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NEW RESULTS ON M-MATRICES, H-MATRICES AND THEIR INVERSE CLASSES*

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Abstract. In this article, some new results on M-matrices, H-matrices and their inverse classes are proved. Specifically, we study when a singular Z-matrix is an M-matrix, convex combinations of H-matrices, almost monotone H-matrices and Cholesky factorizations of H-matrices.

Key words. M-matrix, Inverse M-matrix, H-matrix, Inverse H-matrix, Almost monotone matrix, Cholesky factorization.

AMS subject classifications. 15A48, 15A23, 15A09, 15A18.

1. Introduction.

An H-matrix A is a square complex matrix whose comparison matrix $\mathcal{M}(A)$ is an M-matrix. In the overarching theory and applications of nonnegative matrices, the concepts of M-matrix and its generalizations to Z-matrix and H-matrix are indeed ubiquitous because of their natural occurrence in linear systems resulting from discretization of differential equations, Markov chains, Leontieff models of the economy, dynamical systems and control theory in engineering, the linear complementarity problem in optimization, among many other areas. As comprehensive references for the theory and applications of these matrix classes, we mention [2] and [11].

Even with all the attention H-matrices and related classes have received, there are still unexplored aspects of their theory regarding the eigenstructure, mapping properties, invertibility, factorizations and their transformations. In this regard, the present work addresses, among others, the following questions and considerations: When is a singular Z-matrix equal to a singular M-matrix based on Schur complementation? Study convex combinations of H-matrices, almost monotone H-matrices and Cholesky factorizations of H-matrices.

Our presentation proceeds as follows: Section 2 contains notation, definitions and results to be cited or generalized. New results on M-matrices and inverse M-matrices are found in subsection 3.1. H-matrices and inverse H-matrices are considered in subsection 3.2. Finally, Cholesky factorizations for certain subclasses of H-matrices are discussed in subsection 3.3.

2. Preliminaries and terminology.

Let $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) denote the space of all real (complex) matrices of order $m \times n$. When $n = 1$, we denote these simply by $\mathbb{R}^m$ or $\mathbb{C}^m$. The transpose of A will be denoted by A^T, the range of A by $R(A)$ and the null space of A by $N(A)$. The dimension of $R(A)$ (rank of A) is denoted by $\mathrm{rk}(A)$.

A matrix $A = (a_{ij})$ is said to be *nonnegative* ($A \geq 0$) if $a_{ij} \geq 0$ for all i, j. If $a_{ij} > 0$ for each i, j then A is said to be *positive* ($A > 0$). Similar terminology and notation is used for vectors $x = (x_i) \in \mathbb{R}^n$. Array

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nonnegativity naturally induces a partial order, for example, for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{m\times n}$, $A \geq B$ means $a_{ij} \geq b_{ij}$, for all i, j.

Recall that a square matrix A is said to be *reducible*, if it is permutationally similar to $\left(\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right)$, where X and Z are both square matrices. If a matrix is not reducible, then it is said to be an *irreducible* matrix. The *Frobenius normal form* of a reducible matrix A is given by a block triangular matrix $PAP^T = (R_{ij})$, $i, j = 1, 2, \ldots, p$, in which each square diagonal block R_{ii} is either irreducible or a 1×1 null matrix, and P is a permutation matrix. If A is a symmetric reducible matrix, then one has $PAP^T = \operatorname{diag}(R_{11}, R_{22}, \ldots, R_{pp})$.

2.1. Schur complements. Let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, whose elements are in the ascending order. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, then $A[\alpha, \beta]$ denotes the submatrix of A that lies in the rows and columns indexed by the subsets α and β, respectively. When α or β is empty, the corresponding submatrix is considered vacuous; if it is square, by convention it has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$ and refer to it as a *principal* submatrix of A.

Given $\alpha \subseteq \{1, 2, \ldots, n\}$, we let α^c denote the complement of α in $\{1, 2, \ldots, n\}$. Next, let $A[\alpha]$ be invertible. Then, the *Schur complement* of $A[\alpha]$ in A is denoted and defined by

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c].$$

For a given matrix A, let $A[\alpha]$ be invertible for some α. Then, A is invertible if and only if $A/A[\alpha]$ is invertible. This follows from the formula

$$\det(A) = \det(A[\alpha]) \det(A/A[\alpha]).$$

The next result will be used quite frequently in our discussion.

Lemma 2.1. [1, Lemma 2.1]
If the principal submatrix $A[\alpha]$ of A and the corresponding Schur complement $A/A[\alpha]$ in A are invertible, then

$$(A^{-1})[\alpha^c] = (A/A[\alpha])^{-1}.$$

2.2. Background on M-matrices and inverse M-matrices. An excellent source for a detailed study of nonnegative matrices and M-matrices is [2]. We summarize some facts to be used here frequently and include some key results for reference.

Given a square matrix B, let $\sigma(B)$ denote the *spectrum* of B and let $\rho(B)$ denote the *spectral radius* of B, that is,

$$\rho(B) = \max\{|\lambda| : \lambda \in \sigma(B)\}.$$

The Perron-Frobenius theorem states that for every square nonnegative matrix B, $\rho(B) \in \sigma(B)$ and there is a nonnegative eigenvector of B corresponding to $\rho(B)$.

A real square matrix whose off-diagonal entries are nonpositive is called a *Z-matrix*. Any Z-matrix A may be represented as $A = sI - B$, where B is a nonnegative matrix and s is a real number. In such a representation, if in addition one also has $s \geq \rho(B)$, then A is called an *M-matrix*. If $s > \rho(B)$, then by the Perron-Frobenius theorem, A is an invertible M-matrix. If $s = \rho(B)$, then A is a singular M-matrix.

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It is well-known that a Z-matrix A is a nonsingular M-matrix if and only if A^{-1} exists and $A^{-1} \geq 0$; if A is additionally an irreducible Z-matrix, then A is a nonsingular M-matrix if and only if A is *inverse positive*, that is, A^{-1} exists and $A^{-1} > 0$.

Of interest to us herein are two additional necessary and sufficient conditions for a Z-matrix A to be an invertible M-matrix: (1) that A is *positive stable*, that is, the real part of each eigenvalue of A is positive; (2) that A is *semipositive*, that is, there exists a vector $x > 0$ such that $Ax > 0$. Such a vector x will be referred to as a *semipositivity vector* for x.

