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FLAT PORTIONS ON THE BOUNDARY OF THE NUMERICAL RANGE OF A 5×5 COMPANION MATRIX*

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Abstract. The number of flat portions on the boundary of the numerical range of 5×5 companion matrices, both unitarily reducible and unitarily irreducible cases, is examined. The complete characterization on the number of flat portions of a 5×5 unitarily reducible companion matrix is given. Also under some suitable conditions, it is shown that a unitarily irreducible 5×5 companion matrix cannot have four flat portions on the boundary of its numerical range. This gives a partial affirmative answer to the conjecture given in [3] for $n = 5$. Numerical examples are provided to illustrate the results.

Key words. Numerical range, Companion matrix.

AMS subject classifications. 15A60.

1. Introduction.

The numerical range $W(A)$ of an $n \times n$ matrix A is the subset of the complex plane $\mathbb{C}$ defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

It is well-known that $W(A)$ is a convex (Toeplitz–Hausdorff Theorem), compact subset of $\mathbb{C}$. Other basic properties of the numerical range can be found in [11, 12].

In particular, it is interesting to locate the flat portions (if any) on the boundary $\partial W(A)$ of the numerical range and to indicate a bound for the number of flat portions $f(A)$ for several classes of matrices. A matrix A is *unitarily reducible* if it is unitarily similar to a block diagonal matrix with at least two diagonal blocks A_j. In this case, $W(A)$ is the convex hull of $W(A_j)$. The numerical range $W(A)$ will have flat portions on its boundary $\partial W(A)$, unless one of the $W(A_j)$ contains all others. For a normal matrix A, the blocks A_j can be made one-dimensional and $W(A)$ is nothing but the convex hull of the spectrum $\sigma(A)$. Therefore, $f(A)$ is at most n for a normal matrix A of order n.

For $n = 2$, $f(A) = 0$ when A is unitarily irreducible (i.e., not unitarily reducible) as $W(A)$ is an elliptical disc. Also, for a 2×2 normal matrix A, $f(A) = 1$ where A is different from a scalar multiple of identity, since $W(A)$ is a line segment and finally $f(\lambda I) = 0$. For $n = 3$, from the classification given by Kippenhahn [15] and Keeler et al. [13], it is easily followed that $f(A)$ is at most 2 for a non-normal unitarily reducible matrix A and at most 1 for a unitarily irreducible matrix. For a 4×4 matrix A, Brown and Spitkovsky [1] have established that the sharp bound for $f(A)$ on the boundary of the numerical range is 4, while for the unitarily irreducible case $f(A)$ is at most 3.

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An $n \times n$ ($n \geq 2$) companion matrix is of the form

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{1.1}$$

The characteristic polynomial of (1.1) is given by

$$\det(A - zI) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

A special case of a companion matrix is the Jordan block J_n when all a_j's are zero. Gau and Wu [8] have given a criterion for a unitarily reducible companion matrix in terms of its eigenvalues as follows.

Theorem 1.1 ([8]). *An $n \times n$ ($n \geq 2$) companion matrix A is unitarily reducible if and only if $\sigma(A) = \{a\omega_j : j \in J_1\} \cup \{\frac{1}{\bar{a}}\omega_j : j \in J_2\}$ for some $a \in \mathbb{C} \setminus \{0\}$ and partition $J_1 \cup J_2$ of $\{1, \dots, n\}$, where both J_1 and J_2 are non-empty; $\omega_1, \dots, \omega_n$ being the set of all nth roots of 1. If this condition holds, then A is unitarily similar to $A_1 \oplus A_2$ with $\sigma(A_1) = \{a\omega_j : j \in J_1\}$ and $\sigma(A_2) = \{\frac{1}{\bar{a}}\omega_j : j \in J_2\}$.*

Also, a companion matrix unitarily equivalent to the direct sum of three or more matrices must be unitary (cf. [8], Corollary 1.3). Moreover, if an $n \times n$ unitarily reducible companion matrix has spectral radius one, then it is unitary, and its numerical range is a regular n-sided polygon (cf. [8], Corollary 1.2).
In recent years, properties of the numerical range of S_n - matrices have been thoroughly studied by many mathematicians (see [4], [5], [6]). An $n \times n$ complex matrix A is said to be of class S_n if the eigenvalues of A are all in the open unit disc $\mathbb{D}$ and $\operatorname{rank}(I_n - A^*A) = 1$. An $n \times n$ complex matrix B is said to be of class S_n^{-1} if all eigenvalues of B have modulus greater than one and $\operatorname{rank}(I_n - B^*B) = 1$. For any matrix C in S_n or S_n^{-1}, $\partial W(C)$ contains no line segment (see [5], [4], [10]). For a unitarily reducible companion matrix (not unitary), Gau [10] has given the following result with the help of S_n and S_n^{-1} matrices.

Corollary 1.2 ([10]). *Let A (not unitary) be an $n \times n$ unitarily reducible companion matrix. Then, A is unitarily equivalent to a direct sum $B \oplus C$ with $B \in S_k$ and $C \in S_{n-k}^{-1}$, $1 \leq k \leq n-1$.*

Moreover, Gau and Wu [9] have shown that for a companion matrix A, the number of line segments on $\partial W(A)$ is at most the size of the matrix. In 2012, Eldred et al. [3] have given the necessary and sufficient conditions for the existence of flat portions for companion matrices as follows.

Theorem 1.3 ([3]). *Let A be given by (1.1). Then for $W(A)$ to have a flat portion on the boundary, it is necessary that*

$$\sum_{j=0}^{n-2} a_j \omega^{n-j} sin\frac{\pi(j+1)}{n} = sin\frac{\pi}{n}, \tag{1.2}$$

and

$$\operatorname{Re}(a_{n-1}\omega) = \sum_{j=2}^{n-1} \frac{|\gamma_j|^2}{cos\frac{\pi}{n} - cos\frac{\pi j}{n}} - cos\frac{\pi}{n} \tag{1.3}$$

for some ω with $|\omega| = 1$ and

$$\gamma_j = \frac{1}{\sqrt{2n}}\left(sin\frac{\pi j(n-1)}{n} - \sum_{k=0}^{n-2} a_k \omega^{n-k} sin\frac{\pi j(k+1)}{n}\right) \text{ for } j = 2, \dots, n-1. \tag{1.4}$$

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If the conditions (1.2) *and* (1.3) *hold, then the potential flat portion passes through the point* $\bar{\omega}\cos\frac{\pi}{n}$ *and has the slope* $\pi/2 - \arg\omega$.

