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KANTOROVICH TYPE INEQUALITIES FOR ORDERED LINEAR SPACES*

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Abstract. In this paper Kantorovich type inequalities are derived for linear spaces endowed with bilinear operations $\circ_1$ and $\circ_2$. Sufficient conditions are found for vector-valued maps Φ and Ψ and vectors x and y under which the inequality

$$\Phi(x) \circ_2 \Phi(y) \leq \frac{C+c}{2\sqrt{Cc}}\, \Psi(x \circ_1 y)$$

is satisfied. Complementary inequalities are also given. Some results of Dragomir [J. Inequal. Pure Appl. Math., 5 (3), Art. 76, 2004] and Bourin [Linear Algebra Appl., 416:890–907, 2006] are generalized. The inequalities are applied to C^*-algebras and unital positive maps.

Key words. Kantorovich type inequality, Linear space, Bilinear operation, Preorder, C^*-algebra, Unital positive map, Matrix.

AMS subject classifications. 06F20, 15A45, 15A42, 15A48.

1. Introduction. Let A be an $n \times n$ positive definite matrix such that $0 < mI_n \leq A \leq MI_n$ for some scalars $0 < m < M$. The Kantorovich inequality asserts that (cf. [16, pp. 89-90], [20, p. 28])

$$z^*Az \cdot z^*A^{-1}z \leq \frac{(M+m)^2}{4Mm}\, (z^*z)^2, \tag{1.1}$$

where $z \in \mathbb{C}^n$ is a column vector and * means conjugate transpose. The constant $\kappa = \frac{(M+m)^2}{4Mm}$ is called *Kantorovich constant* [21, p. 688]. Note that $\sqrt{\kappa} = \frac{M+m}{2\sqrt{Mm}}$ is the ratio of the arithmetic to geometric mean of M and m.

Let V be a linear space over $\mathbb{C}$ or $\mathbb{R}$ equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. Dragomir [11, Theorem 2.2] proved the following Kantorovich type inequality:

$$\|x\|\|y\| \leq \frac{|C+c|}{2\sqrt{\mathrm{Re}\,(C\bar{c})}}\, |\langle x, y \rangle| \quad \text{for } x, y \in V, \tag{1.2}$$

provided scalars c, C satisfy $\mathrm{Re}\,(C\bar{c}) > 0$ and

$$0 \leq \mathrm{Re}\,\langle x - cy, Cy - x \rangle \tag{1.3}$$

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(cf. [12, Theorem 1]). As observed in [12, p. 225], (1.2) generalizes Pólya-Szegö, Greub-Reinboldt and Cassels inequalities.

Inequality (1.2) is a reverse of Schwarz's inequality

$$|\langle x, y\rangle| \le \|x\|\|y\| \quad \text{for } x, y \in V. \tag{1.4}$$

A consequence of (1.4) and (1.2) is the following result of Bourin [5, Theorem 2.9]:

$$\sum_{j=1}^{n} a_{[j]} b_{[j]} \le \frac{M+m}{2\sqrt{Mm}} \sum_{j=1}^{n} a_j b_j, \tag{1.5}$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are n-tuples of positive numbers with $0 < m \le \frac{a_j}{b_j} \le M$, $j = 1, \ldots, n$, and, in addition, $a_{[1]} \ge \ldots \ge a_{[n]}$ and $b_{[1]} \ge \ldots \ge b_{[n]}$ are the entries of a and b, respectively, arranged in nonicreasing order.

For other Kantorovich type inequalities, the reader is referred to [2, 5, 6, 7, 16, 18, 20, 21].

In this paper we study Kantorovich type inequalities in the framework of linear spaces equipped with binary operations $\circ_1$ and $\circ_2$. We provide conditions on two (vector-valued) maps Φ and Ψ and vectors x and y implying the validity of the inequality

$$\Phi(x) \circ_2 \Phi(y) \le \frac{C+c}{2\sqrt{Cc}}\, \Psi(x \circ_1 y). \tag{1.6}$$

Complementary inequalities are also derived.