Note also that a real matrix A is an M-matrix if and only if $A + \epsilon I$ is an invertible M-matrix for all $\epsilon > 0$. While this fact is useful in a few instances to prove results on singular M-matrices, this does not apply in many crucial situations. This is due to the reason that singular M-matrices behave significantly differently from their invertible counterpart. One instance of this is illustrated in Theorem 2.2.

A matrix is called an *inverse M-matrix* if it is invertible and its inverse is an M-matrix. Since the inverse of an invertible M-matrix is nonnegative, inverse M-matrices form an important subclass of the set of all nonnegative matrices. We refer the reader to [9], [10] and the monograph [11] for more details on inverse M-matrices.

In the next result, we collect some of the important properties of singular, irreducible M-matrices. Let us recall that a square matrix A is called *almost monotone* if $Ax \geq 0 \Longrightarrow Ax = 0$. An example of an almost monotone matrix is given by $A = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$.

Theorem 2.2. [2, Theorem 4.16]
Let $A \in \mathbb{R}^{n\times n}$ be a singular, irreducible M-matrix. Then the following statements hold.

(a) *Every proper principal submatrix of A is a nonsingular M-matrix.*
(b) $\mathrm{rk}(A) = n - 1$.
(c) *There exists $x > 0$ such that $Ax = 0$.*
(d) *A is almost monotone.*

2.3. H-matrices and inverse H-matrices. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. The *comparison matrix* of A, denoted by $\mathcal{M}(A) = (m_{ij})$, is defined by

$$m_{ij} = \begin{cases} |a_{ij}|, & i = j. \\ -|a_{ij}|, & i \neq j. \end{cases}$$

Observe that $\mathcal{M}(A)$ is a Z-matrix. A is called an *H-matrix* if $\mathcal{M}(A)$ is an M-matrix.

The set of *equimodular matrices* associated with A, denoted by $\Omega(A)$, is defined by

$$\Omega(A) = \{B \in \mathcal{C}^{n\times n} : \mathcal{M}(A) = \mathcal{M}(B)\}.$$

Note that both A and $\mathcal{M}(A)$ are in $\Omega(A)$. The authors of [3] show that there are three distinguishing types of H-matrices that one must take into account, in any consideration. While we shall not discuss all these three classes, we shall be concerned with the class $\mathcal{H}_I$ that consists of all those matrices A for which the comparison matrix $\mathcal{M}(A)$ is invertible, that is,

$$\mathcal{H}_I = \{A \in \mathbb{C}^{n\times n} : \mathcal{M}(A) \text{ is invertible}\}.$$

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Let us just add the rather well-known fact that, whenever the comparison matrix is invertible, all the matrices belonging to the equimodular class are invertible.

An invertible matrix $A \in \mathbf{C}^{n\times n}$ is called an *inverse H-matrix*, if $\mathcal{M}(A^{-1})$ is an invertible M-matrix. In the notation given earlier, this means that $A^{-1} \in \mathcal{H}_I$. This is stronger than requiring a nonsingular matrix A be an H-matrix. For instance, $A = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix} \right)$, is invertible, whereas $\mathcal{M}(A) = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix} \right)$ is singular. Also observe that $\mathcal{M}(A^{-1}) = \frac{1}{2}\mathcal{M}(A)$ is singular.

The class of inverse H-matrices has been introduced and investigated in [5]. It follows that an inverse M-matrix is an inverse H-matrix ([5, Remark 1.6]). Some classes of nested matrices have been identified to be inverse H-matrices (Theorem 2.5, Theorem 2.7 and Theorem 2.9 [5]).

Remark 2.3. Let A be a triangular matrix with non-zero diagonal entries. Then, A is an H-matrix. This follows from the fact that all the diagonal entries of $\mathcal{M}(A)$ are positive and coincide with the eigenvalues of $\mathcal{M}(A)$.

Remark 2.4. All diagonal entries of an H-matrix whose comparison matrix is invertible are non-zero. This is because the diagonal entries of the invertible $\mathcal{M}(A)$ are necessarily positive. There certainly exist H-matrices with zero diagonal entries, which must be reducible. Indeed, an irreducible H-matrix, has the property that all its diagonal entries are non-zero [4, Theorem 3].

H-matrices are intimately connected to diagonally dominant matrices and matrices with positive principal minors as reviewed below.

The matrix $A \in \mathbb{C}^{n\times n}$ is called *diagonally dominant* (DD) if

$$|a_{ii}| \geq \sum_{i\neq j} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$

If all the inequalities above hold strictly, A is called *strictly diagonally dominant* (SDD). A is said to be *(strictly) generalized diagonally dominant* ((S)GDD), if there exists a positive diagonal matrix D such that AD is (S)DD.

Remark 2.5. It is well-known that $\mathcal{M}(A)$ is an invertible M-matrix if and only if A is GSDD. We also have that an irreducible $A \in \mathbb{C}^{n\times n}$ is an H-matrix if only if A is GDD [4, Theorem 4]. These two facts have been used to obtain numerical characterizations of H-matrices by iteratively seeking the diagonal scaling D; see [12].

A complex square matrix each of whose principal minor is positive is called a *P-matrix*. For a P-matrix A, every real eigenvalue of every principal submatrix of A is positive. The converse is also true. That is, if every real eigenvalue of every principal submatrix of a matrix A is positive, then A is a P-matrix. Another necessary and sufficient condition for a real matrix A to be a P-matrix is that, for every signature matrix S, there exists $x > 0$ such that $SASx > 0$ [13, Theorem 1]. Recall that a matrix $A = (a_{ij})$ is said to be *quasidominant* if there exists a positive vector $d = (d_1, d_2, \ldots, d_n)^T$ such that for each i, the following inequality holds:

$$d_i a_{ii} > \sum_{j\neq i} d_j |a_{ij}|.$$

It is known that a square matrix A is a quasidominant if and only if there exists $x > 0$ such that $SASx > 0$, for every signature matrix S; see [13, Theorem 2]. Let A be a real matrix with positive diagonal entries.

