

## THOMPSON ISOMETRIES ON POSITIVE OPERATORS: THE 2-DIMENSIONAL CASE\*

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**Abstract.** In this paper, a former result of the first author is completed. The structure is determined of all surjective isometries of the space of invertible positive operators on the 2-dimensional Hilbert space equipped with the Thompson metric or the Hilbert projective metric.

**Key words.** Thompson metric, Hilbert projective metric, Isometry, Invertible positive operators.

**AMS subject classifications.** 46B28, 47B49.

**1. Introduction and Statement of the Results.** In [3], the first author of the present paper described the structure of all isometries of the space of invertible positive operators on a Hilbert space  $H$  equipped with the Thompson metric or the Hilbert projective metric under the condition  $\dim H \geq 3$ . Apparently, it is a natural question to ask what happens in the case when  $\dim H = 2$ . Indeed, the referee of [3] posed that question which the author could not answer at that time; he only conjectured that the same conclusion should hold as in higher dimensions. The aim of this note is to present a proof of that conjecture.

We begin with recalling some definitions and notation. The general definitions of the Thompson metric and the Hilbert projective metric are as follows. Let  $X$  be a real normed space and  $K \neq \emptyset$  be a closed convex cone in  $X$  which satisfies  $K \cap (-K) = \{0\}$ . For any  $x, y \in X$  we write  $x \leq y$  if  $y - x \in K$ . Clearly,  $\leq$  is a partial order on  $X$ . There is an equivalence relation  $\sim$  on  $K \setminus \{0\}$  which is defined by  $x \sim y$  if and only if we have positive real numbers  $t$  and  $s$  such that  $tx \leq y \leq sx$ . The equivalence classes induced by  $\sim$  are called components. Let  $C$  be a component and for any  $x, y \in C$  denote by  $M(x/y)$  the quantity  $\inf\{t > 0 \mid x \leq ty\}$ . The Thompson metric  $d_T$  on  $C$  is defined by

$$d_T(x, y) = \log \max\{M(x/y), M(y/x)\} \quad (x, y \in C)$$

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where  $\log$  stands for the natural logarithm (for basic properties of  $d_T$  we refer to the original source [4]). Thompson introduced this metric as a modification of the Hilbert projective metric  $d_H$  which is defined on the same component by the formula

$$d_H(x, y) = \log M(x/y)M(y/x) \quad (x, y \in C).$$

Actually,  $d_H$  is not a true metric, only a pseudo-metric. Indeed, for any  $x, y \in C$  we have  $d_H(x, y) = 0$  if and only if there is a positive number  $\alpha$  such that  $x = \alpha y$ .

It is well known that in any unital  $C^*$ -algebra  $\mathcal{A}$ , the set  $\mathcal{A}^+$  of all positive elements of  $\mathcal{A}$  is a nonempty closed convex cone in the normed linear space of all self-adjoint elements of  $\mathcal{A}$  which satisfies  $\mathcal{A}^+ \cap (-\mathcal{A}^+) = \{0\}$ . Moreover, the invertible elements in  $\mathcal{A}^+$  form a component in  $\mathcal{A}^+ \setminus \{0\}$  which we denote by  $\mathcal{A}_{-1}^+$ . The Thompson metric on this component is in an intimate connection with the natural Finsler geometry of  $\mathcal{A}_{-1}^+$  which has many applications in several fields of mathematics (see the introduction of [3] and the references therein).

As already mentioned, in [3] the first author described the structure of all bijective isometries of the space of all invertible positive operators acting on a Hilbert space endowed with the Thompson metric or the Hilbert projective metric under the condition  $\dim H \geq 3$ . The aim of this note is to extend those results for the remaining case  $\dim H = 2$ .

In what follows  $H$  denotes a complex Hilbert space and  $B(H)$  stands for the  $C^*$ -algebra of all bounded linear operators on  $H$ . As in [3], it is easy to see that for any invertible bounded linear or conjugate-linear operator  $S$  on  $H$  and for any function  $\tau : B(H)_{-1}^+ \rightarrow ]0, \infty[$  the transformations

$$A \mapsto SAS^*, \quad A \mapsto SA^{-1}S^* \quad (A \in B(H)_{-1}^+)$$

and

$$A \mapsto \tau(A)SAS^*, \quad A \mapsto \tau(A)SA^{-1}S^* \quad (A \in B(H)_{-1}^+)$$

are isometries corresponding to  $d_T$  and  $d_H$ , respectively. The results in [3] tell us that if  $\dim H \geq 3$ , then there are no other kinds of bijective isometries corresponding to those distances. The following statements assert that the same conclusions hold also in the 2-dimensional case.