THEOREM 1.4 ([3]). *Let the conditions* (1.2) *and* (1.3) *hold for some matrix* A *given by* (1.1) *and* ω *having absolute value 1. Then,* $\partial W(A)$ *has a flat portion passing through* $\bar{\omega}\cos\frac{\pi}{n}$ *if and only if at least one of the scalar products* $\langle \mathrm{Im}(wA)x_1, x_2\rangle$ *and* $\langle \mathrm{Im}(wA)x_2, x_2\rangle$ *differs from zero where*

$$x_1 = \Omega^{-1}\begin{bmatrix} v_1 \\ 0 \end{bmatrix},\ x_2 = \Omega^{-1}\begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}\xi \text{ and } \mathrm{Im}(\omega A) = \frac{(\omega A)-(\omega A)^*}{2i},$$

with Ω, v_1, V *given by*
$\Omega = diag[1, \omega, \dots, \omega^{n-1}]$, $v_1 = [\sin\frac{\pi}{n}, \dots, \sin\frac{\pi(n-1)}{n}]^T$, $V = \sqrt{\frac{2}{n}}\left[\sin\frac{\pi jk}{n}\right]_{k,j=1}^{n-1}$ *and*
$\xi = [0, \xi_2, \dots, \xi_{n-1}, 1]^T$, $\xi_j = \frac{\bar{\gamma}_j}{\cos\frac{\pi}{n} - \cos\frac{\pi j}{n}}$, $j = 2, \dots, n-1$.

The number of flat portions on $\partial W(A)$ of the matrix (1.1) coincides with the number of distinct unimodular solutions ω of (1.2), (1.3) such that the "if and only if" conditions of Theorem 1.4 are satisfied. For $n = 4$, Eldred et al. [3] have proved that a 4×4 companion matrix cannot have three flat portions on the boundary of its numerical range.
A companion matrix of order 5 is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix}. \tag{1.5}$$

In this article, we look into the presence of flat portions on the boundary of the numerical range for a 5×5 companion matrix A given in (1.5). Section 2 contains the results obtained for 5×5 unitarily reducible companion matrices (not unitary). In this case, we show that the possible values of $f(A)$ are 0, 2 and 4. In Section 3, we deal with the unitarily irreducible companion matrices where we partially answer the conjecture given in [3] for $n = 5$.

2. Unitarily reducible 5×5 companion matrices.

Let us first take the companion matrix A (not unitary) defined in (1.5) be unitarily reducible. Then for $n = 5$, combining Theorem 1.1 and Corollary 1.2, we can conclude that such a matrix $A \in M_5(\mathbb{C})$ is unitarily equivalent to the direct sum $B \oplus C$ $\big(B \in S_k$ and $C \in S_{5-k}^{-1}$ for $k = 1, 2, 3, 4\big)$ with

$$\sigma(B) = \left\{a\omega^j : j \in J_1\right\} \text{ and } \sigma(C) = \left\{\frac{1}{\bar{a}}\omega^j : j \in J_2\right\}, \tag{2.6}$$

respectively, such that $|a| < 1$, ω ($\neq 1$) being a primitive 5^{th} root of unity and J_1, J_2 form a partition of $\{1, \dots, 5\}$. Thus, $W(A) = W(B \oplus C) = \mathrm{conv}\{W(B), W(C)\}$, where "conv" denotes the convex hull of the sets $W(B)$ and $W(C)$.

Also, from the results given in [6] and [10], the matrices B and C are unitarily equivalent to the upper triangular matrices $P_1 \in S_k$ and $P_2 \in S_{5-k}^{-1}$ ($k = 1, 2, 3, 4$), respectively. Hence, $W(B) = W(P_1)$ and $W(C) = W(P_2)$. Thus we have

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$$W(A) = \text{conv}\{W(B), W(C)\} = \text{conv}\{W(P_1), W(P_2)\}. \tag{2.7}$$

Now we have the following results.

THEOREM 2.1. *Let A be a unitarily reducible 5×5 companion matrix (not unitary) defined in (1.5) with the set of eigenvalues $a\omega^{j_1}$, $a\omega^{j_2}$, $\frac{1}{\bar{a}}\omega^{j_3}$, $\frac{1}{\bar{a}}\omega^{j_4}$, $\frac{1}{\bar{a}}\omega^{j_5}$, where $a(\neq 0) \in \mathbb{C}$ and $|a| < 1$, ω $(\neq 1)$ denotes a primitive 5^{th} root of unity and $\{j_1, j_2\}, \{j_3, j_4, j_5\}$ form a partition of $\{1, 2, 3, 4, 5\}$. Then for some suitable $\phi \in \mathbb{R}$, the numerical range $W(e^{i\phi}A)$ is symmetric about the real axis.*

Proof. Clearly, A is unitarily equivalent to the direct sum $B \oplus C$ where $B \in S_2$ and $C \in S_3^{-1}$ with $\sigma(B) = \{a\omega^{j_1},\ a\omega^{j_2}\}$ and $\sigma(C) = \{\frac{1}{\bar{a}}\omega^{j_3},\ \frac{1}{\bar{a}}\omega^{j_4},\ \frac{1}{\bar{a}}\omega^{j_5}\}$ as in (2.6). Then $\partial W(B)$ is an ellipse with foci at $a\omega^{j_1}$, $a\omega^{j_2}$ and $\partial W(C)$ is an oval with foci at $\frac{1}{\bar{a}}\omega^{j_3}$, $\frac{1}{\bar{a}}\omega^{j_4}$, $\frac{1}{\bar{a}}\omega^{j_5}$.
Let $a = re^{i\theta}$, where $\theta = \arg(a)$ and $r \in \mathbb{R}$. So, $ae^{-i\theta} = r$ and $\frac{1}{\bar{a}}e^{-i\theta} = \frac{1}{r}$. The eigenvalues of the matrix $e^{-i\theta}A$ are now of the form:

$$r\omega^{j_1},\ r\omega^{j_2},\ \tfrac{1}{r}\omega^{j_3},\ \tfrac{1}{r}\omega^{j_4},\ \tfrac{1}{r}\omega^{j_5}.$$