2. Results.

Throughout this paper, unless otherwise stated, for $i = 1, 2$,

$$V_i \text{ and } X_i \text{ are linear spaces over } \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R},$$

and

$$\circ_i : V_i \times V_i \to X_i \text{ is an } \mathbb{F}\text{-bilinear binary operation.}$$

For example, $\circ_i$ can be interpret as a real inner product if $X_i = \mathbb{R}$, or as an algebra multiplication if $V_i = X_i$ is a distributive algebra.

In addition, we assume that $L_i \subset X_i$ is a convex cone inducing cone preorder $\le_i$ on X_i by

$$y \le_i x \text{ iff } x - y \in L_i.$$

We also assume that

$$0 \le_i x \circ_i x, \quad \text{i.e., } x^2 = x \circ_i x \in L_i, \quad \text{for } x \in V_i. \tag{2.1}$$

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We denote

$$\operatorname{Sym}(u,w) = \frac{1}{2}(u \circ_2 w + w \circ_2 u) \quad \text{for } u,w \in V_2. \tag{2.2}$$

The following theorem is inspired by [11, Theorem 2.2] (cf. [12, Theorem 1]).

Theorem 2.1. *Under the above notation and assumptions, let $\Phi : \mathcal{A} \to V_2$ and $\Psi : \mathcal{B} \to X_2$ be maps, where $\mathcal{A} \subset V_1$ and $\mathcal{B} \subset X_1$ are nonempty sets. Let $x, y \in \mathcal{A}$ and $C, c \in \mathbb{F}$ with $Cc > 0$ and $C + c > 0$ be such that*

(i)

$$0 \leq_1 (x - cy) \circ_1 (Cy - x), \tag{2.3}$$

(ii) $x \circ_1 y = y \circ_1 x$,
(iii) $L_1 \subset \mathcal{B}$ *and* $\alpha x \circ_1 y \in \mathcal{B}$ *for* $\alpha \in \{1, C + c\}$.

Assume that

$$\Phi(v) \circ_2 \Phi(v) \leq_2 \Psi(v \circ_1 v) \quad \textit{for } v \in \{x, y\}, \tag{2.4}$$

$$b \leq_1 a \quad \textit{implies} \quad \Psi(b) \leq_2 \Psi(a) \quad \textit{for } a, b \in L_1, \tag{2.5}$$

$$\Psi(\alpha a) = \alpha \Psi(a) \quad \textit{for } \alpha = C + c \textit{ and } a = x \circ_1 y, \tag{2.6}$$

$$\Psi(x \circ_1 x) + \alpha \Psi(y \circ_1 y) \leq_2 \Psi(x \circ_1 x + \alpha y \circ_1 y) \quad \textit{for } \alpha = Cc. \tag{2.7}$$

Then the following Kantorovich type inequality holds:

$$\operatorname{Sym}[\Phi(x), \Phi(y)] \leq_2 \frac{C + c}{2\sqrt{Cc}} \, \Psi(x \circ_1 y). \tag{2.8}$$

In particular, if $\Phi(x)$ and $\Phi(y)$ commute with respect to $\circ_2$, then

$$\Phi(x) \circ_2 \Phi(y) \leq_2 \frac{C + c}{2\sqrt{Cc}} \, \Psi(x \circ_1 y). \tag{2.9}$$

Remark 2.2. In some cases Theorem 2.1 can be simplified.

(a). If Ψ is linear then conditions (2.6)-(2.7) hold automatically and are superfluous in the statement of Theorem 2.1. If in addition Ψ is positive (i.e. $\Psi(L_1) \subset L_2$) then (2.5) can be dropped out.

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(b). If $\Phi = \Psi$ then condition (2.4) represents a Kadison type inequality (see (2.19)). On the other hand, if $\Phi(x) = [\Psi(x^2)]^{1/2}$ then (2.4) holds automatically (cf. Corollary 2.5 and Theorem 2.7, part II).

(c). Condition (2.4) is necessary for (2.8) and (2.9) to hold. In fact, if $x = y$ then (2.3) is met for $c = C = 1$. In this case, each of (2.8) and (2.9) reduces to (2.4).