**THEOREM 1.1.** *Let  $H$  be a complex Hilbert space with  $\dim H = 2$ . Suppose that  $\Phi : B(H)_{-1}^+ \rightarrow B(H)_{-1}^+$  is a surjective isometry with respect to the metric  $d_T$ . Then there exists an invertible linear or conjugate-linear operator  $S$  on  $H$  such that  $\Phi$  is either of the form*

$$\Phi(A) = SAS^* \quad (A \in B(H)_{-1}^+)$$

or of the form

$$\Phi(A) = SA^{-1}S^* \quad (A \in B(H)_{-1}^+).$$

The result concerning the Hilbert projective metric reads as follows.

**THEOREM 1.2.** *Let  $H$  be a complex Hilbert space with  $\dim H = 2$ . Suppose that  $\Phi : B(H)_{-1}^+ \rightarrow B(H)_{-1}^+$  is a surjective isometry with respect to the Hilbert projective metric. Then there exists an invertible linear or conjugate-linear operator  $S$  on  $H$  and a function  $\tau : B(H)_{-1}^+ \rightarrow ]0, \infty[$  such that  $\Phi$  is of the form*

$$\Phi(A) = \tau(A)SAS^* \quad (A \in B(H)_{-1}^+).$$

**REMARK 1.3.** One may wonder why the inverse operation does not show up in the formulation of Theorem 1.2. The fact is that it does in some hidden way. To see this, let  $\{e_1, e_2\}$  be an orthonormal basis of the 2-dimensional space  $H$ . Define the map  $U : H \rightarrow H$  by

$$Ux = \langle e_2, x \rangle e_1 - \langle e_1, x \rangle e_2 \quad (x \in H).$$

It is easy to see that  $U$  is an antiunitary operator with the property that

$$P^\perp = UPU^* \quad (P \in P_1(H)).$$

One can deduce that

$$A^{-1} = \frac{1}{\det A} UAU^* \quad (A \in B(H)_{-1}^+)$$

implying that every transformation on  $B(H)_{-1}^+$  of the form

$$A \mapsto \tau(A)SA^{-1}S^*$$

with invertible linear or conjugate-linear operator  $S : H \rightarrow H$  and function  $\tau : B(H)_{-1}^+ \rightarrow ]0, \infty[$  is also of the form that appears in Theorem 1.2.

**2. Proofs.** We start with some necessary notation. The spectrum of an operator  $A \in B(H)$  is denoted by  $\sigma(A)$ . Let  $P_1(H)$  stand for the set of all rank-1 projections on  $H$  and denote by  $\text{tr}$  the trace functional. As usual,  $I$  stands for the identity operator on  $H$ . For  $P \in P_1(H)$  we write  $P^\perp = I - P$ . The diameter of any subset  $K$  of  $\mathbb{R}$  is denoted by  $\text{diam } K$ .

We present the proof of our first result Theorem 1.1.

*Proof.* We first recall the following formula relating to the Thompson metric:

$$(2.1) \quad d_T(A, B) = \|\log A^{-1/2} B A^{-1/2}\| \quad (A, B \in B(H)_{-1}^+).$$

A proof can be found, e.g., in [3].

By an observation made in the paragraph preceding the formulation of Theorem 1.1, we deduce that the transformation

$$A \mapsto \Phi(I)^{-1/2} \Phi(A) \Phi(I)^{-1/2} \quad (A \in B(H)_{-1}^+)$$

is a surjective isometry with respect to the Thompson metric which has the additional property that it sends  $I$  to  $I$ . Hence, regarding the statement of Theorem 1.1 we may and do assume that  $\Phi$  already has this property, i.e.,  $\Phi(I) = I$ .

In the proof of Theorem 1 in [3] we proved that any surjective Thompson isometry preserves the geometric mean of invertible positive operators and then that it also preserves the Jordan triple product, i.e., satisfies

$$(2.2) \quad \Phi(ABA) = \Phi(A)\Phi(B)\Phi(A) \quad (A, B \in B(H)_{-1}^+).$$

Next it was proved at the beginning of the proof of Theorem 1 in [2] that (2.2) implies that

$$(2.3) \quad AB = BA \iff \Phi(A)\Phi(B) = \Phi(B)\Phi(A) \quad (A, B \in B(H)_{-1}^+)$$

and

$$(2.4) \quad \Phi(AB) = \Phi(A)\Phi(B) \quad (A, B \in B(H)_{-1}^+, AB = BA).$$

One can easily check that those arguments in [2, 3] do not require that  $\dim H \geq 3$ , and hence the above observations remain valid also in the present case  $\dim H = 2$ .