Let $\omega = e^{\frac{2k\pi i}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ for some $k = 1, 2, 3, 4$. For this particular value of k, consider $\psi = \frac{2}{5}k\pi(2j_1 + 2j_2)$. Thus $e^{i\psi} = e^{\frac{2}{5}k\pi i(2j_1+2j_2)} = \omega^{2(j_1+j_2)}$ and hence the eigenvalues of the matrix $e^{i\psi}e^{-i\theta}A$ are as follows:

$$r\omega^{3j_1+2j_2},\ r\omega^{2j_1+3j_2},\ \tfrac{1}{r}\omega^{j_3+2(j_1+j_2)},\ \tfrac{1}{r}\omega^{j_4+2(j_1+j_2)},\ \tfrac{1}{r}\omega^{j_5+2(j_1+j_2)}.$$

Since $\omega^{3j_1+2j_2}\cdot\omega^{2j_1+3j_2} = \omega^{5(j_1+j_2)} = 1$, therefore the values $r\omega^{3j_1+2j_2}$ and $r\omega^{2j_1+3j_2}$ are conjugate to each other (i.e., any pair of the form $\{r\omega^2, r\omega^3\}$ or $\{r\omega, r\omega^4\}$). Moreover, these two eigenvalues correspond to the foci of an ellipse which is the boundary of the numerical range of $e^{i\psi}e^{-i\theta}B \in S_2$. So, $W(e^{i\psi}e^{-i\theta}B)$ is symmetric with respect to the real axis.
Now the remaining three eigenvalues $\frac{1}{r}\omega^{j_3'}$, $\frac{1}{r}\omega^{j_4'}$, $\frac{1}{r}\omega^{j_5'}$ (where $j_m' = j_m + 2j_1 + 2j_2,\ m = 3, 4, 5$) are the foci of an oval which is the boundary of the numerical range of $e^{i\psi}e^{-i\theta}C \in S_3^{-1}$. Among these three eigenvalues, one takes the value $\frac{1}{r}$ and remaining two eigenvalues are conjugate to each other as $\{j_1, j_2\}, \{j_3, j_4, j_5\}$ form a partition of $\{1, 2, 3, 4, 5\}$.
By Theorem 2.4 of [10], any matrix in S_3^{-1} with the eigenvalues $\frac{1}{r}\omega^{j_3'}$, $\frac{1}{r}\omega^{j_4'}$, $\frac{1}{r}\omega^{j_5'}$ (taken in order as above) has an upper triangular matrix representation as follows

$$C' = \begin{bmatrix} \frac{1}{r}\omega^{j_3'} & \frac{1-r^2}{r^2} & \frac{1-r^2}{r^3}\bar{\omega}^{j_4'} \\ 0 & \frac{1}{r}\omega^{j_4'} & \frac{1-r^2}{r^2} \\ 0 & 0 & \frac{1}{r}\omega^{j_5'} \end{bmatrix}.$$

Then $C' \in S_3^{-1}$. Without any loss of generality, we may take $\omega^{j_4'} = 1$ and thus $\omega^{j_3'}$ and $\omega^{j_5'}$ are conjugate to each other. Therefore C' takes the form as follows

$$C_1 = \begin{bmatrix} \frac{1}{r}\omega^{j_3'} & \frac{1-r^2}{r^2} & \frac{1-r^2}{r^3} \\ 0 & \frac{1}{r} & \frac{1-r^2}{r^2} \\ 0 & 0 & \frac{1}{r}\omega^{j_5'} \end{bmatrix} \in S_3^{-1}.$$

Our aim is to show that the numerical range $W(C_1)$ of C_1 is symmetric with respect to the real axis; that is, $z \in W(C_1)$ implies $\bar{z} \in W(C_1)$.

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To prove this, let us consider $z = \alpha + i\beta \in W(C_1)$. Then, there exists $x = [f, g, h]^T \in \mathbb{C}^3$ with $\|x\| = 1$ such that

$$\alpha + i\beta = x^* C_1 x = \frac{1}{r}\omega^{j_5'}|h|^2 + \frac{1}{r}\omega^{j_3'}|f|^2 + \frac{1}{r}|g|^2 + \frac{1-r^2}{r^2}(\bar{g}h + \bar{f}g) + \frac{1-r^2}{r^3}\bar{f}h.$$

Take $x' = [h, g, f]^T$. Then, $\|x'\| = 1$ and we have

$$(x')^* C_1 x' = \frac{1}{r}\omega^{j_5'}|f|^2 + \frac{1}{r}\omega^{j_3'}|h|^2 + \frac{1}{r}|g|^2 + \frac{1-r^2}{r^2}(g\bar{h} + f\bar{g}) + \frac{1-r^2}{r^3}f\bar{h}.$$

Since $\frac{1}{r}\omega^{j_3'}$ and $\frac{1}{r}\omega^{j_5'}$ are conjugate to each other, so $(x')^* C_1 x' = \alpha - i\beta = \bar{z} \in W(C_1)$. Therefore, $W(C_1)$ is symmetric with respect to the real axis.
Also from Theorem 2.7 of [10], we know that any two matrices in S_3^{-1} having same set of eigenvalues (counting multiplicities) are unitarily equivalent. Therefore, $W(e^{i\psi}e^{-i\theta}C)$ is also symmetric with respect to the real axis.
Hence, the convex hull of $W(e^{i\psi}e^{-i\theta}B)$ and $W(e^{i\psi}e^{-i\theta}C)$, that is, $W(e^{i\phi}A)$ where $\phi = \psi - \theta$, is also symmetric with respect to the real axis. ∎

Theorem 2.2. *Let A be a unitarily reducible 5×5 companion matrix (not unitary) defined in (1.5) with the set of eigenvalues $a\omega^{j_1}$, $a\omega^{j_2}$, $a\omega^{j_3}$, $\frac{1}{\bar{a}}\omega^{j_4}$, $\frac{1}{\bar{a}}\omega^{j_5}$, where $a(\neq 0) \in \mathbb{C}$ and $|a| < 1$, ω $(\neq 1)$ denotes a primitive 5^{th} root of unity and $\{j_1, j_2, j_3\}, \{j_4, j_5\}$ form a partition of $\{1, 2, 3, 4, 5\}$. Then for some suitable $\phi' \in \mathbb{R}$, the numerical range $W(e^{i\phi'}A)$ is symmetric about the real axis.*

Proof. Similar arguments are to be followed as in Theorem 2.1. ∎

Thus, we can conclude that if A is unitarily equivalent to $B \oplus C$ with

1. $B \in S_1$ and $C \in S_4^{-1}$, then $W(B) = \{b\}$ with $|b| < 1$ and $W(C)$ is a convex set with no flat portion. Hence, $W(A) = \text{conv}\{W(B), W(C)\}$ has either 0 or 2 flat portions on its boundary according as b does or does not belong to $W(C)$.
2. $B \in S_2$ and $C \in S_3^{-1}$, then the possible numbers of $f(A)$ are 0, 2 and 4 (by Theorem 2.1).
3. $B \in S_3$ and $C \in S_2^{-1}$, then the possible numbers of $f(A)$ are 0, 2 and 4 (by Theorem 2.2).
4. $B \in S_4$ and $C \in S_1^{-1}$, then $W(B) \subseteq \{z \in \mathbb{C} : |z| < 1\}$ (See pp. 181, [7]) and $W(C) = \{b\}$ with $|b| > 1$. Hence, $\partial W(A)$ has exactly 2 flat portions.

