the entry* $\operatorname{adj}(G)_{ij}$ *is equal to*

$$\begin{cases} d_i - m, & \text{if } i = j \\ w_{ij}^{(2)}, & \text{if } i \neq j. \end{cases}$$

Proof. From (2.1), the matrix of coefficients of x^{n-3} in $\operatorname{adj}(G)$ is $\mathbf{M}_{n-3} = \mathbf{A}^2 - m\mathbf{I}$. The i^{th} diagonal entry of $\mathbf{M}_{n-3}$ is thus $(\mathbf{A}^2)_{ii} - m$, and clearly $(\mathbf{A}^2)_{ii} = d_i$. Moreover, if $i \neq j$, then the ij^{th} entry of $\mathbf{M}_{n-3}$ is equal to $(\mathbf{A}^2)_{ij}$, or, equivalently, $w_{ij}^{(2)}$. ∎

It can be seen, then, that the adjugate matrix contains a rich amount of information about the graph G.

3. Traces of matrices. The trace of a square matrix $\mathbf{M}$, denoted by $\operatorname{tr}(\mathbf{M})$, is the sum of its diagonal entries. Let the two matrices $\mathbf{M}$ and $\mathbf{N}$ be such that $\mathbf{MN}$ is a square matrix and let $\mathbf{M} \circ \mathbf{N}^\mathsf{T}$ denote the entrywise product of $\mathbf{M}$ and $\mathbf{N}^\mathsf{T}$. Then, we have the following result that links the trace of matrices with entrywise products of matrices.

Lemma 3.1 ([7]). $\operatorname{tr}(\mathbf{MN})$ *is equal to the sum of all the entries of* $\mathbf{M} \circ \mathbf{N}^\mathsf{T}$.

Since $\mathbf{M} \circ \mathbf{N}^\mathsf{T} = \mathbf{N} \circ \mathbf{M}^\mathsf{T}$, $\operatorname{tr}(\mathbf{MN}) = \operatorname{tr}(\mathbf{NM})$ for all conformable matrices $\mathbf{M}$ and $\mathbf{N}$.

We now consider the traces of various matrices and matrix products in the following subsections.

3.1. The trace of $\operatorname{adj}(x\mathbf{I} - \mathbf{A})$**.** Jacobi's formula (not to be confused by Jacobi's theorem — see section 5) states that if $\mathbf{B}(x)$ is a matrix whose entries are functions of the variable x, then [14, p.28–29]:

$$\frac{d}{dx}|\mathbf{B}(x)| = \operatorname{tr}\big(\operatorname{adj}(\mathbf{B}(x))\,\mathbf{B}'(x)\big)\,.$$

In the above formula, $\mathbf{B}'(x)$ is the matrix where, for all i and j, its ij^{th} entry is the derivative of the ij^{th} entry of $\mathbf{B}(x)$ with respect to x. If we let $\mathbf{B}(x)$ be the matrix $x\mathbf{I} - \mathbf{A}$, then clearly $\mathbf{B}'(x) = \mathbf{I}$ and we obtain

$$\frac{d}{dx}|x\mathbf{I} - \mathbf{A}| = \operatorname{tr}\big(\operatorname{adj}(x\mathbf{I} - \mathbf{A})\big).$$

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In other words,

$$\operatorname{tr}\big(\operatorname{adj}(x\mathbf{I}-\mathbf{A})\big)=\phi'(G,x), \tag{3.2}$$

where $\phi'(G,x)$ is the derivative of $\phi(G,x)$ with respect to x.

As we state at the start of Section 4, the i^{th} diagonal entry of $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ is equal to $\phi(G-i,x)$. Hence, the above argument provides an alternative proof of the result by Clarke [1], which is also proved in [6, Theorem 2.3.1] by other means.

3.2. The trace of $\mathbf{A}\operatorname{adj}(x\mathbf{I}-\mathbf{A})$. We start from the equation $(x\mathbf{I}-\mathbf{A})(x\mathbf{I}-\mathbf{A})^{-1}=\mathbf{I}$ and expand the brackets:

$$\begin{aligned}x(x\mathbf{I}-\mathbf{A})^{-1}-\mathbf{A}(x\mathbf{I}-\mathbf{A})^{-1}&=\mathbf{I}\\ \mathbf{A}(x\mathbf{I}-\mathbf{A})^{-1}&=x(x\mathbf{I}-\mathbf{A})^{-1}-\mathbf{I}.\\ \mathbf{A}\operatorname{adj}(x\mathbf{I}-\mathbf{A})&=x\operatorname{adj}(x\mathbf{I}-\mathbf{A})-\phi(G,x)\,\mathbf{I}.\end{aligned}$$

By taking the trace on both sides of the last equation and using (3.2), we discover that

$$\operatorname{tr}\big(\mathbf{A}\operatorname{adj}(x\mathbf{I}-\mathbf{A})\big)=x\,\phi'(G,x)-n\,\phi(G,x). \tag{3.3}$$

Note that the right-hand side of (3.3) is zero if and only if G has no edges.

3.3. The trace of $\mathbf{A}^2\operatorname{adj}(x\mathbf{I}-\mathbf{A})$. We start from the equation $(x\mathbf{I}-\mathbf{A})^2(x\mathbf{I}-\mathbf{A})^{-1}=x\mathbf{I}-\mathbf{A}$ this time:

$$\begin{aligned}x^2(x\mathbf{I}-\mathbf{A})^{-1}-2x\,\mathbf{A}(x\mathbf{I}-\mathbf{A})^{-1}+\mathbf{A}^2(x\mathbf{I}-\mathbf{A})^{-1}&=x\mathbf{I}-\mathbf{A}.\\ \mathbf{A}^2(x\mathbf{I}-\mathbf{A})^{-1}&=x\mathbf{I}-\mathbf{A}-x^2(x\mathbf{I}-\mathbf{A})^{-1}+2x\,\mathbf{A}(x\mathbf{I}-\mathbf{A})^{-1}.\\ \mathbf{A}^2\operatorname{adj}(x\mathbf{I}-\mathbf{A})&=x\,\phi(G,x)\,\mathbf{I}-\phi(G,x)\,\mathbf{A}-x^2\operatorname{adj}(x\mathbf{I}-\mathbf{A})+2x\,\mathbf{A}\operatorname{adj}(x\mathbf{I}-\mathbf{A}).\end{aligned}$$