Proof of Theorem 2.1. Since the operation $\circ_1$ is bilinear, (2.3) gives

$$0 \le_1 C\, x \circ_1 y - x \circ_1 x - Cc\, y \circ_1 y + c\, y \circ_1 x,$$

which is equivalent to

$$x \circ_1 x + Cc\, y \circ_1 y \le_1 C\, x \circ_1 y + c\, y \circ_1 x,$$

because $\le_1$ is a cone preorder. Now, **(ii)** implies

$$x \circ_1 x + Cc\; y \circ_1 y \le_1 (C + c) x \circ_1 y. \tag{2.10}$$

By (2.1), $x \circ_1 x + Cc\; y \circ_1 y \in L_1$, because $Cc > 0$ and L_1 is a convex cone. Therefore (2.10) yields $(C + c)x \circ_1 y \in L_1$. Using (2.7), (2.10), (2.5) and (2.6), we derive

$$\Psi(x \circ_1 x) + Cc\; \Psi(y \circ_1 y) \le_2 \Psi(x \circ_1 x + Cc\, y \circ_1 y) \le_2 (C + c)\Psi(x \circ_1 y).$$

Consequently, by (2.4), we obtain

$$\Phi(x) \circ_2 \Phi(x) + Cc\; \Phi(y) \circ_2 \Phi(y) \le_2 (C + c)\Psi(x \circ_1 y). \tag{2.11}$$

Hence, by $Cc > 0$,

$$\frac{1}{\sqrt{Cc}}\Phi(x) \circ_2 \Phi(x) + \sqrt{Cc}\,\Phi(y) \circ_2 \Phi(y) \le_2 \frac{C + c}{\sqrt{Cc}}\; \Psi(x \circ_1 y). \tag{2.12}$$

On the other hand, by (2.1),

$$0 \le_2 \left(\frac{1}{\sqrt[4]{Cc}}\; \Phi(x) - \sqrt[4]{Cc}\; \Phi(y) \right) \circ_2 \left(\frac{1}{\sqrt[4]{Cc}}\; \Phi(x) - \sqrt[4]{Cc}\; \Phi(y) \right).$$

In consequence, by the bilinearity of $\circ_2$,

$$0 \le_2 \frac{1}{\sqrt{Cc}}\; \Phi(x) \circ_2 \Phi(x) - \Phi(x) \circ_2 \Phi(y) - \Phi(y) \circ_2 \Phi(x) + \sqrt{Cc}\; \Phi(y) \circ_2 \Phi(y).$$

Hence

$$\Phi(x) \circ_2 \Phi(y) + \Phi(y) \circ_2 \Phi(x) \le_2 \frac{1}{\sqrt{Cc}}\; \Phi(x) \circ_2 \Phi(x) + \sqrt{Cc}\; \Phi(y) \circ_2 \Phi(y),$$

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because $\leq_2$ is induced by a convex cone. Simultaneously, by (2.2),

$$2\mathrm{Sym}\,[\Phi(x),\Phi(y)] = \Phi(x)\circ_2\Phi(y) + \Phi(y)\circ_2\Phi(x).$$

Therefore we get

$$2\mathrm{Sym}\,[\Phi(x),\Phi(y)] \leq_2 \frac{1}{\sqrt{Cc}}\,\Phi(x)\circ_2\Phi(x) + \sqrt{Cc}\,\Phi(y)\circ_2\Phi(y). \tag{2.13}$$

Combining (2.12) and (2.13), we obtain the required inequality (2.8). □

Remark 2.3. Let H be a real linear space with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. It is not hard to verify that Dragomir's result (1.2) (with $\mathbb{F}=\mathbb{R}$ and $C,c>0$) can be obtained from Theorem 2.1 by setting

$$V_1 = H,\ \ V_2 = X_1 = X_2 = \mathbb{R},\ \ L_1 = L_2 = \mathbb{R}_+,$$

$$x\circ_1 y = \langle x,y\rangle \quad \text{for } x,y\in H, \quad \text{and} \quad \alpha\circ_2\beta = \alpha\beta \quad \text{for } \alpha,\beta\in\mathbb{R},$$

$$\Phi(x) = \|x\| \quad \text{for } x\in H, \quad \text{and} \quad \Psi(\alpha) = |\alpha| \quad \text{for } \alpha\in\mathbb{R}.$$

In this case, (2.11) takes the form of inequality from [12, Lemma 1].