Thus, we have the following result.

Theorem 2.3. *Let A (not unitary) be a unitarily reducible 5×5 companion matrix. Then, the possible numbers of flat portions on the boundary of $W(A)$ are 0, 2 and 4.*

Let A (not unitary) be a unitarily reducible companion matrix with the set of eigenvalues $\sigma(A) = \sigma(B) \cup \sigma(C)$ as described in (2.6). Then, there exist two upper triangular matrices $P_1 \in S_k$ and $P_2 \in S_{5-k}^{-1}$ with $W(A) = \text{conv}\{W(P_1), W(P_2)\}$ (by (2.7)). The following three examples (considering $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$) are constructed to show the existence of reducible companion matrices having 0, 2 and 4 flat portions on the boundary of its numerical range.

Example 2.4 (**No flat portion**). Let a companion matrix A_1 be such that

$$\sigma(A_1) = \left\{\frac{5\omega}{8}, \frac{5\omega^4}{8}, \frac{8\omega^3}{5}, \frac{8}{5}, \frac{8\omega^2}{5}\right\},$$

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that is, $a = \frac{5}{8}$, $J_1 = \{1, 4\}$ and $J_2 = \{3, 5, 2\}$ as in (2.6). Then, A_1 is unitarily reducible by Theorem (1.1). The upper triangular matrices $P_1 \in S_2$ and $P_2 \in S_3^{-1}$ are of the following form

$$P_1 = \begin{bmatrix} \frac{5\omega}{8} & \frac{39}{64} \\ 0 & \frac{5\omega^4}{8} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} \frac{8\omega^3}{5} & \frac{39}{25} & \frac{312}{125} \\ 0 & \frac{8}{5} & \frac{39}{25} \\ 0 & 0 & \frac{8\omega^2}{5} \end{bmatrix},$$

respectively. The numerical ranges $W(P_1), W(P_2)$ and $W(A_1)$ are given in Figures 1 and 2.

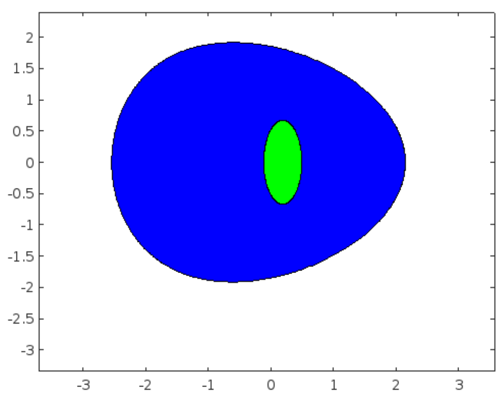

FIGURE 1. $W(P_1)$ *(green) and* $W(P_2)$ *(blue).*

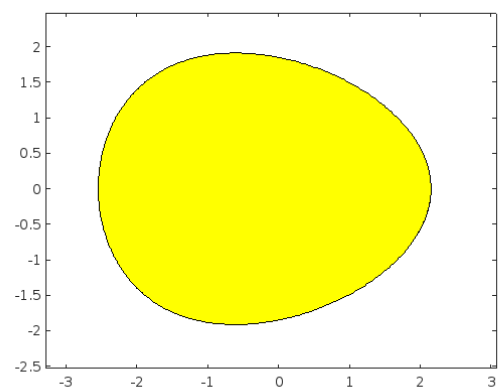

FIGURE 2. $W(A_1)$.

Example 2.5 (**Two flat portions**). Let a companion matrix A_2 be such that

$$\sigma(A_2) = \left\{ \frac{2\omega}{5}, \frac{2\omega^2}{5}, \frac{2\omega^3}{5}, \frac{2\omega^4}{5}, \frac{5}{2} \right\},$$

that is, $a = \frac{2}{5}$, $J_1 = \{1, 2, 3, 4\}$ and $J_2 = \{5\}$ as in (2.6). Then, A_2 is unitarily reducible by Theorem (1.1). The upper triangular matrices $P_1 \in S_4$ and $P_2 \in S_1^{-1}$ are of the following form

$$P_1 = \begin{bmatrix} \frac{2\omega}{5} & \frac{21}{25} & \frac{-42\omega^3}{125} & \frac{84}{625} \\ 0 & \frac{2\omega^2}{5} & \frac{21}{25} & \frac{-42\omega^2}{125} \\ 0 & 0 & \frac{2\omega^3}{5} & \frac{21}{25} \\ 0 & 0 & 0 & \frac{2\omega^4}{5} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} \frac{5}{2} \end{bmatrix},$$

respectively. The numerical ranges $W(P_1), W(P_2)$ and $W(A_2)$ are given in Figures 3 and 4.

Example 2.6 (**Four flat portions**). Let a companion matrix A_3 be such that

$$\sigma(A_3) = \left\{ \frac{7\omega^4}{8}, \frac{7\omega}{8}, \frac{8\omega^2}{7}, \frac{8}{7}, \frac{8\omega^3}{7} \right\},$$

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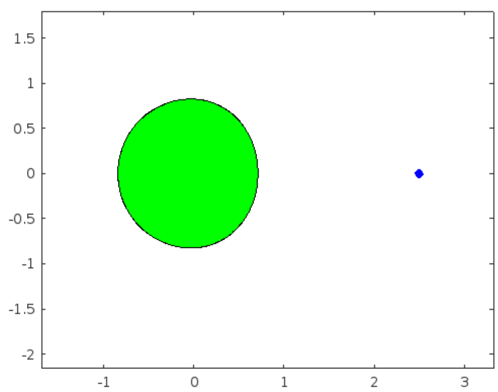