We again take the trace on both sides of the last equation. By using (3.2) and (3.3) and simplifying, we arrive at (3.4) below:

$$\operatorname{tr}\big(\mathbf{A}^2\operatorname{adj}(x\mathbf{I}-\mathbf{A})\big)=nx\,\phi(G,x)-0-x^2\phi'(G,x)+2x(x\,\phi'(G,x)-n\,\phi(G,x)).$$

$$\operatorname{tr}\big(\mathbf{A}^2\operatorname{adj}(x\mathbf{I}-\mathbf{A})\big)=x(x\,\phi'(G,x)-n\,\phi(G,x)). \tag{3.4}$$

What we said before for (3.3) is also true here: the right-hand side of (3.4) is zero if and only if G has no edges.

3.4. The Trace of $\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})$. The technique used in the previous two subsections may be extended to obtain, for all $k\geq 3$, the trace of $\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ in terms of the matrix traces of $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$, $\mathbf{A}\operatorname{adj}(x\mathbf{I}-\mathbf{A}),\ldots,\mathbf{A}^{k-1}\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ by starting from the equation:

$$(x\mathbf{I}-\mathbf{A})^k\,(x\mathbf{I}-\mathbf{A})^{-1}=(x\mathbf{I}-\mathbf{A})^{k-1}.$$

However, since the right-hand side is an expression involving the matrices $\mathbf{I},\mathbf{A},\mathbf{A}^2,\ldots,\mathbf{A}^{k-1}$, the traces of these matrices will form part of the expression for the trace of $\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})$. We recall that, for all j, the trace of $\mathbf{A}^j$ is the number of closed walks of length j in G. For the cases $k=1$ and $k=2$, we only required the matrix traces of $\mathbf{I}$ and $\mathbf{A}$, which are n and 0, respectively.

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A publication of the International Linear Algebra Society
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Because of this, we can obtain a relatively simple closed-form equation for the trace of $\mathbf{A}^k \operatorname{adj}(x\mathbf{I}-\mathbf{A})$ in terms of $\phi(G,x)$ and $\phi'(G,x)$.

THEOREM 3.2. *For any nonnegative integer k,*

$$\operatorname{tr}\Big(\mathbf{A}^k \operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big) = x^k\,\phi'(G,x) - \left(\sum_{j=0}^{k-1} w_j x^{k-1-j}\right)\phi(G,x), \tag{3.5}$$

where w_j is the number of closed walks of length j in G.

Proof. We use induction on k. When $k=0$, the result in (3.5) matches that of (3.2). So suppose (3.5) is true for $k=0,1,\ldots,\ell$ and consider $\operatorname{tr}\Big(\mathbf{A}^{\ell+1}\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big)$. By expanding both sides of the equation $(x\mathbf{I}-\mathbf{A})^{\ell+1}\operatorname{adj}(x\mathbf{I}-\mathbf{A}) = \phi(G,x)\,(x\mathbf{I}-\mathbf{A})^{\ell}$, we obtain

$$\begin{aligned}(-1)^{\ell+1}\operatorname{tr}\Big(\mathbf{A}^{\ell+1}\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big) &= \phi(G,x)\left(\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}w_j x^{\ell-j}\right)\\ &\quad+\left(\sum_{j=0}^{\ell}(-1)^{j+1}\binom{\ell+1}{j}x^{\ell+1-j}\operatorname{tr}\big(\mathbf{A}^j\operatorname{adj}(x\mathbf{I}-\mathbf{A})\big)\right).\end{aligned} \tag{3.6}$$

By the inductive hypothesis, the right-hand side of (3.6) becomes

$$\phi(G,x)\left(\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}w_j x^{\ell-j}\right)+\left(\sum_{j=0}^{\ell}(-1)^{j+1}\binom{\ell+1}{j}x^{\ell+1-j}\left(x^j\,\phi'(G,x)-\left(\sum_{i=0}^{j-1}w_i x^{j-1-i}\right)\phi(G,x)\right)\right),$$

which, after grouping terms in $\phi'(G,x)$ and in $\phi(G,x)$, becomes

$$x^{\ell+1}\,\phi'(G,x)\left(\sum_{j=0}^{\ell}(-1)^{j+1}\binom{\ell+1}{j}\right)+\phi(G,x)\left(\sum_{j=0}^{\ell}(-1)^j\left(\binom{\ell}{j}w_j x^{\ell-j}+\binom{\ell+1}{j}\left(\sum_{i=0}^{j-1}w_i x^{\ell-i}\right)\right)\right).$$

If we substitute $t=-1$ in the identity $(1+t)^{\ell+1}=\sum_{j=0}^{\ell+1}\binom{\ell+1}{j}t^j$, we obtain $\sum_{j=0}^{\ell+1}(-1)^j\binom{\ell+1}{j}=0$. Hence $\sum_{j=0}^{\ell}(-1)^{j+1}\binom{\ell+1}{j}=(-1)^{\ell+1}$.

Let us now focus on the coefficient of x^q in the expression:

$$\sum_{j=0}^{\ell}(-1)^j\left(\binom{\ell}{j}w_j x^{\ell-j}+\binom{\ell+1}{j}\left(\sum_{i=0}^{j-1}w_i x^{\ell-i}\right)\right), \tag{3.7}$$

where $0\le q\le\ell$. This is equal to

$$(-1)^{\ell-q}w_{\ell-q}\left(\binom{\ell}{q}-\binom{\ell+1}{q}+\binom{\ell+1}{q-1}-\binom{\ell+1}{q-2}+\cdots+(-1)^q\binom{\ell+1}{1}\right).$$

Since $\binom{\ell+1}{p}=\binom{\ell}{p}+\binom{\ell}{p-1}$ for all p, the above expression in brackets telescopes to $(-1)^q\binom{\ell+1}{0}=(-1)^q$. Thus, the coefficient of x^q of the expression (3.7) is $(-1)^\ell w_{\ell-q}$ for all q.