As the positive scalar multiples of  $I$  are exactly the elements of  $B(H)_{-1}^+$  which commute with all elements, it follows from (2.3) that there is a function  $f : ]0, \infty[ \rightarrow ]0, \infty[$  such that

$$\Phi(\lambda I) = f(\lambda)I \quad (\lambda > 0).$$

It is clear that  $f(1) = 1$ . Moreover, it follows from (2.1) that

$$(2.5) \quad |\log f(\lambda) - \log f(\mu)| = |\log \lambda - \log \mu| \quad (\lambda, \mu > 0).$$

Let  $F = \log \circ f \circ \exp$ . Then (2.5) implies that  $F : \mathbb{R} \rightarrow \mathbb{R}$  is an isometry. It is well known and in fact very easy to see that there exists a real number  $a$  such that either we have

$$F(x) = a + x \quad (x \in \mathbb{R})$$

or we have

$$F(x) = a - x \quad (x \in \mathbb{R}).$$

As  $f(1) = 1$ , we deduce that either

$$f(\lambda) = \lambda \quad (\lambda > 0)$$

or

$$f(\lambda) = \frac{1}{\lambda} \quad (\lambda > 0).$$

In the first case, let  $\phi = \Phi$ . While in the second case we define  $\phi$  by

$$\phi(A) = \Phi(A)^{-1} \quad (A \in B(H)_{-1}^+).$$

Since the inverse operation is a surjective Thompson isometry (see the paragraph before the formulation of Theorem 1.1), the same holds for  $\phi : B(H)_{-1}^+ \rightarrow B(H)_{-1}^+$ . It is clear that

$$\phi(\lambda I) = \lambda I \quad (\lambda > 0)$$

and  $\phi$  satisfies (2.2), (2.3) and (2.4).

The elements of  $B(H)_{-1}^+$  are exactly the operators of the form  $\lambda P + \mu P^\perp$  where  $P \in P_1(H)$  and  $\lambda, \mu$  are positive real numbers. Let  $P \in P_1(H)$  be fixed. It is easy to see that given two different positive real numbers  $\lambda$  and  $\mu$ , an invertible positive operator commutes with  $\lambda P + \mu P^\perp$  if and only if it is of the form  $\lambda' P + \mu' P^\perp$  where  $\lambda'$  and  $\mu'$  are positive real numbers. This, together with (2.3), yields that there exists a rank-1 projection  $\tilde{P}$  such that for any positive real numbers  $\lambda, \mu$  we have scalars  $\tilde{\lambda}, \tilde{\mu} > 0$  for which

$$\phi(\lambda P + \mu P^\perp) = \tilde{\lambda} \tilde{P} + \tilde{\mu} \tilde{P}^\perp.$$

Therefore, we have functions  $g, h : ]0, \infty[ \rightarrow ]0, \infty[$  such that

$$(2.6) \quad \phi(\lambda P + P^\perp) = g(\lambda) \tilde{P} + h(\lambda) \tilde{P}^\perp \quad (\lambda > 0).$$

It is clear from (2.4) that  $g$  and  $h$  are multiplicative functions.

Since  $\phi$  is multiplicative on the commuting elements of  $B(H)_{-1}^+$  and leaves the scalar operators fixed, we infer that

$$\phi(P + \mu P^\perp) = \frac{\mu}{g(\mu)} \tilde{P} + \frac{\mu}{h(\mu)} \tilde{P}^\perp \quad (\mu > 0).$$

Multiplying this equality with the one in (2.6), we obtain

$$(2.7) \quad \phi(\lambda P + \mu P^\perp) = \mu g\left(\frac{\lambda}{\mu}\right) \tilde{P} + \mu h\left(\frac{\lambda}{\mu}\right) \tilde{P}^\perp \quad (\lambda, \mu > 0).$$

Let  $\lambda, \mu, \delta, \varepsilon > 0$ . Substituting  $A = \lambda P + \mu P^\perp$  and  $B = \delta P + \varepsilon P^\perp$  into the equality (2.1), using (2.7) and the isometric property of  $\phi$  we obtain

$$\max \left\{ \left| \log \frac{\delta}{\lambda} \right|, \left| \log \frac{\varepsilon}{\mu} \right| \right\} = \max \left\{ \left| \log \frac{\varepsilon}{\mu} g \left( \frac{\delta \mu}{\lambda \varepsilon} \right) \right|, \left| \log \frac{\varepsilon}{\mu} h \left( \frac{\delta \mu}{\lambda \varepsilon} \right) \right| \right\}.$$

This yields

$$(2.8) \quad \max\{|\log t|, |\log s|\} = \max \left\{ \left| \log sg \left( \frac{t}{s} \right) \right|, \left| \log sh \left( \frac{t}{s} \right) \right| \right\} \quad (s, t > 0).$$