If X_i is an algebra with unity e_i and convex cone $L_i\subset X_i$ $(i=1,2)$, then a linear map $\Psi: X_1\to X_2$ is said to be a *unital positive map* if $\Psi(e_1)=e_2$ and $\Psi L_1\subset L_2$.

Theorem 2.4. *Under the assumptions before Theorem 2.1, let $V_i = X_i$ and let $(V_i,\circ_i)$ be algebra with unity e_i $(i=1,2)$.*

Let $x\in V_1$ be such that

$$0 \leq_1 (x - ce_1)\circ_1(Ce_1 - x) \tag{2.14}$$

for some scalars $C,c\in\mathbb{F}$ with $Cc>0$ and $C+c>0$.

Assume that $\Psi: V_1\to V_2$ is a positive linear map (i.e., $\Psi L_1\subset L_2$) and $\Phi: V_1\to V_2$ is a unital map (i.e., $\Phi(e_1)=e_2$) satisfying

$$\Phi(x)\circ_2\Phi(x) \leq_2 \Psi(x\circ_1 x) \quad \textit{and} \quad e_2 \leq_2 \Psi(e_1). \tag{2.15}$$

Then we have the inequality

$$\Phi(x) \leq_2 \frac{C+c}{2\sqrt{Cc}}\,\Psi(x). \tag{2.16}$$

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Proof. Set $y = e_1$. Conditions (2.5)-(2.7) are fulfilled, because Ψ is a positive linear map. Moreover, (2.15) gives (2.4). According to Theorem 2.1, we get (2.9) with $y = e_1$ and $\Phi(y) = e_2$. This proves (2.16). □

COROLLARY 2.5. *Under the assumptions of Theorem 2.4 for V_i, X_i, L_i, $\circ_i$ and x, suppose that for each $a \in L_2$ there exists unique vector $b = a^{1/2} \in L_2$ such that $b^2 = b \circ_2 b = a$.*

Assume $\Psi : V_1 \to V_2$ is a unital positive map. If (2.14) is met then we have the inequality

$$[\Psi(x^2)]^{1/2} \leq_2 \frac{C+c}{2\sqrt{Cc}}\, \Psi(x). \tag{2.17}$$

Proof. Define

$$\Phi(v) = [\Psi(v^2)]^{1/2} \quad \text{for } v \in V_1. \tag{2.18}$$

Then Φ is unital, since Ψ is so. It follows from (2.18) that (2.15) holds. Now, by using (2.16), we get (2.17). □

By $\mathbb{M}_p$ and $\mathbb{H}_p$ we denote the linear spaces, respectively, of $p \times p$ complex matrices, and of $p \times p$ Hermitian matrices. The Loewner cone of all $p \times p$ positive semidefinite matrices is denoted by $\mathbb{L}_p$. For matrices $A, B \in \mathbb{M}_p$ we write $B \leq A$ if $A - B \in \mathbb{L}_p$. The symbol I_p stands for the $p \times p$ identity matrix.

Remind that a linear map $\Psi : \mathbb{M}_n \to \mathbb{M}_k$ is said to be a *unital positive map* if $\Psi(I_n) = I_k$ and $\Psi\mathbb{L}_n \subset \mathbb{L}_k$ (see [4, 14]). It is known that

$$[\Psi(A)]^2 \leq \Psi(A^2) \quad \text{for } A \in \mathbb{L}_n \tag{2.19}$$

(Kadison's inequality; see [1], [4, p. 2], [8]).

REMARK 2.6. **(a)** In the matrix setting, (2.17) reduces to a result of Ando [1]. Cf. also [6, Corollaries 2.5 and 2.9] and [17, Corollary 2.6, part (ii), $p = 2$].

(b) Inequality (2.17) generalizes a result of Liu and Neudecker [15, Proposition 5] (see also [6, Lemma 1.1]):

$$(U^*X^2U)^{1/2} \leq \frac{M+m}{2\sqrt{Mm}}\, U^*XU\,, \tag{2.20}$$

where U is an $n \times k$ matrix such that $U^*U = I_k$, and X is an $n \times n$ positive definite matrix satisfying

$$0 < m \leq \lambda_j(X) \leq M,\ j = 1, \ldots, n, \quad \text{for some scalars } m, M. \tag{2.21}$$

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To see this, consider

$$V_1 = X_1 = \mathbb{M}_n, \;\; V_2 = X_2 = \mathbb{M}_k, \;\; L_1 = \mathbb{L}_n, \;\; L_2 = \mathbb{L}_k,$$

with the usual matrix multiplication, and

$$\Psi(A) = U^*AU \quad \text{for } A \in \mathbb{M}_n,$$

where U is an $n \times k$ matrix such that $U^*U = I_k$.