Hence,

$$(-1)^{\ell+1}\operatorname{tr}\Big(\mathbf{A}^{\ell+1}\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big)=(-1)^{\ell+1}\left(x^{\ell+1}\,\phi'(G,x)-\phi(G,x)\left(\sum_{q=0}^{\ell}w_{\ell-q}x^q\right)\right),$$

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which may be rewritten as:

$$\operatorname{tr}\Big(\mathbf{A}^{\ell+1}\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big) = x^{\ell+1}\,\phi'(G,x) - \left(\sum_{j=0}^{\ell} w_j x^{\ell-j}\right)\phi(G,x).$$

The induction is complete. □

From (3.2), the derivative of $\phi(G,x)$ is deducible from the trace of $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$, and hence, by integration, $\phi(G,x)$ itself may be obtained, except perhaps its constant term. If k is a positive integer, then $\phi(G,x)$ is derivable from $\operatorname{tr}\Big(\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big)$ in its entirety if we also have the number of closed walks in G of length $0,1,2,\ldots,k-1$ at our disposal.

Theorem 3.3. *If k is any positive integer, then the characteristic polynomial of G is deducible from the trace of the matrix $\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ and the number of closed walks of length $0,1,2,\ldots,k-1$ in G.*

Proof. Let $\phi(G,x)$ be the polynomial $\sum_{q=0}^{n} c_q x^q$. Then, $\phi'(G,x)=\sum_{q=0}^{n} q\,c_q x^{q-1}$. Clearly, $c_n=1$ and $c_{n-1}=0$. Moreover, $w_0=n$ and $w_1=0$. By Theorem 3.2, the trace of $\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ is equal to

$$\left(\sum_{q=0}^{n} q\,c_q x^{q+k-1}\right) - \left(w_0x^{k-1}+w_1x^{k-2}+\cdots+w_{k-1}\right)\left(\sum_{q=0}^{n} c_q x^q\right). \tag{3.8}$$

Thus, $\operatorname{tr}\Big(\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big)$ is a polynomial of degree at most $n+k-1$. Suppose it is the polynomial $\sum_{q=0}^{n+k-1} a_q x^q$. By comparing the coefficients of x^{n+k-1} and of x^{n+k-2} in (3.8), we discover that $a_{n+k-1}=0$ and $a_{n+k-2}=0$. Hence, the degree of $\operatorname{tr}\Big(\mathbf{A}^k\operatorname{adj}(x\mathbf{I}-\mathbf{A})\Big)$ is at most $n+k-3$. Moreover, by comparing the coefficients of $x^{n+k-3}, x^{n+k-4},\ldots,x^{k-1}$ in (3.8), we obtain the following system of $n-1$ equations in the $n-1$ unknowns $c_{n-2},\ldots,c_0$:

$$\begin{aligned}
a_{n+k-3} &= -2\,c_{n-2} - w_1\,c_{n-1} - w_2\,c_n.\\
a_{n+k-4} &= -3\,c_{n-3} - w_1\,c_{n-2} - w_2\,c_{n-1} - w_3\,c_n.\\
a_{n+k-5} &= -4\,c_{n-4} - w_1\,c_{n-3} - w_2\,c_{n-2} - w_3\,c_{n-1} - w_4\,c_n.\\
\vdots\quad &\quad \vdots \qquad\qquad \vdots\\
a_n &= -(k-1)\,c_{n-k+1} - w_1\,c_{n-k+2} - w_2\,c_{n-k+3} - \cdots - w_{k-1}\,c_n.\\
a_{n-1} &= -k\,c_{n-k} - w_1\,c_{n-k+1} - w_2\,c_{n-k+2} - \cdots - w_{k-1}\,c_{n-1}.\\
a_{n-2} &= -(k+1)\,c_{n-k-1} - w_1\,c_{n-k} - w_2\,c_{n-k+1} - \cdots - w_{k-1}\,c_{n-2}.\\
\vdots\quad &\quad \vdots \qquad\qquad \vdots\\
a_{k-1} &= -n\,c_0 - w_1\,c_1 - w_2\,c_2 - w_3\,c_3 - \cdots - w_{k-1}\,c_{k-1}.
\end{aligned}$$

(The above system of equations assumes that $k>n$. If that is not the case, then all equations would involve $w_1,\ldots,w_{k-1}$.) The coefficient c_{n-2} is obtained from the first of the above system of equations. This value of c_{n-2} in the second equation yields c_{n-3}. Substituting the values of c_{n-2} and c_{n-3} in the third equation yields c_{n-4}. Continuing this process, all the coefficients of $\phi(G,x)$ are eventually obtained. □

4. Back to the polynomial reconstruction problem. By definition of the adjugate matrix, the i^{th} diagonal entry of $\operatorname{adj}(x\mathbf{I}-\mathbf{A})$ is the principal minor of $(x\mathbf{I}-\mathbf{A})$ arising from deleting its i^{th} row and

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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columns — in other words, it is $\phi(G-i,x)$. In Section 5, we shall also be associating the polynomials off the diagonal of $\text{adj}(G)$ with characteristic polynomials of certain graphs related to G.

By Corollary 2.2 and (3.2), the trace of the adjugate matrix is the sum of the characteristic polynomials of the n vertex-deleted subgraphs. Hence,

$$\phi'(G,x) = \sum_{i=1}^{n} \phi(G-i,x). \tag{4.9}$$

We thus have:

Theorem 4.1 ([6, p. 35]). *All the coefficients of $\phi(G,x)$ are recoverable from $\mathcal{PD}(G)$, except possibly the coefficient of x^0.*

We have seen in Theorem 3.3 that $\phi(G,x)$ is recoverable from $\text{tr}\Big(\mathbf{A}^k \,\text{adj}(x\mathbf{I}-\mathbf{A})\Big)$ and $w_0,\ldots,w_{k-1}$. It is known that $w_0,\ldots,w_{n-1}$ may be actually deduced from $\mathcal{PD}(G)$ [2,8]. We now proceed to use Theorem 3.1 to interpret the trace of $\mathbf{A}^k \,\text{adj}(x\mathbf{I}-\mathbf{A})$ for $k=1$ and for $k=2$:

Theorem 4.2. $\text{tr}\,(\mathbf{A}\ \text{adj}(x\mathbf{I}-\mathbf{A})) = 2\sum_{i\sim j}\text{adj}(x\mathbf{I}-\mathbf{A})_{ij}$.