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Then, A is quasidominant if and only if $A \in \mathcal{H}_I$. The following is a simple consequence of this observation: If $A \in \mathbb{R}^{n\times n}$ belongs to $\mathcal{H}_I$ and has positive diagonal entries, then A is a P-matrix.

3. New results. In this section, we obtain new results on M-matrices, H-matrices and their inverse classes.

3.1. M-matrices and inverse M-matrices. Let us start by recalling the following result: Let A be a Z-matrix such that for some $\alpha \subseteq \{1, 2, \dots, n\}$ both the principal submatrix $A[\alpha]$ as well as the Schur complement $A/A[\alpha]$ are invertible M-matrices. Then, A is an invertible M-matrix [1, Theorem 2.2]. In the next result, we obtain an analogue for the case of singular M-matrices, while Theorem 3.2 studies inverse M-matrices.

THEOREM 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a singular Z-matrix. If for some $\alpha \subseteq \{1, 2, \dots, n\}$ $A[\alpha]$ is an invertible M-matrix and $A/A[\alpha]$ is a singular M-matrix, then A itself is an M-matrix.*

Proof. Let $\epsilon > 0$. We show that $A + \epsilon I$ is an invertible M-matrix. Note that for the given subset α, the matrix $A[\alpha] + \epsilon I[\alpha] = (A + \epsilon I)[\alpha]$ is an invertible M-matrix. Then, the Schur complement of $A + \epsilon I$ with respect to the subset α is well defined and is given by

$$(A + \epsilon I)/(A + \epsilon I)[\alpha] = (A[\alpha^c] + \epsilon I[\alpha^c]) - A[\alpha^c, \alpha](A[\alpha] + \epsilon I[\alpha])^{-1}A[\alpha, \alpha^c].$$

Since $A[\alpha^c, \alpha], A[\alpha, \alpha^c]$ are nonpositive matrices and as $(A[\alpha] + \epsilon I[\alpha])^{-1} \geq 0$, it follows that $(A + \epsilon I)/(A + \epsilon I)[\alpha]$ is a Z-matrix. Further, $A/A[\alpha] + \epsilon I[\alpha^c]$ is an invertible M-matrix. Also, since $A[\alpha]$ is an invertible M-matrix and $A[\alpha] \leq (A[\alpha] + \epsilon I[\alpha])$, we have $(A[\alpha] + \epsilon I[\alpha])^{-1} \leq A[\alpha]^{-1}$. Hence, we have

$$\begin{aligned}(A[\alpha^c] &+ \epsilon I[\alpha^c]) - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c] \\ &\leq (A[\alpha^c] + \epsilon I[\alpha^c]) - A[\alpha^c, \alpha](A[\alpha] + \epsilon I[\alpha])^{-1}A[\alpha, \alpha^c],\end{aligned}$$

which implies that

$$(A/A[\alpha] + \epsilon I[\alpha^c]) \leq (A + \epsilon I)/(A + \epsilon I)[\alpha].$$

Since $A/A[\alpha] + \epsilon I[\alpha^c]$ is an invertible M-matrix, it now follows that

$$(A + \epsilon I)/(A + \epsilon I)[\alpha],$$

is an invertible M-matrix. From the remark made earlier, it now follows that $A + \epsilon I$ is an invertible M-matrix. □

THEOREM 3.2. *Let $A \in \mathbb{R}^{n\times n}$ such that A^{-1} is a Z-matrix. If for some $\alpha \subseteq \{1, 2, \dots, n\}$ both $A[\alpha]$ and $A/A[\alpha]$ are inverse M-matrices, then A is an inverse M-matrix.*

Proof. The matrices $A[\alpha]$ and $A/A[\alpha]$ are inverse M-matrices. So, $(A[\alpha])^{-1}$ and $(A/A[\alpha])^{-1}$ are M-matrices. As pointed earlier, $(A^{-1})[\alpha^c] = (A/A[\alpha])^{-1}$ and so $A/A[\alpha] = ((A^{-1})[\alpha^c])^{-1}$. Reversing the roles of A and A^{-1}, as well as α and α^c, we then have $(A[\alpha])^{-1} = A^{-1}/A^{-1}[\alpha^c]$ By applying the result for invertible M-matrices (as stated earlier), it follows that A^{-1} is an invertible M-matrix, that is, A is an inverse M-matrix. □

REMARK 3.3. *It is quite well-known that the sum of two M-matrices is not necessarily an M-matrix. Let $A, B \in \mathbb{R}^{n\times n}$ be invertible M-matrices (so that their diagonal entries are positive). Then, A is said to*

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proportionally dominate B rowwise, if for each i, j we have $\frac{a_{ij}}{a_{ii}} \geq \frac{b_{ij}}{b_{ii}}$. Columnwise proportional dominance is defined analogously. If A proportionally dominates B rowwise or columnwise, then for any numbers $r, s \geq 0$ with $r+s > 0$, the matrix $M_{rs} := rA + sB$ is an M-matrix [6]. Let us also include a more general result. Let A, B be invertible M-matrices possessing a common semipositivity vector, viz., there exists $u > 0$ such that $Au, Bu > 0$. It follows that the convex combination $M_\alpha := \alpha A + (1-\alpha)B$ is an M-matrix for all $\alpha \in [0, 1]$ [16, Theorem 3.4]. More can be said. The matrix M_α is an M-matrix for all $\alpha \in [0, 1]$ if and only if the matrix $B^{-1}A$ has no negative real eigenvalues. As a consequence, we also have that M_α is an M-matrix if and only if $B^{-1}A$ is an M-matrix [7]. These considerations motivate us to consider the question of when the convex combination of inverse M-matrices is again an inverse M-matrix.