FIGURE 3. $W(P_1)$ *(green) and* $W(P_2)$ *(blue).*

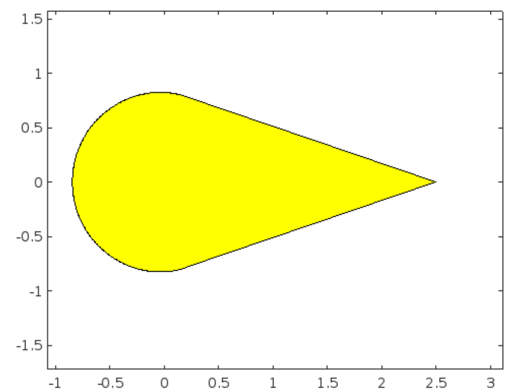

FIGURE 4. $W(A_2)$.

that is, $a = \frac{7}{8}$, $J_1 = \{4, 1\}$ and $J_2 = \{2, 5, 3\}$ as in (2.6). Then, A_3 is unitarily reducible by Theorem (1.1). The upper triangular matrices $P_1 \in S_2$ and $P_2 \in S_3^{-1}$ are of the following form

$$P_1 = \begin{bmatrix} \frac{7\omega^4}{8} & \frac{15}{64} \\ 0 & \frac{7\omega}{8} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} \frac{8\omega^2}{7} & \frac{15}{49} & \frac{120}{343} \\ 0 & \frac{8}{7} & \frac{15}{49} \\ 0 & 0 & \frac{8\omega^3}{7} \end{bmatrix},$$

respectively. The numerical ranges $W(P_1), W(P_2)$ and $W(A_3)$ are given in Figures 5 and 6.

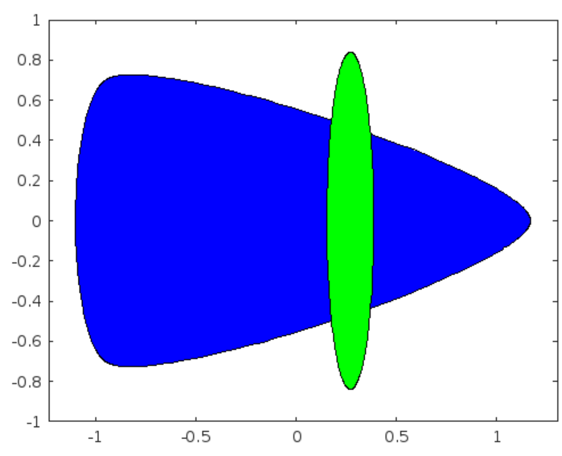

FIGURE 5. $W(P_1)$ *(green) and* $W(P_2)$ *(blue).*

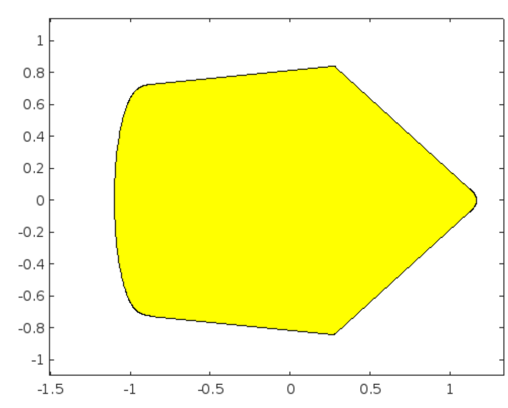

FIGURE 6. $W(A_3)$.

3. Unitarily irreducible 5×5 companion matrices.

From the paper [3], we have a conjecture on the number of flat portions $f(A)$ on the boundary of the numerical range for a companion matrix A as given below:

Conjecture: The equality $f(A) = n - 1$ for an $n \times n$ companion matrix A implies that n is odd and A is unitarily reducible.

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In our case, that is, for $n = 5$, the conjecture may be stated as,

A unitarily irreducible 5×5 *companion matrix cannot have four flat portions on the boundary of its numerical range.*

So, we proceed to the case of a 5×5 unitarily irreducible companion matrix. We assume the boundary of $W(A)$ has a flat portion, and then, the conditions of Theorem 1.3 hold for some unimodular ω. For $n = 5$, equations (1.2) and (1.3) of Theorem 1.3 turn into

$$(a_0\omega^5 + a_3\omega^2)r + (a_1\omega^4 + a_2\omega^3)s = r, \tag{3.8}$$

$$\mathrm{Re}(a_4\omega) = \frac{|\gamma_2|^2}{\frac{1}{2}} + \frac{|\gamma_3|^2}{\frac{\sqrt{5}}{2}} + \frac{|\gamma_4|^2}{\frac{\sqrt{5}+1}{2}} - \frac{\sqrt{5}+1}{4}, \tag{3.9}$$

where,

$$r = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4},\ \ s = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4},$$

and

$$\begin{aligned}
\gamma_2 &= \frac{1}{\sqrt{10}}(-s - a_0 s\omega^5 - a_1 r\omega^4 + a_2 r\omega^3 + a_3 s\omega^2),\\
\gamma_3 &= \frac{1}{\sqrt{10}}(s - a_0 s\omega^5 + a_1 r\omega^4 + a_2 r\omega^3 - a_3 s\omega^2),\\
\gamma_4 &= \frac{1}{\sqrt{10}}(-r - a_0 r\omega^5 + a_1 s\omega^4 - a_2 s\omega^3 + a_3 r\omega^2).
\end{aligned}$$

On simplifying by using (3.8), we get

$$\begin{aligned}
\gamma_2 &= \frac{-\omega^3}{\sqrt{2}\sqrt{10-2\sqrt{5}}}(2a_0\omega^2 + \sqrt{5}a_1\omega + a_2),\\
\gamma_3 &= \frac{5\omega^3}{\sqrt{10}\sqrt{10-2\sqrt{5}}}(a_1\omega + a_2),\\
\text{and } \gamma_4 &= \frac{-2\omega^3}{\sqrt{10}}(a_0 r\omega^2 + a_2 s).
\end{aligned}$$

Substituting the values of γ_2, γ_3 and γ_4 in equation (3.9), we have

$$\begin{aligned}
\mathrm{Re}(a_4\omega) &= \Big(\frac{1+\sqrt{5}}{4}\Big)|a_0|^2 + \Big(\frac{3+\sqrt{5}}{4}\Big)|a_1|^2 + \Big(\frac{1+\sqrt{5}}{4}\Big)|a_2|^2 - \Big(\frac{1+\sqrt{5}}{4}\Big)\\
&\quad + \mathrm{Re}\bigg(\Big(\frac{2a_0\bar{a}_1}{\sqrt{5}-1} + a_0\bar{a}_2\omega + \frac{2a_1\bar{a}_2}{\sqrt{5}-1}\Big)\omega\bigg).
\end{aligned}$$