Proof. The entry $\mathbf{A}_{ij}$ is 1 if $i\sim j$ and is 0 otherwise. The result follows by noting that $\mathbf{A}_{ij}=\mathbf{A}_{ji}$ and applying Lemma 3.1. ∎

By Theorem 2.1, the entries of $\text{adj}(x\mathbf{I}-\mathbf{A})$ referred to in Theorem 4.2 are exactly those of degree $n-2$.

Theorem 4.3. *$\phi(G,x)$ is recoverable from the multiset $\mathcal{S}$ containing all the polynomials of degree $n-2$ above the main diagonal of* $\text{adj}(G)$. *Indeed,*

$$\phi(G,x) = x^n + \sum_{k=0}^{n-2} a_k\, x^k,$$

where, for all $0\le k\le n-2$, a_k is the product of $\frac{2}{k-n}$ and the coefficient of x^k in the sum of all the polynomials in $\mathcal{S}$.

Proof. From Theorem 4.2 and (3.3),

$$2\sum_{i\sim j}\text{adj}(x\mathbf{I}-\mathbf{A})_{ij} = x\,\phi'(G,x) - n\,\phi(G,x). \tag{4.10}$$

By Theorem 2.1, the entry $\text{adj}(x\mathbf{I}-\mathbf{A})_{ij}$ is of degree $n-2$ whenever there is a walk of length one — that is, an edge — between vertices i and j in G. Hence, n is two more than the degree of any polynomial in $\mathcal{S}$. Since the right-hand side of (4.10) is equal to $\sum_{k=0}^{n-2}(k-n)a_k\,x^k$, the result follows. ∎

Remark 4.4. *In Section 2, we had observed that we may recover G itself given the matrix* $\text{adj}(G)$ *by locating its entries of degree $n-2$. In this respect, Theorem 4.3 looks pointless at first sight, as it only recovers $\phi(G,x)$ and not G from the entries of degree $n-2$ in* $\text{adj}(G)$. *However, the entry positions of the polynomials of degree $n-2$ in* $\text{adj}(G)$ *are crucial to the reconstruction of G. In Theorem 4.3, we are only given the polynomials of degree $n-2$ in* $\text{adj}(G)$ *as a multiset of m elements, without any ordering or anything that lets us infer the original locations of these polynomials in* $\text{adj}(G)$. *By Theorem 4.3, we can still recover $\phi(G,x)$ from these m polynomials taken from* $\text{adj}(G)$ *without knowing from which entry positions they were taken from.*

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A publication of the International Linear Algebra Society
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THEOREM 4.5. $\operatorname{tr}\left(\mathbf{A}^2 \operatorname{adj}(x\mathbf{I}-\mathbf{A})\right) = \left(\sum_{i=1}^n d_i\,\phi(G-i,x)\right) + 2\sum_{\substack{i\sim j\sim k\\ i\neq k}} w_{ik}^{(2)} \operatorname{adj}(x\mathbf{I}-\mathbf{A})_{ik}$.

Proof. The entry $(\mathbf{A}^2)_{ij}$ is the number of walks of length two in G between vertices i and j. If $i=j$, then $(\mathbf{A}^2)_{ii} = d_i$. If $i \neq j$, then $(\mathbf{A}^2)_{ij} = w_{ij}^{(2)}$. The result then follows using Lemma 3.1, recalling that the entries on the diagonal of $\operatorname{adj}(G)$ are $\phi(G-1,x),\ldots,\phi(G-n,x)$. ∎

By Theorem 2.3, only the entries of $\operatorname{adj}(G)$ having degree $n-1$, $n-2$ or $n-3$ are possibly among those described on the right-hand side of the equation in Theorem 4.5. Those of degree $n-1$ are on the diagonal of $\operatorname{adj}(G)$, by Corollary 2.2; these are the polynomials in the first summation on the right-hand side of Theorem 4.5. Again by Corollary 2.2, the entries of $\operatorname{adj}(G)$ of degree $n-2$ in the second summation of this equation would correspond to vertices i and k that are joined by an edge, and they would also have $w_{ik}^{(2)}$ walks of length two between them. Thus, these edges would be on at least one triangle of vertices $i\sim j\sim k\sim i$ for some vertex j. Finally, the polynomials of degree $n-3$ in the second summation of Theorem 4.5 correspond to vertices i and k in G that are not joined by an edge but that would have $w_{ik}^{(2)}$ walks of length two between them.

Moreover, by Theorem 2.3, if i,j,k are distinct vertices in G such that $i\sim j\sim k$, then $w_{ik}^{(2)}$ is the coefficient of x^{n-3} in the polynomial $\operatorname{adj}(G)_{ik}$. Furthermore, the vertex degrees $d_1,\ldots,d_n$ are deducible from the coefficients of x^{n-3} of $\phi(G-1,x),\ldots,\phi(G-n,x)$ as well, in the following manner.

LEMMA 4.6 ([3,12]). *The degree sequence $d_1,\ldots,d_n$ of a graph G is deducible from $\mathcal{PD}(G)$.*

Proof. By the Sachs coefficient theorem [16], the coefficient of x^{n-3} in $\phi(G-i,x)$ is $-m_i$, where m_i is the number of edges in $G-i$. However, by Theorem 2.3, the same coefficient is equal to d_i-m; hence $d_i = m-m_i$. This means that if we deduce m, then we are done. But m is revealed by evaluating the polynomial $\sum_{i=1}^n \phi(G-i,x)$, which is equal to $\phi'(G,x)$ by (4.9). The coefficient of x^{n-3} in this polynomial is $(d_1-m)+(d_2-m)+\cdots+(d_n-m) = 2m-nm = m(2-n)$. This provides us with the value of m we require. ∎

We thus have the following result.