Theorem 3.4. *Let A, B be inverse M-matrices such that AB^{-1} (or BA^{-1}) is a positive diagonal matrix. Then, for any $0 < \lambda < 1$, the convex combination $\lambda A + (1-\lambda)B$ is an inverse M-matrix.*

Proof. Let A and B be inverse M-matrices such that AB^{-1} is a positive diagonal matrix. First, observe that for $0 < \lambda < 1$, $\lambda A + (1-\lambda)B \geq 0$, since $A, B \geq 0$. Also,

$$(\lambda A + (1-\lambda)B)^{-1} = B^{-1}(\lambda AB^{-1} + (1-\lambda)I)^{-1}.$$

Since AB^{-1} is a positive diagonal matrix, $D = (\lambda AB^{-1} + (1-\lambda)I)^{-1}$ is also a positive diagonal matrix. Consider

$$(\lambda A + (1-\lambda)B)^{-1} = B^{-1}D.$$

Since B^{-1} is a Z-matrix (as it is an M-matrix), the product $B^{-1}D$ is a Z-matrix, showing that $(\lambda A + (1-\lambda)B)^{-1}$ is a Z-matrix. Thus, $(\lambda A + (1-\lambda)B)^{-1}$ is an M-matrix.
If BA^{-1} is positive, then one uses:

$$(\lambda A + (1-\lambda)B)^{-1} = A^{-1}(\lambda I + (1-\lambda)BA^{-1})^{-1}.$$

□

Remark 3.5. *If one relaxes the hypotheses in Theorem 3.4, by just assuming A to be nonnegative, then the conclusion still holds. Thus, the following assertion is true: Let A be an nonnegative matrix and B be an inverse M-matrix such that AB^{-1} is a positive diagonal matrix. Then, for any $0 < \lambda < 1$, the convex combination $\lambda A + (1-\lambda)B$ is an inverse M-matrix. The corresponding proposition, when the roles of A and B are interchanged, also holds true.*

Corollary 3.6. *Let A, B be inverse M-matrices such that AB^{-1} (or BA^{-1}) is an M-matrix. Suppose that $A^{-1} \leq B^{-1}$. Then, for any $0 < \lambda < 1$ the convex combination $\lambda A + (1-\lambda)B$ is an inverse M-matrix.*

Proof. Since $A \geq 0$, the inequality $A^{-1} \leq B^{-1}$ yields $AB^{-1} \geq I$. Also, since AB^{-1} is an M-matrix, its off-diagonal entries are nonpositive. This means that AB^{-1} is a positive diagonal matrix. By Theorem 3.4, it now follows that $\lambda A + (1-\lambda)B$ is an inverse M-matrix. An entirely similar argument applies when we assume that $B^{-1}A$ is an M-matrix. □

Let us illustrate Theorem 3.4 by an example.

Example 3.7. Let $A = \begin{pmatrix} 2 & 1 & 2 \\ 6 & 5 & 8 \\ 4 & 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1/2 & 1 \\ 3 & 5/2 & 4 \\ 4/3 & 1 & 2 \end{pmatrix}$. Their inverses are given by $A^{-1} = \begin{pmatrix} 3/2 & 0 & -1/2 \\ -1 & 1 & -1 \\ -1/2 & -1/2 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 3 & 0 & -3/2 \\ -2 & 2 & -3 \\ -1 & -1 & 3 \end{pmatrix}$ Then, A and B are inverse M-matrices and $AB^{-1} = D$,

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where $D = \begin{pmatrix} 2&0&0\\0&2&0\\0&0&3 \end{pmatrix}$. For $0 < t < 1$,

$$tA + (1-t)B = \begin{pmatrix} t+1 & (t+1)/2 & t+1 \\ 3(t+1) & 5(t+1)/2 & 4(t+1) \\ (8t+4)/3 & 2t+1 & 4t+2 \end{pmatrix}$$

is a nonnegative matrix and

$$(tA + (1-t)B)^{-1} = \begin{pmatrix} 3/(t+1) & 0 & -3/(4t+2) \\ -2/(t+1) & 2/(t+1) & -3/(2t+1) \\ -1/(t+1) & -1/(t+1) & 3/(2t+1) \end{pmatrix}$$

is a Z-matrix, proving that $tA + (1-t)B$ is an inverse M-matrix.

3.2. H-matrices and inverse H-matrices. We begin with the question of when the convex combination of two H-matrices is again an H-matrix, with Remark 3.3 providing the motivation. We present a framework for an affirmative answer, and this is the result of Theorem 3.8. It is also pertinent to point to the result [16, Theorem 3.16]: Let $A, B \in \mathcal{H}_I$ with real entries. Further, let A, B both be positive stable (meaning that the eigenvalues lie in the open right half of the complex plane). Then, for any $t \in [0,1]$, the matrix $tA + (1-t)B$ is a positive stable H-matrix if $t\mathcal{M}(A) + (1-t)\mathcal{M}(B)$ is an M-matrix for all $t \in [0,1]$. Finally, it must be mentioned that a real matrix $A \in \mathcal{H}_I$ is a positive stable matrix if and only if all its diagonal entries are positive [7]. In our result, we are not assuming that the comparison matrix is invertible.

Let us recall the following [1], [6]: Let A be an invertible M-matrix and B be a Z-matrix satisfying $A \le B$. Then, B is an invertible M-matrix. This result holds for singular M-matrices, too. Let A be a singular M-matrix and B be a Z-matrix satisfying $A \le B$. Then, for any $\epsilon > 0$, we get $A + \epsilon I \le B + \epsilon I$. Also, $A + \epsilon I$ is an invertible M-matrix. Then, $B + \epsilon I$ is an invertible M-matrix, showing that B is an M-matrix.

Theorem 3.8. *Let A and B be two H-matrices with positive diagonal entries such that $\mathcal{M}(A) \le \mathcal{M}(B)$. Then for any $t \in [0,1]$, the matrix $tA + (1-t)B$ is an H-matrix.*

Proof. First, we show that if F and G are matrices with nonnegative diagonal entries, then $\mathcal{M}(F+G) \ge \mathcal{M}(F) + \mathcal{M}(G)$. Let $F = (f_{ij})$ and $G = (g_{ij})$. Since $|f_{ij} + g_{ij}| \le |f_{ij}| + |g_{ij}|$, for all i, j one has the following inequality for the off-diagonal entries:

$$-|f_{ij} + g_{ij}| \ge -|f_{ij}| - |g_{ij}|.$$

As the diagonal entries of F and G are nonnegative, for all i we get

$$|f_{ii} + g_{ii}| = |f_{ii}| + |g_{ii}|,$$

proving the claim.
Now, the diagonal entries of $tA + (1-t)B$ are positive. Then, from what we proved just now, we have

$$\begin{aligned} \mathcal{M}(tA + (1-t)B) &\ge \mathcal{M}(tA) + \mathcal{M}((1-t)B) \\ &= t\mathcal{M}(A) + (1-t)\mathcal{M}(B) \\ &\ge \mathcal{M}(A). \end{aligned}$$

Since $\mathcal{M}(A)$ is an M-matrix, $\mathcal{M}(tA + (1-t)B)$ is an M-matrix. Now we conclude that $tA + (1-t)B$ is an H-matrix. ∎

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Let us illustrate Theorem 3.8 by an example.