Thus,

$$\Big(\frac{1+\sqrt{5}}{4}\Big)(|a_0|^2 + |a_2|^2 - 1) + \Big(\frac{3+\sqrt{5}}{4}\Big)|a_1|^2 = \mathrm{Re}\bigg(\bigg(a_4 - \frac{2}{\sqrt{5}-1}\Big(a_0\bar{a}_1 + a_1\bar{a}_2\Big) - a_0\bar{a}_2\omega\bigg)\omega\bigg). \tag{3.10}$$

Now, consider three cases as follows,

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1. When $a_0 = 0 = a_2$, (3.10) reduces to

$$\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right) = \mathrm{Re}(a_4\omega). \tag{3.11}$$

Then, (3.11) is a tautology if

$$a_4 = 0,\ \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right) = 0. \tag{3.12}$$

It has no unimodular solution if $|a_4| < \left|\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right|$ and its (automatically unimodular) solutions are given by

$$\omega = \frac{\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right)}{a_4} \pm i\frac{\sqrt{|a_4|^2 - \left(\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right)^2}}{a_4}, \tag{3.13}$$

in the remaining case

$$0 \neq |a_4| \geq \left|\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right|.$$

2. When $a_0 = 0$, (3.10) reduces to

$$\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 = \mathrm{Re}\left(\left(a_4 - \frac{2}{\sqrt{5}-1}a_1\bar{a_2}\right)\omega\right). \tag{3.14}$$

Then, (3.14) is a tautology if

$$a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1} = 0,\ \left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 = 0. \tag{3.15}$$

It has no unimodular solution if $\left|a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}\right| < \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|$ and its (automatically unimodular) solutions are given by

$$\omega = \frac{\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2}{a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}} \pm i\frac{\sqrt{|a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}|^2 - \left(\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right)^2}}{a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}}, \tag{3.16}$$

in the remaining case

$$0 \neq \left|a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}\right| \geq \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|.$$

3. When $a_2 = 0$, (3.10) reduces to

$$\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 = \mathrm{Re}\left(\left(a_4 - \frac{2}{\sqrt{5}-1}a_0\bar{a_1}\right)\omega\right). \tag{3.17}$$

Then, (3.17) is a tautology if

$$a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1} = 0,\ \left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 = 0. \tag{3.18}$$

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It has no unimodular solution if

$$\left|a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}\right| < \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|,$$

and its (automatically unimodular) solutions are given by

$$\omega = \frac{\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2}{a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}} \pm i\frac{\sqrt{\left|a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}\right|^2 - \left(\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right)^2}}{a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}}, \tag{3.19}$$

in the remaining case

$$0 \neq \left|a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}\right| \geq \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|.$$

Thus, we have the following lemma.

Lemma 3.1. *Let A be a companion matrix as defined in (1.5). Assume $W(A)$ has a flat portion on its boundary. Then, the followings hold.*

1. *If $a_0 = 0 = a_2$, then $|a_4| \geq \left|\left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right|$ and (3.8) has a unimodular solution ω. Moreover, this ω must coincide with one of the values given by (3.13), unless (3.12) holds.*
2. *If $a_0 = 0$, then $\left|a_4 - \frac{2a_1\bar{a_2}}{\sqrt{5}-1}\right| \geq \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_2|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|$ and (3.8) has a unimodular solution ω. Moreover, this ω must coincide with one of the values given by (3.16), unless (3.15) holds.*
3. *If $a_2 = 0$, then $\left|a_4 - \frac{2a_0\bar{a_1}}{\sqrt{5}-1}\right| \geq \left|\left(\frac{1+\sqrt{5}}{4}\right)(|a_0|^2-1) + \left(\frac{3+\sqrt{5}}{4}\right)|a_1|^2\right|$ and (3.8) has a unimodular solution ω. Moreover, this ω must coincide with one of the values given by (3.19), unless (3.18) holds.*

Theorem 3.2. *Let A be a unitarily irreducible companion matrix as defined in (1.5), where $a_0a_2 = 0$. Then $f(A) \neq 4$.*

Proof. If possible, let $f(A) = 4$. Then, equation (3.8) has at least four distinct unimodular solutions, say $\alpha_1, \alpha_2, \alpha_3$ and α_4.

Case 1: First take $a_0 = 0 = a_2$. We see that $f(A) = 4$ is possible only when (3.12) holds. Also for $a_0 = 0 = a_2$, (3.8) turns into

$$a_1 s\omega^4 + a_3 r\omega^2 = r.$$

Since $f(A) = 4$ (by our assumption), so $a_1 \neq 0$. By Vieta's formulae, $\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{-r}{a_1 s}$ and therefore $|a_1|^2 = \frac{r^2}{s^2} = \frac{3-\sqrt{5}}{2}$ which contradicts the tautology (3.12), that is, $|a_1|^2 = \frac{\sqrt{5}-1}{2}$. Thus, if $a_0 = 0 = a_2$, then $f(A) \neq 4$.

Case 2: Now take $a_0 = 0$. Observe that $f(A) = 4$ is possible only when (3.15) holds.
Therefore, equation (3.8) takes the form as follows,

$$a_1 s\omega^4 + a_2 s\omega^3 + a_3 r\omega^2 = r. \tag{3.20}$$

Since $f(A) = 4$ (by our assumption), $a_1 \neq 0$. Applying Vieta's formula on (3.20), we get,

$$\sum_{i=1}^{4}\alpha_i = -\frac{a_2}{a_1} \quad \text{and} \quad \sum_{i=1}^{4}\frac{1}{\alpha_i} = 0.$$

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This implies $a_2 = 0$. Hence in this case, the tautology (3.15) reduces to (3.12) and a contradiction arises as in *Case* 1.
So, if $a_0 = 0$, then $f(A) \neq 4$.
Case 3: Finally taking $a_2 = 0$, it is clear that $f(A) = 4$ is possible only when (3.18) holds.
From (3.8) we get

$$a_0 r\omega^5 + a_1 s\omega^4 + a_3 r\omega^2 - r = 0. \tag{3.21}$$

We take $a_0 \neq 0$ as $a_0 = 0$ leads to the contradiction as in *Case* 1.
If possible, let the fifth root of (3.21) be α_5. By Vieta's formulae, we have

$$\begin{aligned} \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5} &= 0, \\ \text{i.e., } \bar{\alpha_1} + \bar{\alpha_2} + \bar{\alpha_3} + \bar{\alpha_4} &= -\frac{1}{\alpha_5}. \end{aligned}$$