THEOREM 4.7. *$\phi(G,x)$ is recoverable from the multiset consisting of all the polynomial entries on the diagonal of $\operatorname{adj}(G)$, together with the entries above the diagonal of $\operatorname{adj}(G)$ whose coefficient of x^{n-3} is nonzero.*

Proof. By Theorem 4.5 and (3.4),

$$\left(\sum_{i=1}^n d_i\,\phi(G-i,x)\right) + 2\sum_{\substack{i\sim j\sim k\\ i\neq k}} w_{ik}^{(2)} \operatorname{adj}(x\mathbf{I}-\mathbf{A})_{ik} = x^2\phi'(G,x) - nx\,\phi(G,x). \tag{4.11}$$

From the explanation prior to the statement of this theorem, as well as by applying Theorem 2.3 and Lemma 4.6, the polynomials conveyed on the left-hand side of (4.11), together with the values of all the d_i's and the $w_{ik}^{(2)}$'s, are all possible to obtain from the multiset described in the theorem statement. So let the polynomial on the left-hand side of (4.11) be $\sum_{k=0}^{n-1} b_k x^k$ for appropriate coefficients $b_0,\ldots,b_{n-1}$. The right-hand side of (4.11) is $\sum_{k=0}^{n-2}(k-n)a_k\,x^{k+1}$, where, for all $0\le k\le n-2$, a_k is the coefficient of x^k in $\phi(G,x)$. Hence, $a_k = \frac{b_{k+1}}{k-n}$ for all k between 0 and $n-2$. This successfully recovers $\phi(G,x)$, because the remaining coefficients of $\phi(G,x)$, those of x^n and of x^{n-1}, are known to be 1 and 0, respectively. ∎

Theorem 4.7 has a more direct significance to the PRP than Theorem 4.3, because the entries of $\operatorname{adj}(G)$ of degree $n-1$ that are located on its main diagonal comprise $\mathcal{PD}(G)$, and these entries are among those in the multiset described in the theorem.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

In the proof of Theorem 4.7, the coefficient of x^0 in the polynomial of the left-hand side of (4.11) is not used to obtain $\phi(G,x)$. Indeed, this coefficient must be zero, since it is zero on the right-hand side of (4.11). This provides us with the following corollary.

Corollary 4.8. *Let $\mathbf{A}-i$ be the adjacency matrix of $G-i$ for any vertex i in G having degree d_i. Then,*

$$\sum_{i=1}^{n} d_i\,|\mathbf{A}-i| = 2\sum_{\substack{i\sim j\sim k\\ i\neq k}} w_{ik}^{(2)}\,\mathrm{adj}(\mathbf{A})_{ik}.$$

We remark that the left-hand side of the formula in Corollary 4.8 is accessible from $\mathcal{PD}(G)$, while the right-hand side is not. Thus, given $\mathcal{PD}(G)$, the above corollary provides a small hint as to the off-diagonal entries of $\mathrm{adj}(\mathbf{A})$ and the number of walks of length two between distinct vertices in G, which may possibly be useful in the context of the PRP.

We can actually do better than Theorem 4.7, since, apart from the polynomials of degree $n-1$ of the adjugate matrix, we do not actually need the entire polynomials of the remaining polynomials from $\mathrm{adj}(x\mathbf{I}-\mathbf{A})$ having a nonzero coefficient of x^{n-3} in order to reconstruct $\phi(G,x)$, but only a couple of coefficients from each.

Theorem 4.9. *$\phi(G,x)$ is recoverable from $\mathcal{PD}(G)$ together with the p ordered pairs $\left(c_{n-3}^{(1)}, c_1^{(1)}\right),\ldots,\left(c_{n-3}^{(p)}, c_1^{(p)}\right)$, where the polynomials $\sum_{i=0}^{n-2} c_i^{(1)}x^i,\ldots,\sum_{i=0}^{n-2} c_i^{(t)}x^i$ are the p entries above the main diagonal of $\mathrm{adj}(G)$ having $c_{n-3}^{(i)}\neq 0$.*

Proof. Using (4.9) and Theorem 4.1, we first deduce $\phi'(G,x)$ from $\mathcal{PD}(G)$, from which all the coefficients of $\phi(G,x)$, except a_0, the coefficient of x^0, are revealed to us. If we consider only the x terms on both sides of (4.11), we obtain

$$\left(\sum_{i=1}^{n} d_i\,b_1^{(i)}\right) + 2\sum_{i=1}^{p} c_{n-3}^{(i)}\,c_1^{(i)} = -na_0,$$

where $b_i^{(i)}$ is the coefficient of x of $\phi(G-i,x)$. Recall that the values of d_i, for $i\in\{1,2,\ldots,n\}$, may be obtained by Lemma 4.6. Since n is the number of polynomials in $\mathcal{PD}(G)$, a_0 is deduced, as required. □

5. Writing entries of $\mathrm{adj}(G)$ in terms of characteristic polynomials of graphs. As we have seen, the diagonal entries of the adjugate matrix are the characteristic polynomials of the vertex-deleted subgraphs of G. Apart from the result conveyed in Theorem 2.1, however, the entries off the diagonal of $\mathrm{adj}(G)$ have still not been described in terms of characteristic polynomials of graphs. We now proceed to do so.

One relatively popular formula worth mentioning, which has been recently used successfully by Sciriha and her coauthors to research the conductance or insulation of hydrocarbon molecules [9–11, 15, 19], is a corollary to Jacobi's theorem. Jacobi's theorem [14, p. 26] states that if we have a $(p+q)\times(p+q)$ matrix $\mathbf{M}$ and we consider the upper left $p\times p$ submatrix $\mathbf{N}$ of $\mathbf{M}$ and the lower right $q\times q$ submatrix $\mathbf{P}$ of $\mathrm{adj}(\mathbf{M})$, then $|\mathbf{P}| = |\mathbf{M}|^{q-1}\,|\mathbf{N}|$. If we apply Jacobi's theorem to the matrix $\mathbf{M}=x\mathbf{I}-\mathbf{A}$, letting $p=n-2$ and $q=2$, then we deduce the following result.