Example 3.9. Let

$$A = \begin{pmatrix} 5 & -3 & 2 \\ 1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & 2 & -1 \\ 1 & 5 & 1 \\ 0 & -2 & 7 \end{pmatrix}.$$

Then, one may verify that A and B are H-matrices (having positive diagonal entries) with $\mathcal{M}(A) \leq \mathcal{M}(B)$. For $0 < t < 1$, the matrix

$$tA + (1-t)B = \begin{pmatrix} -t+6 & -5t+2 & 3t-1 \\ 1 & -t+5 & -3t+1 \\ t & 4t-2 & -4t+7 \end{pmatrix}.$$

Let $C := tA + (1-t)B$. The expression for the comparison matrix of C takes one of the following mutually exclusive and collectively exhaustive forms.
For $0 < t < \frac{1}{3}$, it is given by $\mathcal{M}(C) = \left(\begin{smallmatrix} -t+6 & 5t-2 & 3t-1 \\ -1 & -t+5 & 3t-1 \\ -t & 4t-2 & -4t+7 \end{smallmatrix}\right)$.

For $\frac{1}{3} < t < \frac{2}{5}$, the matrix $\mathcal{M}(C) = \left(\begin{smallmatrix} -t+6 & 5t-2 & -3t+1 \\ -1 & -t+5 & -3t+1 \\ -t & 4t-2 & -4t+7 \end{smallmatrix}\right)$.

For the case $\frac{2}{5} < t < \frac{1}{2}$, we have $\mathcal{M}(C) = \left(\begin{smallmatrix} -t+6 & -5t+2 & -3t+1 \\ -1 & -t+5 & -3t+1 \\ -t & 4t-2 & -4t+7 \end{smallmatrix}\right)$,

while for $\frac{1}{2} < t < 1$, it is $\mathcal{M}(C) = \left(\begin{smallmatrix} -t+6 & -5t+2 & -3t+1 \\ -1 & -t+5 & -3t+1 \\ -t & -4t+2 & -4t+7 \end{smallmatrix}\right)$.

In all the cases, $\mathcal{M}(C)$ has a positive row sum, and so $tA + (1-t)B$ is an H-matrix.

The next result addresses the question of the extent to which Theorem 2.2 can be generalized for H-matrices. We show that the first two properties have exact analogues for H-matrices. However, the other two properties do not hold, as illustrated by the matrix $A = \left(\begin{smallmatrix} 2 & 3 \\ 4 & 6 \end{smallmatrix}\right)$.

Theorem 3.10.

(a) *Every principal submatrix of an H-matrix is an H-matrix.*
Let $A \in \mathbb{R}^{n\times n}$ be a singular irreducible H-matrix. Then
(b) *Every proper principal submatrix of A is an invertible H-matrix.*
(c) $\mathrm{rk}(A) = n - 1$.

Proof. (a) Let A be an H-matrix so that $\mathcal{M}(A)$ is an M-matrix. Let $A[\alpha]$ be a principal submatrix of A. Then, the corresponding comparison matrix of $A[\alpha]$ is $\mathcal{M}(A)[\alpha]$, which is a principal submatrix of $\mathcal{M}(A)$. Hence $\mathcal{M}(A)[\alpha]$ is an M-matrix and so $A[\alpha]$ is an H-matrix.
(b) Since A is a singular, irreducible H-matrix, we have that $\mathcal{M}(A)$ is a singular, irreducible M-matrix. Let $A[\alpha]$ be any proper principal submatrix of A so that (as observed above), $\mathcal{M}(A)[\alpha]$ is a principal submatrix of $\mathcal{M}(A)$. Hence using Theorem 2.2, we can say that $\mathcal{M}(A)[\alpha]$ is a nonsingular M-matrix. Hence $A[\alpha]$ is a nonsingular H-matrix.
(c) From (b), in particular, one has, $\mathrm{rk}(A) = n - 1$ proving (c). ∎

As was mentioned earlier, a singular irreducible M-matrix is almost monotone (Theorem 2.2). This motivates us to ask the question as to which singular irreducible H-matrices are almost monotone. Below, we give an interesting answer.

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THEOREM 3.11. *Let A be a singular irreducible real H-matrix with positive diagonal entries. If A is almost monotone, then A is an M-matrix.*

Proof. Since A is an irreducible H-matrix with positive diagonal entries, there exists a positive vector $d = (d_1, d_2, \ldots, d_n)^T$ such that, for all $i = 1, 2, \ldots, n$ we have

$$a_{ii}d_i = |a_{ii}|d_i \geq \sum_{i \neq j} |a_{ij}|d_j \geq \sum_{i \neq j} a_{ij}d_j.$$

From this equation, one concludes that each row sum of AD is nonnegative, where $D = \operatorname{diag}(d_1, d_2, \ldots, , d_n)$. In other words, $(AD)e \geq 0$, where $e^T = (1, 1, \ldots, 1)$. Since A is almost monotone, one has $A(De) = 0$. Hence, for all $i \in \{1, 2, \ldots, n\}$,

$$a_{ii}d_i = -a_{i1}d_1 - \ldots - a_{ii-1}d_{i-1} - a_{ii+1}d_{i+1} - \ldots - a_{in}d_n.$$

Substituting this in the above inequality, we get

$$-\sum_{i \neq j} a_{ij}d_j \geq \sum_{i \neq j} |a_{ij}|d_j.$$

Rewriting the above, we have

$$\sum_{i \neq j} (a_{ij} + |a_{ij}|)d_j \leq 0.$$

Since all the d_j's are positive, it follows that $a_{ij} \leq -|a_{ij}| \leq 0$, showing that all the off-diagonals of A are nonpositive. Thus, A is a Z-matrix and so $A = \mathcal{M}(A)$, proving that A is an M-matrix. ☐

EXAMPLE 3.12. In Theorem 3.11, the assumption that the diagonal entries are positive is indispensable. The matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

is a singular H-matrix. Note that A has a negative diagonal entry (and so it is not an M-matrix), and it is almost monotone.