Let $z = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and thus $\bar{z} = -\frac{1}{\alpha_5}$. So, $\bar{z} \neq 0$ and hence $z \neq 0$. Also, we have

$$\begin{aligned} \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = \frac{1}{a_0}, \text{ i.e., } \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{-\bar{z}} &= \frac{1}{a_0}, \\ \text{i.e., } |z| &= |a_0|. \end{aligned}$$

Thus,

$$\begin{aligned} z - \frac{1}{\bar{z}} = \sum_{i=1}^{5} \alpha_i &= \frac{-a_1 s}{a_0 r}, \\ \text{i.e., } \frac{\left||z|^2 - 1\right|}{|z|} &= \frac{|a_1| s}{|a_0| r}. \\ \text{Therefore, } |a_1|^2 &= \frac{r^2\left||z|^2 - 1\right|^2}{s^2}. \end{aligned}$$

From (3.18), we get

$$\begin{aligned} \Big(\frac{1+\sqrt{5}}{4}\Big)(|z|^2 - 1) + \Big(\frac{3+\sqrt{5}}{4}\Big)\Big(\frac{r^2\left||z|^2 - 1\right|^2}{s^2}\Big) = 0, \\ \text{i.e., } \big(|z|^2 - 1\big)\Big((1+\sqrt{5}) + 2\big(|z|^2 - 1\big)\Big) = 0. \end{aligned}$$

Hence, $|z|^2 = 1$ (as $|z|^2 = \frac{1-\sqrt{5}}{2}$, a contradiction) which implies $|a_0| = 1$. From (3.18), it follows that $a_1 = 0$ and consequently $a_4 = 0$. Therefore, in this case (i.e., when (3.18) is true), (1.5) reduces to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & -a_3 & 0 \end{bmatrix} \text{ with } |a_0| = 1.$$

Thus, (3.21) becomes

$$a_0\omega^5 + a_3\omega^2 = 1. \tag{3.22}$$

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Since $|\alpha_5| = \frac{1}{|z|} = 1$, we can say that equation (3.22) has five unimodular solutions. If the fifth root is not distinct, then α_5 will be a double root of (3.22), and thus, $a_0\alpha_5^5 + a_3\alpha_5^2 - 1 = 0 = 5a_0\alpha_5^4 + 2a_3\alpha_5$. This gives $|a_0| = \frac{2}{3}$, a contradiction. Hence, the conditions of Lemma 3.1 hold for five distinct unimodular values of ω, but this does not give the guarantee of having five flat portions on the boundary of the numerical range of A. So, further investigation is needed in this case.
Since all the roots of (3.22) lie on the unit circle, Theorem (A) of [2] implies $a_3 = 0$. Thus, the characteristic equation of the matrix A is finally reduced to $z^5 + a_0 = 0$ (with $|a_0| = 1$). This implies that the eigenvalues of A are 5^{th} roots of unity. Thus, the matrix A becomes unitarily reducible (by Corollary 1.2 of [8]), which is a contradiction.
Hence, if A is a 5×5 unitarily irreducible companion matrix as in (1.5), where at least one of $a_0,\ a_2$ is zero, then A cannot have 4 flat portions on the boundary of its numerical range. □

Remark 3.3. Here, in the proof, equation (3.10) is in the form $\mathrm{Re}\big((a + b\omega)\omega\big) = c$ with $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$. To solve this, we have considered the condition "at least one of $a_0,\ a_2$ is zero" to reduce it to the form $\mathrm{Re}(a\omega) = c$. The conjecture given in [3] for a unitarily irreducible 5×5 companion matrix is now open only for the case when both a_0 and a_2 are non-zero.

We are now going to show the existence of irreducible companion matrices which have 0, 1, 2, 3 flat portions on the boundary of its numerical range.

Example 3.4 (**No flat portion**). We give an example when A is unitarily irreducible 5×5 companion matrix where $f(A) = 0$. Let $a_0 = a_1 = a_2 = a_3 = 0$ and $a_4 = -(1 + i)$ so that we have:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1+i \end{bmatrix}. \tag{3.23}$$

Here, conditions of Corollary 2.5 of [3] are satisfied, and thus, A has no flat portion on the boundary of its numerical range as shown in Figure 7.

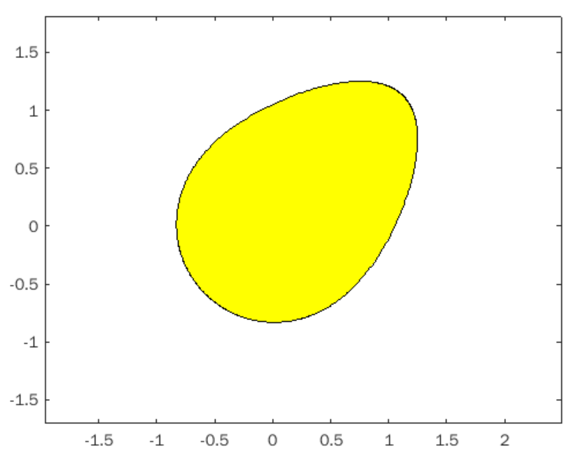

FIGURE 7. *Numerical range $W(A)$ given in (3.23).*

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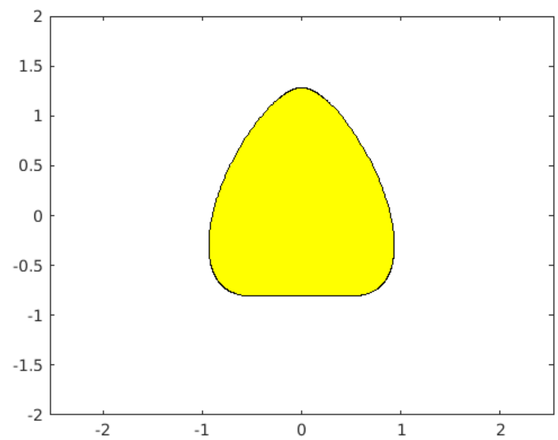

FIGURE 8. *Numerical range $W(A)$ given in* (3.24).