We now interpret Theorem 2.1 in the framework of C^*-algebras V_i, $i = 1, 2$, and unital positive maps. Here, for given $x, y \in V_i$, $y \leq x$ means $x - y = a^*a$ for some $a \in V_i$.

THEOREM 2.7. *For $i = 1, 2$, let $V_i = X_i$ be a C^*-algebra with unity e_i and convex cone $L_i = \{a^*a : a \in V_i\}$ of all nonnegative elements of V_i.*

Let $x, y \in V_1$ be two elements such that $x^*y = y^*x$ and

$$(x - cy)^*(Cy - x) \geq 0 \quad \textit{for some positive scalars } C, c. \tag{2.22}$$

Assume that $\Psi : V_1 \to V_2$ is a unital positive map.

(I). *If*

$$(\Psi(v))^*\Psi(v) \leq \Psi(v^*v) \quad \textit{for } v \in \{x, y\}, \tag{2.23}$$

then we have the inequality

$$\frac{1}{2}[(\Psi(x))^*\Psi(y) + (\Psi(y))^*\Psi(x)] \leq \frac{C + c}{2\sqrt{Cc}}\, \Psi(x^*y). \tag{2.24}$$

If, in addition, $\Psi(x)$ and $\Psi(y)$ are two commuting self-adjoint elements of V_2, then (2.24) becomes

$$\Psi(x)\Psi(y) \leq \frac{C + c}{2\sqrt{Cc}}\, \Psi(x^*y). \tag{2.25}$$

(II). *We have the inequality*

$$\begin{aligned} &\frac{1}{2}\Big([\Psi(x^*x)]^{1/2}\,[\Psi(y^*y)]^{1/2} + [\Psi(y^*y)]^{1/2}\,[\Psi(x^*x)]^{1/2}\Big) \\ &\qquad\qquad\qquad\qquad \leq \frac{C + c}{2\sqrt{Cc}}\, \Psi(x^*y). \end{aligned} \tag{2.26}$$

If, in addition, $[\Psi(x^*x)]^{1/2}$ and $[\Psi(y^*y)]^{1/2}$ are two commuting elements of V_2, then we have the inequality

$$[\Psi(x^*x)]^{1/2}\,[\Psi(y^*y)]^{1/2} \leq \frac{C + c}{2\sqrt{Cc}}\, \Psi(x^*y). \tag{2.27}$$

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Proof. Put

$$u \circ_i v = u^* v \quad \text{for } u, v \in V_i,\ i = 1, 2.$$

Then $\circ_i$ is bilinear over $\mathbb{F} = \mathbb{R}$, and (2.1) is satisfied. Since Ψ is a unital positive map, conditions (2.5)-(2.7) are fulfilled.

(I). Take $\Phi = \Psi$. Then (2.4) is met by (2.23). In consequence, by Theorem 2.1, inequalities (2.8) and (2.9) hold with $\Phi = \Psi$. Therefore (2.24) and (2.25) are valid.

(II). Choose $\Phi(v) = [\Psi(v^* v)]^{1/2}$ for $v \in V_1$. Then (2.4) holds automatically, and (2.26) and (2.27) follow directly from (2.8) and (2.9), respectively. ∎

In the matrix setting if $\Phi = \Psi$ is a unital positive map, then condition (2.23) of Theorem 2.7 reduces to Kadison's inequality (2.19). In general, Ψ and Φ need not be linear maps (see Remark 2.3).

We now discuss inequalities (2.14) and (2.22) which are crucial conditions for Theorems 2.4 and 2.7, respectively, to hold.