It is clear that if $A \in \mathcal{H}_I$, then any principal submatrix of A also satisfies this property. In the next result, we have an analogue for inverse H-matrices.

THEOREM 3.13. *Every principal submatrix of an inverse H-matrix is also an inverse H-matrix.*

Proof. Let A be an inverse H-matrix and $A[\alpha]$ be a principal submatrix of A whose rows and columns are indexed by $\alpha \subseteq \{1, 2, \ldots, n\}$. Then, one has $(A[\alpha])^{-1} = A^{-1}/A^{-1}[\alpha^c]$. It was shown in [3] that the Schur complement of an H-matrix whose comparison matrix is invertible is also an H-matrix with invertible comparison matrix, hence one has $(A[\alpha])^{-1}$ is an H-matrix whose comparison matrix is invertible. Thus, $A[\alpha]$ is an inverse H-matrix. ☐

In [5], a closure property of inverse H-matrices is shown to hold for a subclass of inverse H-matrices. This subclass is a proper subclass of class of inverse H-matrices with positive diagonal entries. In the next result we show that this extends to the larger class.

THEOREM 3.14. *Let A be an inverse H-matrix with positive diagonal entries. Then $A + D$ is an inverse H-matrix for every nonnegative diagonal matrix D.*

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Proof. Take $A = B^{-1}$, where $B \in \mathcal{H}_I$. Then, there exists a positive diagonal matrix D_1 such that BD_1 is strictly row diagonally dominant. First, for simplicity, we consider the case $D = \alpha I, \alpha > 0$. Then,

$$\begin{aligned}(\alpha I + A)^{-1} D_1 &= (\alpha I + B^{-1})^{-1} D_1 \\ &= (\alpha D_1^{-1} + (BD_1)^{-1})^{-1}.\end{aligned}$$

Since BD_1 is strictly row diagonally dominant, $\alpha D_1^{-1} + (BD_1)^{-1}$ is strictly column diagonally dominant. So, $(\alpha I + A)^{-1} D_1$ is a strictly row diagonally dominant matrix. Hence, $\alpha I + A \in \mathcal{H}_I$, whenever $\alpha > 0$. We next demonstrate the desired result assuming that D is a positive diagonal matrix. Then, since $A \in \mathcal{H}_I$, $D^{-1}A \in \mathcal{H}_I$. This implies $D^{-1}A + I \in \mathcal{H}_I$. Now $D(D^{-1}A + I) = A + D \in \mathcal{H}_I$. A continuity argument yields the result for a nonnegative diagonal matrix. ∎

It is well-known that, if $A \in \mathbb{R}^{n \times n}$ belongs to $\mathcal{H}_I$ and has positive diagonal entries, then A is a P-matrix [11, Proposition 4.5.11]. The following result shows that an analogous result for inverse H-matrices also holds.

THEOREM 3.15. *Let $A \in \mathbb{R}^{n \times n}$. If A is an inverse H-matrix such that the diagonal entries of A^{-1} are positive, then A is a P-matrix.*

Proof. By definition, $\mathcal{M}(A^{-1})$ is an invertible M-matrix. By the fact that the diagonal entries of A^{-1} are positive, it follows that A^{-1} is a P-matrix. Since the inverse of P-matrix is always a P-matrix [11, Theorem 4.3.2], A is also a P-matrix. ∎

EXAMPLE 3.16. In Theorem 3.15, the assumption that the diagonal entries of A^{-1} are positive cannot be dispensed with. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ 1/5 & -1/2 & -1/10 \\ 1/5 & 0 & 2/5 \end{pmatrix}$. Now one can observe that the row sum of the comparison matrix of A^{-1} are positive and so A is an inverse H-matrix. However, A is not a P-matrix. Note that A^{-1} has a negative diagonal entry.

3.3. Cholesky factorization for H-matrices.

In this section, we pursue the problem of obtaining Cholesky factorizations for some classes of H-matrices. The motivation for such a consideration comes from the following two results.

THEOREM 3.17. [8, Theorem 1] *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric M-matrix. Then, there exists a triangular M-matrix G such that A has the representation $A = GG^T$.*

THEOREM 3.18. [15, Theorem 2] *If A is a symmetric, singular M-matrix, then there exists a permutation matrix P such that $PAP^T = GG^T$, where G is a triangular, singular M-matrix.*

A verbatim analogue of Theorem 3.17 for an invertible H-matrix is false. Let $A = \begin{pmatrix} -2 & 2 \\ 2 & 3 \end{pmatrix}$. Then, $A \in \mathcal{H}_I$. If $A = GG^T$ then,

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

In particular, this yields $a_{11}^2 = -2$, an absurdity. Observe that, for an invertible symmetric matrix which has the representation as in Theorem 3.17, the diagonals must be positive. Interestingly, the moment this extra condition is imposed, one obtains the desired representation. Next, we show this.

THEOREM 3.19. *Let A be a real symmetric matrix with positive diagonal entries. If $A \in \mathcal{H}_I$, then A has a representation $A = GG^T$, where G is a triangular matrix, with $G \in \mathcal{H}_I$.*

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Proof. We prove this result by induction on the order of the matrix. Consider the case when $n = 2$. Let $A = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \in \mathcal{H}_I$ with $a, d > 0$. We claim that there exist $a_{11}, a_{12}, a_{22} \in \mathbb{R}$ such that

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 & a_{11}a_{12} \\ a_{12}a_{11} & a_{12}^2 + a_{22}^2 \end{pmatrix}.$$

Thus, we have the following requirements:

$$a_{11} = \pm\sqrt{a},\ a_{12} = \pm b/\sqrt{a} \text{ and } a_{22}^2 = d - a_{12}^2 = (ad - b^2)/a.$$