Example 3.5 (**One flat portion**). We provide an explicit example when A is unitarily irreducible 5×5 companion matrix and $f(A)=1$. Let $a_0=a_1=a_4=0, a_2=i$ and $a_3=\frac{\sqrt{5}-1}{2}$ so that we have:

$$A=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&-i&\frac{1-\sqrt{5}}{2}&0\end{bmatrix}. \tag{3.24}$$

Here, equation (3.10) is satisfied and (3.8) has one unimodular solution i and two non-unimodular solutions $\sqrt{\frac{5\sqrt{5}-7}{8}}+\frac{1-\sqrt{5}}{4}i, -\sqrt{\frac{5\sqrt{5}-7}{8}}+\frac{1-\sqrt{5}}{4}i$. For $\omega=i$, we have $\langle \mathrm{Im}(\omega A)x_1,x_2\rangle=1.06331i\neq 0$, where x_1 and x_2 are determined by Theorem 1.4. Thus, the matrix A given by (3.24) has one flat portion on $\partial W(A)$ as shown in Figure 8.

Example 3.6 (**Two flat portions**). We provide an example where A is a unitarily irreducible 5×5 companion matrix such that $f(A)=2$. Let $a_0=a_1=a_4=0, a_2=i$ and $a_3=-\sqrt{\frac{3+\sqrt{5}}{2}}$ so that we have:

$$A=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&-i&\sqrt{\frac{3+\sqrt{5}}{2}}&0\end{bmatrix}. \tag{3.25}$$

Here, equation (3.10) is satisfied and (3.8) has two distinct unimodular solutions $\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}-\frac{1+\sqrt{5}}{4}i$, $-\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}-\frac{1+\sqrt{5}}{4}i$ and non-unimodular solution $\frac{\sqrt{5}-1}{2}i$.
For $\omega=\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}-\frac{1+\sqrt{5}}{4}i$, $\langle \mathrm{Im}(\omega A)x_1,x_2\rangle=0.452254+0.734732i\neq 0$
and for $\omega=-\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}-\frac{1+\sqrt{5}}{4}i$, $\langle \mathrm{Im}(\omega A)x_1,x_2\rangle=-0.452254+0.734732i\neq 0$, where x_1 and x_2 are determined by Theorem 1.4. Thus, the matrix A given by (3.25) has two flat portions on $\partial W(A)$ as shown in Figure 9.

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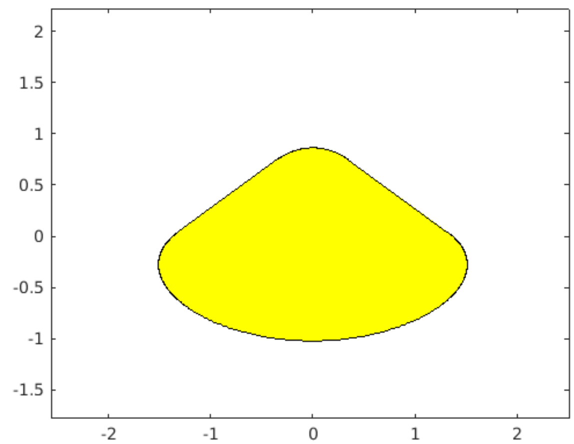

FIGURE 9. *Numerical range $W(A)$ given in (3.25).*

Example 3.7 (**Three flat portions**). We provide an explicit example when A is unitarily irreducible 5×5 companion matrix and $f(A)=3$.
Let $a_1=\sqrt{\frac{2(\sqrt{5}-1)}{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}}$, $a_2=\sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}}$,
$a_3=1-\left(\frac{\sqrt{2(\sqrt{5}-1)}-\sqrt{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}}{\sqrt{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}}\right)\left(\frac{1+\sqrt{5}}{2}\right)$ and $a_4=\frac{2\sqrt{2}\sqrt{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}}{\sqrt{\sqrt{5}-1}\left(\sqrt{5}-3+\sqrt{22+2\sqrt{5}}\right)}$ so that we have:

$$A=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&-a_1&-a_2&-a_3&-a_4\end{bmatrix}. \tag{3.26}$$

Here, equation (3.8) has three distinct unimodular solutions and (3.15) is satisfied. Three unimodular roots of (3.8) are $-1,\ y\pm i\sqrt{1-y^2}$ where

$$y=\frac{1}{2}\left(1-\sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}-1)}}-\sqrt{\frac{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}+1)}}\right)$$

and non-unimodular root is $\sqrt{\frac{\sqrt{5}+\sqrt{22+2\sqrt{5}}-3}{2(1+\sqrt{5})}}$. For $\omega=-1$, $\langle \mathrm{Im}(\omega A)x_1,x_2\rangle=1.16193i\neq 0$, for

$$\begin{aligned}\omega=&\frac{1}{2}\left(1-\sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}-1)}}-\sqrt{\frac{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}+1)}}\right)\\&+i\sqrt{1-\left[\frac{1}{2}\left(1-\sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}-1)}}-\sqrt{\frac{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}+1)}}\right)\right]^2},\end{aligned}$$

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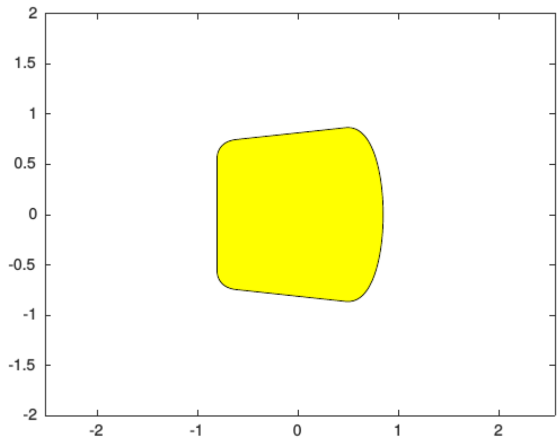

FIGURE 10. *Numerical range $W(A)$ given in* (3.26).

$\langle \mathrm{Im}(\omega A)x_1, x_2\rangle = -0.165937 + 1.28128i \neq 0$ and for

$$\omega = \frac{1}{2}\left(1 - \sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}-1)}} - \sqrt{\frac{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}+1)}}\right) - i\sqrt{1-\left[\frac{1}{2}\left(1 - \sqrt{\frac{\sqrt{5}-7+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}-1)}} - \sqrt{\frac{\sqrt{5}-3+\sqrt{22+2\sqrt{5}}}{2(\sqrt{5}+1)}}\right)\right]^2},$$

$\langle \mathrm{Im}(\omega A)x_1, x_2\rangle = 0.165937 + 1.28128i \neq 0$, where x_1 and x_2 are determined by Theorem 1.4. Thus, the matrix A given by (3.26) has three flat portions on $\partial W(A)$ as shown in Figure 10.

NOTE 3.8. *All numerical ranges are plotted using the program given by C. Cowen and E. Harel, available at http://www.math.iupui.edu/~ccowen/Downloads/33NumRange.html. Numerical calculations have been done in this article using Mathematica.*

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