LEMMA 2.8. *Let V_1 be a C^*-algebra with unity e_1 and convex cone $L_1 = \{a^* a : a \in V_1\}$. Suppose that for each hermitian element $x \in V_1$ there exist real scalars $\lambda_j = \lambda_{j,x}$ and nonzero hermitian elements $a_j = a_{j,x} \in L_1$ $j = 1, \ldots, n$, such that*

(i) $x = \lambda_1 a_1 + \ldots + \lambda_n a_n$,
(ii) $e_1 = a_1 + \ldots + a_n$,
(iii) $a_j a_l = a_j$ *if* $j = l$, *and* $a_j a_l = 0$ *if* $j \neq l$,
(iv) $x \in L_1$ *implies* $\lambda_1, \ldots, \lambda_n \geq 0$.

Let $c, C \in \mathbb{R}$ and let $x, y \in V_1$ be two commuting hermitian elements with invertible y.

Consider conditions

$$ce_1 \leq xy^{-1} \leq Ce_1, \tag{2.28}$$

$$c \leq \lambda_{j,xy^{-1}} \leq C \quad \text{for } j = 1, \ldots, n, \tag{2.29}$$

$$(xy^{-1} - ce_1)(Ce_1 - xy^{-1}) \geq 0, \tag{2.30}$$

$$(x - cy)(Cy - x) \geq 0. \tag{2.31}$$

Then (2.28) $\Rightarrow$ (2.29) $\Rightarrow$ (2.30) $\Rightarrow$ (2.31).

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Proof. By (i) and (ii) applied to hermitian element xy^{-1} we have

$$xy^{-1} - ce_1 = (\lambda_1 - c)a_1 + \ldots + (\lambda_n - c)a_n, \tag{2.32}$$

$$Ce_1 - xy^{-1} = (C - \lambda_1)a_1 + \ldots + (C - \lambda_n)a_n. \tag{2.33}$$

If (2.28) holds, then $xy^{-1} - ce_1 \in L_1$ and $Ce_1 - xy^{-1} \in L_1$. So, using (iv) and (2.32)-(2.33), we obtain

$$\lambda_j - c \geq 0 \quad \text{and} \quad C - \lambda_j \geq 0 \quad \text{for } j = 1, \ldots, n,$$

where $\lambda_j = \lambda_{j,xy^{-1}}$. This gives (2.29).

On the other hand, by (2.32)-(2.33) and (iii), we have

$$(xy^{-1} - ce_1)(Ce_1 - xy^{-1}) = (\lambda_1 - c)(C - \lambda_1)a_1 + \ldots + (\lambda_n - c)(C - \lambda_n)a_n.$$

In consequence, (2.29) forces (2.30) by $a_j \in L_1$, $j = 1, \ldots, n$.

To see the implication (2.30) $\Rightarrow$ (2.31), it is sufficient to pre- and post-multiply (2.30) by $y^* = y$, and use the commutativity of x and y. ☐

Clearly, employing Lemma 2.8 for $y = e_1$, we obtain the implications

$$ce_1 \leq x \leq Ce_1 \;\Rightarrow\; c \leq \lambda_j(x) \leq C \;\Rightarrow\; 0 \leq (x - ce_1)(Ce_1 - x). \tag{2.34}$$

Lemma 2.8 gives possibility to produce Kantorovich type inequalities with various variants of assumptions on x and y (see [7, Theorems 2.1 and 2.4, Corollaries 2.2 and 2.3]).

We now return to Theorem 2.7 and inequality (2.27).

Corollary 2.9. *For $i = 1, 2$, let V_i, X_i, L_i and e_i be as in Theorem 2.7.*

Let $x \in L_1$ be an invertible element such that

$$(x - ce_1)(Ce_1 - x) \geq 0 \quad \textit{for some positive scalars } C, c. \tag{2.35}$$

Assume that $\Psi : V_1 \to V_2$ is a unital positive map. For any integer p, if $\Psi(x^{\frac{p+1}{2}})$ and $\Psi(x^{\frac{p-1}{2}})$ are two commuting elements of V_2, then we have the inequality

$$[\Psi(x^{p+1})]^{1/2}\,[\Psi(x^{p-1})]^{1/2} \leq \frac{C + c}{2\sqrt{Cc}}\,\Psi(x^p). \tag{2.36}$$

Proof. It follows from Lemma 2.8 that (2.35) implies

$$(x^{\frac{p+1}{2}} - cx^{\frac{p-1}{2}})(Cx^{\frac{p-1}{2}} - x^{\frac{p+1}{2}}) \geq 0.$$