Theorem 5.1 ([19]). *If $i\neq j$, then the square of the entry $\mathrm{adj}(x\mathbf{I}-\mathbf{A})_{ij}$ is equal to*

$$\phi(G-i,x)\,\phi(G-j,x) - \phi(G,x)\,\phi(G-i-j,x). \tag{5.12}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

By Theorems 2.1 and 5.1, the entry $\mathrm{adj}(x\mathbf{I}-\mathbf{A})_{ij}$ is the square root of (5.12) that has a positive leading coefficient. In this manner, the entry $\mathrm{adj}(G)_{ij}$ is a square root of a polynomial expression in terms of $\phi(G,x)$, $\phi(G-i,x)$, $\phi(G-j,x)$ and $\phi(G-i-j,x)$.

In this paper, we shall be using another formula for $\mathrm{adj}(G)_{ij}$ that, to our knowledge, is little known in the algebraic graph theory community. The formula used in this paper will not contain any square roots.

We start with Cauchy's formula for the determinant of a rank-one perturbation [14, Equation 0.8.5.11]. It states that for any square matrix $\mathbf{M}$ and any vectors $\mathbf{u}$ and $\mathbf{v}$ of conformable sizes:

$$\left|\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal\right| = |\mathbf{M}|\left(1+\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{u}\right). \tag{5.13}$$

Consider $\left|\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal+\mathbf{v}\mathbf{u}^\intercal\right|$. By (5.13), this is equal to

$$\left|\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal\right|\left(1+\mathbf{u}^\intercal\left(\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal\right)^{-1}\mathbf{v}\right).$$

To evaluate $\left(\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal\right)^{-1}$, we resort to a version of the Sherman–Morrison–Woodbury formula [14, Equation 0.7.4.2]:

$$\left(\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal\right)^{-1} = \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^\intercal\mathbf{M}^{-1}}{1+\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{u}}.$$

Hence,

$$\left|\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal+\mathbf{v}\mathbf{u}^\intercal\right| = |\mathbf{M}|\left(1+\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{u}\right)\left(1+\mathbf{u}^\intercal\mathbf{M}^{-1}\mathbf{v} - \frac{\left(\mathbf{u}^\intercal\mathbf{M}^{-1}\mathbf{u}\right)\left(\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{v}\right)}{1+\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{u}}\right).$$

Assuming that $\mathbf{M}$ is a symmetric matrix, we can simplify the above to

$$\left|\mathbf{M}+\mathbf{u}\mathbf{v}^\intercal+\mathbf{v}\mathbf{u}^\intercal\right| = |\mathbf{M}|\left(\left(1+\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{u}\right)^2 - \left(\mathbf{u}^\intercal\mathbf{M}^{-1}\mathbf{u}\right)\left(\mathbf{v}^\intercal\mathbf{M}^{-1}\mathbf{v}\right)\right). \tag{5.14}$$

Let $\mathbf{M}=x\mathbf{I}-\mathbf{A}$, $\mathbf{u}=\mathbf{e}_j$ and $\mathbf{v}=\pm\mathbf{e}_i$. Substituting these matrices in (5.14), we get

$$\left|x\mathbf{I}-\left(\mathbf{A}\pm\mathbf{e}_i\mathbf{e}_j^\intercal\pm\mathbf{e}_j\mathbf{e}_i^\intercal\right)\right| = \phi(G,x)\left(\left(1\mp(x\mathbf{I}-\mathbf{A})^{-1}_{ij}\right)^2 - \left((x\mathbf{I}-\mathbf{A})^{-1}_{ii}\right)\left((x\mathbf{I}-\mathbf{A})^{-1}_{jj}\right)\right). \tag{5.15}$$

Let us expand the term $\left(1\mp(x\mathbf{I}-\mathbf{A})^{-1}_{ij}\right)^2$ in (5.15):

$$\begin{aligned}\left(1\mp(x\mathbf{I}-\mathbf{A})^{-1}_{ij}\right)^2 &= 1\mp\frac{2\,\mathrm{adj}(x\mathbf{I}-\mathbf{A})_{ij}}{|x\mathbf{I}-\mathbf{A}|} + \frac{\left(\mathrm{adj}(x\mathbf{I}-\mathbf{A})_{ij}\right)^2}{\left(|x\mathbf{I}-\mathbf{A}|\right)^2}\\ &= 1\mp\frac{2\,\mathrm{adj}(x\mathbf{I}-\mathbf{A})_{ij}}{\phi(G,x)} + \frac{\phi(G-i,x)\,\phi(G-j,x)-\phi(G,x)\,\phi(G-i-j,x)}{\left(\phi(G,x)\right)^2},\end{aligned}$$

by Theorem 5.1. Moreover, $(x\mathbf{I}-\mathbf{A})^{-1}_{ii}$ and $(x\mathbf{I}-\mathbf{A})^{-1}_{jj}$ are $\frac{\phi(G-i,x)}{\phi(G,x)}$ and $\frac{\phi(G-j,x)}{\phi(G,x)}$, respectively.

Note that if the vertices i and j in G are joined by an edge, then $\mathbf{A}-\mathbf{e}_i\mathbf{e}_j^\intercal-\mathbf{e}_j\mathbf{e}_i^\intercal = \underset{i\nsim j}{\mathbf{A}}$, the adjacency matrix of the graph obtained from G after removing the edge joining i and j. On the other hand, if there is no edge joining vertices i and j in G, then $\mathbf{A}+\mathbf{e}_i\mathbf{e}_j^\intercal+\mathbf{e}_j\mathbf{e}_i^\intercal = \underset{i\sim j}{\mathbf{A}}$, the adjacency matrix of the graph obtained from G after joining i and j by an edge.