Since, $a > 0$ and $\det(A)/a > 0$, the numbers a_{11}, a_{12} and a_{22} are well defined. This establishes the basis step.
Let us suppose that the theorem is true for matrices of order $n - 1$. We can write

$$A = \begin{pmatrix} A_{n-1} & b \\ b^T & a_{nn} \end{pmatrix},$$

where $b \in \mathbb{R}^{n-1}$ (being written as a column vector) and $A_{n-1} \in \mathcal{H}_I$ is symmetric, with positive diagonal entries. By the induction hypothesis, there is a triangular matrix $G_{n-1} \in \mathcal{H}_I$ such that

$$G_{n-1}G_{n-1}^T = A_{n-1}.$$

Clearly, G_{n-1} is nonsingular. Let c be such that $G_{n-1}c = b$. We must show that there exists x such that

$$\begin{pmatrix} G_{n-1} & 0 \\ c^T & x \end{pmatrix} \begin{pmatrix} G_{n-1}^T & c \\ 0 & x \end{pmatrix} = \begin{pmatrix} A_{n-1} & b \\ b^T & a_{nn} \end{pmatrix}. \tag{1}$$

Thus, x is required to satisfy the condition

$$c^T c + x^2 = a_{nn}. \tag{2}$$

Note that

$$\begin{aligned} a_{nn} - c^T c &= a_{nn} - b^T (G_{n-1}G_{n-1}^T)^{-1} b \\ &= a_{nn} - b^T A_{n-1}^{-1} b \\ &= A/A_{n-1}, \end{aligned}$$

is the Schur complement of A_{n-1} in A, which is positive. Thus, equation (2) has a solution given by $x = \sqrt{a_{nn} - c^T c}$. Define

$$G := \begin{pmatrix} G_{n-1} & 0 \\ c^T & \sqrt{a_{nn} - c^T c} \end{pmatrix}.$$

Then, the triangular matrix $G \in \mathcal{H}_I$, due to the fact that every triangular matrix with non-zero diagonal entry belongs to $\mathcal{H}_I$. Equation (1) provides the sought after representation for the matrix A, completing the proof. $\square$

Let us illustrate Theorem 3.19 by an example.

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Example 3.20. Consider

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Clearly, A is an invertible, symmetric, real H-matrix with positive diagonal entries. We can write

$$A = \begin{pmatrix} A_2 & b \\ b^T & a_{33} \end{pmatrix},$$

where $A_2 = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ and $b = (1, 0)^T$. Now we write $A_2 = G_2 G_2^T$, where

$$G_2 = \begin{pmatrix} 2 & 0 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Therefore, $c = G_2^{-1} b = (\frac{1}{2}, -\frac{1}{2\sqrt{2}})^T$ and $\sqrt{a_{33} - c^T c} = \sqrt{\frac{5}{8}}$.

It may be verified that $A = GG^T$, where

$$G = \begin{pmatrix} G_2 & 0 \\ c^T & \sqrt{a_{33} - c^T c} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ \frac{1}{2} & -\frac{1}{2\sqrt{2}} & \sqrt{\frac{5}{8}} \end{pmatrix}.$$

Clearly $G \in \mathcal{H}_I$.

Theorem 3.21. *Let A be a real symmetric, invertible, irreducible, H-matrix such that $\mathcal{M}(A)$ is singular. Let the diagonal entries of A be positive. Then, A can be factorized as $A = GG^T$, where G is a triangular, invertible H-matrix.*

Proof. Similar to the proof of Theorem 3.19. □

For the case of reducible matrices, we have:

Theorem 3.22. *Let A be a symmetric, invertible, real H-matrix such that $\mathcal{M}(A)$ is singular. Suppose that the diagonal entries of A are positive. Then, there exists a permutation matrix P such that $PAP^T = GG^T$, where G is a triangular, invertible, real H-matrix.*

Proof. There exists a permutation matrix P such that

$$PAP^T = \operatorname{diag}(A_1, A_2, \ldots, A_k),$$

where, for each i, A_i is an irreducible matrix. From the given conditions, one has A_i is a symmetric, irreducible real H-matrix with positive diagonal entries. If $\mathcal{M}(A_i)$ is invertible then by Theorem 3.19, there exists G_i, a triangular, invertible real H-matrix such that $A_i = G_i G_i^T$. Else if $\mathcal{M}(A_i)$ is a singular matrix then by Theorem 3.21 there exists G_i, a triangular, invertible real H-matrix such that $A_i = G_i G_i^T$. Thus, $PAP^T = GG^T$, where $G = \operatorname{diag}(G_1, G_2, \ldots, G_k)$ is a invertible, real H-matrix. □

Theorem 3.23. *Let A be a real symmetric, singular H-matrix represented as a block matrix*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

where A_{11} is a nonsingular matrix with positive diagonal entries. If $\operatorname{rk}(A) = \operatorname{rk}(A_{11})$, *then there exists a triangular real H-matrix G such that $A = GG^T$.*

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Proof. Since A_{11} is a symmetric, nonsingular real H-matrix with positive diagonal entries, by Theorem 3.19, one has $A_{11} = G_1 G_1^T$, where G_1 is a triangular, nonsingular real H-matrix. Since rk(A)=rk(A_{11}), one has

$$A_{22} = A_{12}^T (G_1^T)^{-1} G_1^{-1} A_{12}.$$

Thus,

$$\begin{aligned} A &= \begin{pmatrix} G_1 G_1^T & A_{12} \\ A_{12}^T & A_{12}^T (G_1^T)^{-1} G_1^{-1} A_{12} \end{pmatrix} \\ &= \begin{pmatrix} G_1 & 0 \\ A_{12}(G_1^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} G_1^T & G_1^{-1} A_{12} \\ 0 & 0 \end{pmatrix} \\ &= GG^T. \end{aligned}$$

Since every triangular matrix is an H-matrix, G is a (singular) real H-matrix. ∎

Here is an illustrative example.