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That is (2.22) holds for $x^{\frac{p+1}{2}}$ and $x^{\frac{p-1}{2}}$. Applying (2.27), we obtain (2.36). □

Example 2.10. The Kantorovich inequality (1.1) can be derived from Corollary 2.9 applied to the map

$$\Psi(A) = z^*Az \quad \text{for } A \in \mathbb{M}_n,$$

where $z \in \mathbb{C}^n$ with $z^*z = 1$. Indeed, Ψ is a unital positive map from $\mathbb{M}_n$ to $\mathbb{C}$. Here

$$V_1 = X_1 = \mathbb{M}_n, \;\; L_1 = \mathbb{L}_n, \;\; V_2 = X_2 = \mathbb{C}, \;\; L_2 = \mathbb{R}_+.$$

For $A > 0$, let $0 < c < C$ be scalars such that the spectrum of A lies in the interval $[c, C]$. Then (2.36) with $x = A$ and $p = 0$ becomes (1.1).

In a similar way, from (2.36) one can obtain the Schopf's inequality [20, p. 31]:

$$z^*A^{p+1}z \cdot z^*A^{p-1}z \le \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}(z^*A^pz)^2,$$

where p is an integer, and λ_1 and λ_n are the largest and smallest eigenvalues of an $n \times n$ positive definite matrix A.

In the proof of Theorem 2.1, a key fact leading to (2.8) and (2.9) is inequality (2.10). (2.10) is a consequence of the bilinearity of the operation $\circ_1$. So, in order to get (2.9), it is possible to use (2.10) instead of the bilinearity of $\circ_1$. In fact, in the literature there are inequalities of types (2.10) and (2.9) with non-bilinear $\circ_1$.

Example 2.11. Consider the following spaces and cones

$$V_1 = X_1 = \mathbb{M}_n, \;\; L_1 = \mathbb{L}_n, \;\; V_2 = X_2 = \mathbb{R}, \;\; L_2 = \mathbb{R}_+.$$

Define maps as follows

$$\Phi(A) = (z^*Az)^{1/2} \quad \text{for } A \in \mathcal{A} = \mathbb{L}_n, \tag{2.37}$$

and

$$\Psi(A) = z^*Az \quad \text{for } A \in \mathcal{B} = \mathbb{L}_n, \tag{2.38}$$

where $z \in \mathbb{C}^n$ with $z^*z = 1$.

Take $\circ_2$ to be the usual multiplication on $\mathbb{R}$. Let $\circ_1$ be the binary operation of *geometric mean* [21, p. 689]:

$$A \circ_1 B = G(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \quad \text{for } 0 < A, B \in \mathbb{L}_n.$$

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With the aid of the version of Theorem 2.1 based on (2.10), we shall show how to obtain the inequality [21, Theorem 2.2]:

$$(z^*Az)^{1/2}(z^*Bz)^{1/2} \leq \frac{C+c}{2\sqrt{Cc}}\, z^*G(A,B)z \tag{2.39}$$

for $0 < A, B \in \mathbb{L}_n$ with $0 < cI_n \leq A, B \leq CI_n$ and $0 < c < C$.

To do this, we use the result [13, 21]:

$$\frac{1}{2}(A+B) \leq \frac{C+c}{2\sqrt{Cc}}\, G(A,B)$$

for $0 < A, B \in \mathbb{L}_n$ with $0 < cI_n \leq A, B \leq CI_n$ and $0 < c < C$. Because $G(A, \alpha B) = \alpha^{1/2}G(A,B)$ for $\alpha > 0$ [21, p. 689], substituting CcB instead of B leads to

$$A + CcB \leq (C+c)\, G(A,B),$$

which is of the form (2.10).

Furthermore, $G(A,B) = G(B,A)$ [21, p. 689]. Clearly, conditions (2.5)-(2.7) are satisfied. Since $G(A,A) = A$ [21, p. 689], it is readily seen that (2.4) is met.

By the discussion before this example, we get (2.9). It is not hard to check that (2.9), with Φ and Ψ defined by (2.37) and (2.38), can be rewritten as (2.39).

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