In view of all the above, (5.15) becomes the following two equations, depending on whether $i \overset{G}{\nsim} j$ or $i \overset{G}{\sim} j$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

THEOREM 5.2. *If $i \neq j$, then*

$$2\,\mathrm{adj}(x\mathbf{I} - \underset{i\nsim j}{\mathbf{A}})_{ij} = \phi(\underset{i\nsim j}{G}, x) - \phi(\underset{i\sim j}{G}, x) - \phi(G-i-j, x); \text{ and}$$

$$-2\,\mathrm{adj}(x\mathbf{I} - \underset{i\sim j}{\mathbf{A}})_{ij} = \phi(\underset{i\sim j}{G}, x) - \phi(\underset{i\nsim j}{G}, x) - \phi(G-i-j, x).$$

REMARK 5.3. *For any graph G, $\phi(G-i-j,x) = \phi(\underset{i\sim j}{G} - i - j, x) = \phi(\underset{i\nsim j}{G} - i - j, x)$. Moreover, recall that if $i \overset{G}{\sim} j$, then $\underset{i\sim j}{G}$ is isomorphic to G, while if $i \overset{G}{\nsim} j$, then $\underset{i\nsim j}{G}$ is isomorphic to G.*

The following two corollaries emerge directly from Theorem 5.2.

COROLLARY 5.4. $\phi(G-i-j,x) = \mathrm{adj}(\underset{i\sim j}{G})_{ij} - \mathrm{adj}(\underset{i\nsim j}{G})_{ij}$.

COROLLARY 5.5. $\phi(\underset{i\nsim j}{G}, x) - \phi(\underset{i\sim j}{G}, x) = \mathrm{adj}(\underset{i\sim j}{G})_{ij} + \mathrm{adj}(\underset{i\nsim j}{G})_{ij}$.

The two corollaries just presented provide remarkably simple expressions for the characteristic polynomial of the two-vertex-deleted subgraph $G-i-j$ and for the difference between the characteristic polynomials of $\underset{i\nsim j}{G}$ and $\underset{i\sim j}{G}$ in terms of the ij^{th} entries of the adjugate matrices of the latter two graphs. This illustrates how closely related the entries of the adjugate matrix are to the characteristic polynomials of these graphs.

We now revisit our main results of Section 4 and apply Theorem 5.2 to them.

THEOREM 5.6. *If $m < 3$ or $m > n$, then $\phi(G,x)$ is uniquely determined from the multiset $\mathcal{S}$ consisting of the m polynomials $\left\{\phi(\underset{u\nsim v}{G}, x) + \phi(G-u-v, x)\right\}$ for all pairs of vertices (u,v) in G that are joined by an edge. If $3 \leq m \leq n$, then all the coefficients of $\phi(G,x)$ bar the coefficient of x^{n-m} are recoverable from $\mathcal{S}$.*

Proof. Note that m is the number of elements in $\mathcal{S}$, while n is the degree of each element in $\mathcal{S}$. From (4.10) and Theorem 5.2,

$$\sum_{u\sim v}\Big(-\phi(G,x) + \phi(\underset{u\nsim v}{G}, x) + \phi(G-u-v,x)\Big) = x\,\phi'(G,x) - n\,\phi(G,x).$$

Equivalently,

$$\sum_{u\sim v}\Big(\phi(\underset{u\nsim v}{G}, x) + \phi(G-u-v,x)\Big) = x\,\phi'(G,x) + (m-n)\,\phi(G,x). \tag{5.16}$$

Let the coefficients of $x^n, x^{n-1}, x^{n-2}, x^{n-3}, \ldots, x^0$ of $\phi(G,x)$ be $1, 0, -m, a_{n-3}, \ldots, a_0$. If, for all k between 0 and $n-3$, the coefficient of x^k in the polynomial $\phi(\underset{u\nsim v}{G}, x) + \phi(G-u-v,x)$ is b_k, then by comparing coefficients in (5.16), $b_k = (m-n+k)a_k$ for all k. Thus, a_k is recoverable from b_k if $k \neq n-m$. If $m < 3$ or $m > n$, then this inequality will always be satisfied and $\phi(G,x)$ is recovered completely. Otherwise, all the coefficients of $\phi(G,x)$ are recovered except for the coefficient of x^{n-m}. ∎

It is interesting that Theorem 5.6 fails to reconstruct one coefficient from $\phi(G,x)$ if G has few edges. We remark that graphs having less than $n-1$ edges are necessarily disconnected. Results on the PRP for disconnected graphs were reported in [8, 20]. A graph having $n-1$ edges is either disconnected or a tree. An early positive result on the PRP is that if G is a tree, then $\phi(G,x)$ is recoverable from $\mathcal{PD}(G)$ [5].

From the proof of the above theorem, we obtain the following corollary.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

Corollary 5.7. *If $3 \le m \le n$, then the coefficients of x^{n-m} in the two polynomials $\sum_{u\sim v} \phi(\underset{u\nsim v}{G}, x)$ and $\sum_{u\sim v} \phi(G-u-v, x)$ have opposite signs.*

Applying Theorems 5.2 to 4.7 leads to the following result.

Theorem 5.8. *Let G be a graph on at least four vertices and let V be the set in which each element is a pair of distinct vertices (u, v) such that there is at least one walk of length two between u and v in G. Let S_1 be the set:*

$$\left\{\phi(\underset{u\sim v}{G}, x) + \phi(G-u-v, x) \mid (u,v) \in V, u \overset{G}{\nsim} v\right\},$$

and S_2 be the set:

$$\left\{\phi(\underset{u\nsim v}{G}, x) + \phi(G-u-v, x) \mid (u,v) \in V, u \overset{G}{\sim} v\right\}.$$

If the number of walks of length two between all pairs of vertices in G that are joined by an edge is different from the number of walks of length two between all pairs of vertices in G that are not joined by an edge, then $\phi(G, x)$ is uniquely determined from the multiset $\mathcal{PD}(G) \cup S_1 \cup S_2$.

Proof. We first obtain $\phi'(G, x)$ from $\mathcal{PD}(G)$ using (4.9). Thus, only a_0, the coefficient of x^0 in $\phi(G, x)$, is left for us to deduce.