Example 3.24. Consider

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ A_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } A_{22} = 2.$$

Every proper principal minor of $\mathcal{M}(A)$ is positive, while its determinant is zero. Here $\text{rk}(A) = \text{rk}(A_{11}) = 2$. If $G = \left(\begin{smallmatrix} \sqrt{2} & 0 \\ 1/\sqrt{2} & \sqrt{3/2} \end{smallmatrix}\right)$, then $A_{11} = GG^T$. Now

$$G^{-1} A_{12} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & \sqrt{2/3} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -\sqrt{3/2} \end{pmatrix},$$

and $A_{12}^T (G^T)^{-1} = \left(\begin{smallmatrix} 1/\sqrt{2} & -\sqrt{3/2} \end{smallmatrix}\right)$. One may verify that

$$A = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & \sqrt{3/2} & 0 \\ 1/\sqrt{2} & -\sqrt{3/2} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & -\sqrt{3/2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Corollary 3.25. *Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric, singular, irreducible H-matrix, whose first $n-1$ diagonal entries are positive. Then, there exists a triangular real H-matrix G such that $A = GG^T$.*

Proof. The matrix A may be partitioned as

$$A = \begin{pmatrix} A_{11} & b \\ b^T & a_{nn} \end{pmatrix},$$

where A_{11} is a real symmetric H-matrix with positive diagonal entries. Since A is a singular irreducible H-matrix, by Theorem 3.10, all its proper principal submatrices are nonsingular. Hence, rk(A_{11})=rk(A). Now, the proof follows from Theorem 3.23. ∎

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Corollary 3.26. *Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric, singular, irreducible, H-matrix with $n-1$ positive diagonal entries. Then, there exists a permutation matrix P such that $PAP^T = GG^T$, where G is a triangular, singular real H-matrix.*

Proof. From the hypotheses on the matrix A, it follows that there exists a permutation matrix P such that PAP^T is a symmetric, singular, irreducible, real H-matrix with first $n-1$ diagonal entries positive. Now, the proof follows from the previous corollary. ∎

Theorem 3.27. *Let A be a real symmetric, singular, H-matrix whose diagonal entries are nonnegative. Then there exists a permutation matrix P such that $PAP^T = GG^T$, where G is a triangular, singular, real H-matrix.*

Proof. There exists a permutation matrix P such that

$$PAP^T = \operatorname{diag}(A_1, A_2, \ldots, A_k),$$

where A_i is either an irreducible matrix or the zero matrix of order 1. Further, each A_i is a real symmetric, H-matrix. We know that each diagonal entry of an irreducible H-matrix is non-zero. If A_i is irreducible, then all its diagonal entries are positive. Hence, there exists G_i, a triangular real H-matrix such that $A_i = G_i G_i^T$. If A_i is a zero matrix then we can choose $G_i = 0$. Thus, $PAP^T = GG^T$, where $G = \operatorname{diag}(G_1, G_2, \ldots, G_k)$ is a singular, real H-matrix. ∎

Here is a numerical example.

Example 3.28. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

If we take

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then we get

$$PAP^T = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Thus, $PAP^T = GG^T$, where

$$G = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & \sqrt{3/2} & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & -\sqrt{3/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The concluding result shows that an analogue of Theorem 3.19 holds for inverse H-matrices. In fact, it is a consequence of that result.

THEOREM 3.29. *Let A be a symmetric, real inverse H-matrix such that A^{-1} has all its diagonal entries positive. Then, A has a representation $A = GG^T$, where G is a triangular, real inverse H-matrix.*

Proof. The matrix A^{-1} is a symmetric, real H-matrix with positive diagonal entries. By Theorem 3.19, we can write $A^{-1} = GG^T$, where G is a triangular, invertible real H-matrix. Then, $A = (G^T)^{-1}G^{-1}$. Since the inverse of a triangular matrix is again a triangular matrix, the proof is complete. $\square$

Here is an example illustrating Theorem 3.29.

EXAMPLE 3.30. Consider

$$A = \begin{pmatrix} 2/9 & -1/3 \\ -1/3 & 1 \end{pmatrix}.$$

Hence,

$$A^{-1} = \begin{pmatrix} 9 & 3 \\ 3 & 2 \end{pmatrix},$$

an H-matrix with positive diagonal entries. Set

$$G^T = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}.$$

Then, G is a triangular, inverse H-matrix and $A = GG^T$.

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REFERENCES

[1] T. Ando. Inequalities for M-matrices. *Linear Multilinear Algebra*, 8:91–316, 1980.
[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Classics in Applied Mathematics. SIAM, Philadelphia, PA, 1994.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 729-744, November 2022.

[3] R. Bru, C. Corral, I. Gimenez, and J. Mas. Schur complement of general H-matrices. *Numer. Linear Algebra Appl.*, 16:35–947, 2000.

[4] R. Bru, C. Corral, I. Gimenez, and J. Mas. Classes of general H-matrices. *Linear Algebra Appl.*, 429:2358–2366, 2008.

[5] M. Chatterjee and K.C. Sivakumar. Inverse H-matrices. *Linear Algebra Appl.*, 572:1–31, 2019.

[6] K. Fan. Inequalities for the sum of two M-matrices. In: *Inequalities*. Academic Press, New York, 105–117, 1967.

[7] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.

[8] D.H. Jacobson. Factorization of symmetric M-matrices. *Linear Algebra Appl.*, 9:275–278, 1974.

[9] C.R. Johnson. Inverse M-matrices. *Linear Algebra Appl.*, 47:195–216, 1982.

[10] C.R. Johnson and R.L. Smith. Inverse M-matrices II. *Linear Algebra Appl.*, 435:195–216, 2011.

[11] C.R. Johnson, R.L. Smith, and M.J. Tsatsomeros. *Matrix Positivity*. Cambridge University Press, Cambridge, 2020.

[12] B. Li, L. Li, M. Harada, H. Niki, and M. Tsatsomeros. An Iterative Criterion for H-matrices. *Linear Algebra Appl.*, 271:179–190, 1988.

[13] P.J. Moylan. Matrices with positive principal minors. *Linear Algebra Appl.*, 17:53–58, 1977.

[14] A.M. Ostrowski. Über die Determinanten mit überwiegegender Hauptdiagonale. *Comment. Math. Helv.*, 10:69–96, 1937.

[15] R.L. Smith. Factorization of symmetric singular M-Matrices. *Linear Algebra Appl.*, 23:253–259, 1979.

[16] D.M. Stipanovic and D.D. Siljak. Stability of polytopic systems via convex M-matrices and parameter-dependent Liapunov functions. *Nonlin. Anal.*, 40:589–609, 2000.