From (4.11) and Theorem 5.2,

$$\left(\sum_{i=1}^{n} d_i\, \phi(G-i, x)\right) + \sum_{\substack{u\sim w\sim v \\ u\neq v,\, u\nsim v}} w_{uv}^{(2)} \left(\phi(G, x) - \phi(\underset{u\sim v}{G}, x) - \phi(G-u-v, x)\right)$$
$$- \sum_{\substack{u\sim w\sim v \\ u\neq v,\, u\sim v}} w_{uv}^{(2)} \left(\phi(G, x) - \phi(\underset{u\nsim v}{G}, x) - \phi(G-u-v, x)\right) = x^2\, \phi'(G, x) - nx\, \phi(G, x).$$

By Lemma 4.6, the coefficient of x^{n-3} in $\phi'(G, x)$ is $-m(n-2)$, so m is deducible from $\mathcal{PD}(G)$, and so are the vertex degrees d_i, $1 \le i \le n$.

Note that if $(u, v) \in V$ and $u \overset{G}{\nsim} v$, then the coefficient of x^{n-2} in the polynomial $\phi(\underset{u\sim v}{G}, x)+\phi(G-u-v, x)$ is $-m$, while if $(u, v) \in V$ and $u \overset{G}{\sim} v$, then the coefficient of x^{n-2} in the polynomial $\phi(\underset{u\nsim v}{G}, x)+\phi(G-u-v, x)$ is $-m+2$. Hence, the polynomials of degree n in $S_1 \cup S_2$ arising from the pairs of vertices that are connected by an edge in G are distinguishable from those that are not.

Let H_{uv} be the graph $\underset{u\sim v}{G}$ if $u \overset{G}{\nsim} v$ or the graph $\underset{u\nsim v}{G}$ if $u \overset{G}{\sim} v$. The coefficient of x^{n-3} in the expression $\phi(G, x) - \phi(H_{uv}, x) - \phi(G-u-v, x)$ is $\pm 2\, w_{uv}^{(2)}$, where the sign is positive if $u \overset{G}{\nsim} v$ and is negative otherwise. Since the coefficient of x^{n-3} in $\phi(G-u-v, x)$ is 0 for all vertices u and v, the value $\pm 2\, w_{uv}^{(2)}$ is equal to the coefficient of x^{n-3} in $\phi(G, x)$ minus the coefficient of x^{n-3} in $\phi(H_{uv}, x)$. (Observe that since $n > 3$, the coefficient of x^{n-3} in $\phi(G, x)$ may be obtained from $\phi'(G, x)$ by integration.) Hence, the value of each $w_{uv}^{(2)}$ may be evaluated for all the polynomials in $S_1 \cup S_2$.

Now let $s = \sum_{\substack{u\sim w\sim v \\ u\neq v,\, u\nsim v}} w_{uv}^{(2)} - \sum_{\substack{u\sim w\sim v \\ u\neq v,\, u\sim v}} w_{uv}^{(2)}$. This number is the difference between the number of walks of length two between vertices that are joined by an edge and the number of walks of length two between

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 697-711, October 2022.

vertices that are not joined by an edge. Then, we have

$$\left(\sum_{i=1}^{n} d_i\,\phi(G-i,x)\right) - \sum_{\substack{u\sim w\sim v\\ u\neq w,\, u\nsim w}} w_{uv}^{(2)}\left(\phi(\underset{u\sim v}{G},x)+\phi(G-u-v,x)\right) + \sum_{\substack{u\sim w\sim v\\ u\neq w,\, u\sim w}} w_{uv}^{(2)}\left(\phi(\underset{u\nsim v}{G},x)+\phi(G-u-v,x)\right) = x^2\,\phi'(G,x)-(nx+s)\,\phi(G,x). \tag{5.17}$$

Let $b_0^{(i)}$ be the coefficient of x^0 in $\phi(G-i,x)$ for all i and let $c_0^{(uv)}$ be the coefficient of x^0 in the polynomial $\phi(H_{uv},x)+\phi(G-u-v,x)$ for all u and v. If we substitute $x=0$ in (5.17), we obtain

$$\left(\sum_{i=1}^{n} d_i\, b_0^{(i)}\right) - \sum_{\substack{u\sim w\sim v\\ u\neq v,\, u\nsim v}} w_{uv}^{(2)}\,c_0^{(uv)} + \sum_{\substack{u\sim w\sim v\\ u\neq v,\, u\sim v}} w_{uv}^{(2)}\,c_0^{(uv)} = -s\,a_0.$$

Thus, if $s\neq 0$, then we recover a_0 from the above equation, as required. ∎

6. Conclusion. By considering the trace of $\mathbf{A}^k\,\mathrm{adj}(x\mathbf{I}-\mathbf{A})$ for various values of k, new ways of reconstructing $\phi(G,x)$ from various multisets of characteristic polynomials — some including $\mathcal{PD}(G)$, some excluding it — were presented in this paper. We hope that, by illustrating these techniques and showing the myriad of properties that $\mathrm{adj}(G)$ possesses, we have provided a new perspective on the PRP, together with further tools to aid in the possible resolution of this problem.

As far as we know, the only other paper that finds $\phi(G,x)$ from a superset of $\mathcal{PD}(G)$ is [12] by Hagos; it shows that $\phi(G,x)$ can be recovered from the pair $(\mathcal{PD}(G),\mathcal{PD}(\overline{G}))$, where $\overline{G}$ is the complement of G. However, technically, the paper [12] does not extend the set $\mathcal{PD}(G)$ to a larger set, but keeps the two multisets $\mathcal{PD}(G)$ and $\mathcal{PD}(\overline{G})$ separate, so that each element in $\mathcal{PD}(G)$ is known to be a characteristic polynomial of a vertex-deleted subgraph of G, not of $\overline{G}$, and vice versa. We did not do this in this paper; our multiset was always a single set, rather than pairs of separate multisets.

In future work, one may be interested in using Theorems 3.2 and 3.3 with $k\geq 3$ to deduce other ways of obtaining $\phi(G,x)$ from different multisets consisting of entries taken from the adjugate matrix.

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