Substituting  $s = 1$  into this equation, we see

$$|\log g(t)| \leq |\log t|, \quad |\log h(t)| \leq |\log t| \quad (t > 0).$$

Consequently, for the additive functions  $G = \log \circ g \circ \exp$  and  $H = \log \circ h \circ \exp$  we have

$$|G(t) - G(s)| = |G(t - s)| \leq |t - s|, \quad \text{and} \quad |H(t) - H(s)| \leq |t - s|$$

for every  $t, s \in \mathbb{R}$ . This gives us that  $G, H$  are continuous and hence linear. Consequently, they are scalar multiples of the identity. This implies that  $g, h$  are power functions which means that we have real numbers  $a$  and  $b$  such that

$$g(x) = x^a \quad (x > 0)$$

and

$$h(x) = x^b \quad (x > 0).$$

Substituting  $s = 1, t = e$  respectively  $s = e, t = 1$  into (2.8), we obtain that

$$\max\{|a|, |b|\} = \max\{|a - 1|, |b - 1|\} = 1.$$

From this we infer that either  $a = 1$  and  $b = 0$ , or  $a = 0$  and  $b = 1$ . The equation (2.7) implies that in the first case we have

$$\phi(\lambda P + \mu P^\perp) = \lambda \tilde{P} + \mu \tilde{P}^\perp \quad (\lambda, \mu > 0)$$

while in the second case we have

$$\phi(\lambda P + \mu P^\perp) = \mu \tilde{P} + \lambda \tilde{P}^\perp \quad (\lambda, \mu > 0).$$

In both cases we can conclude

$$(2.9) \quad \sigma(\phi(A)) = \sigma(A) \quad (A \in B(H)_{-1}^+).$$

This implies that  $\phi$  preserves the trace of operators

$$(2.10) \quad \operatorname{tr} \phi(A) = \operatorname{tr} A \quad (A \in B(H)_{-1}^+).$$

The elements of  $B(H)_{-1}^+$  with spectrum  $\{1, 2\}$  are exactly the operators of the form  $I + P$  where  $P \in P_1(H)$ . By the spectrum preserving property (2.9) of  $\phi$  this yields that for any  $P \in P_1(H)$  there exists a unique element  $\psi(P)$  of  $P_1(H)$  such that

$$\phi(I + P) = I + \psi(P).$$

Clearly, the map  $\psi : P_1(H) \rightarrow P_1(H)$  is bijective. Let  $P, Q \in P_1(H)$ . Easy computation shows that

$$\operatorname{tr}(I + P)(I + Q)(I + P) = 6 + 3 \operatorname{tr} PQ$$

and

$$\operatorname{tr}(I + \psi(P))(I + \psi(Q))(I + \psi(P)) = 6 + 3 \operatorname{tr} \psi(P)\psi(Q).$$

Referring to the equality (2.2) and the trace preserving property of  $\phi$  we obtain

$$(2.11) \quad \operatorname{tr} PQ = \operatorname{tr} \psi(P)\psi(Q).$$

Bijjective transformations on the set of all rank-1 projections on a Hilbert space of any dimension with the above property (2.11) are said to preserve the transition probability and they play important role in the mathematical foundations of quantum mechanics. In fact, there they are usually called quantum mechanical symmetry transformations. A fundamental theorem of Wigner describes the structure of all those maps. For the present case it says that the transformation  $\psi$  is of the form

$$(2.12) \quad \psi(P) = UPU^* \quad (P \in P_1(H))$$

with some unitary or antiunitary operator  $U$  on  $H$ . For Wigner's theorem and some of its generalizations we refer, e.g., the book [1] (p. 12 in the Introduction and Section 2.1). From (2.12) it follows that

$$\phi(I + P) = U(I + P)U^* \quad (P \in P_1(H)).$$

Define the map  $\tilde{\phi}$  by

$$\tilde{\phi}(A) = U^* \phi(A) U \quad (A \in B(H)_{-1}^+).$$

It is clear that  $\tilde{\phi} : B(H)_{-1}^+ \rightarrow B(H)_{-1}^+$  is a surjective Thompson isometry with the additional property that  $\tilde{\phi}(I + P) = I + P$  ( $P \in P_1(H)$ ). Clearly,  $\tilde{\phi}$  has all the

preserver properties that are possessed by  $\phi$ , in particular it satisfies (2.2), (2.3), (2.9) and (2.10).

Pick any  $P \in P_1(H)$ . By the commutativity preserving property of  $\tilde{\phi}$  and the equality

$$\tilde{\phi}(2P + P^\perp) = 2P + P^\perp$$

we obtain the following: For any pair of positive real numbers  $\lambda, \mu$  we have positive real numbers  $\alpha, \beta$  such that

$$\tilde{\phi}(\lambda P + \mu P^\perp) = \alpha P + \beta P^\perp.$$

Let  $T = \lambda P + \mu P^\perp$ . We compute on the one hand

$$\mathrm{tr}(I + P)T(I + P) = \mathrm{tr} \tilde{\phi}((I + P)T(I + P)) = \mathrm{tr} \tilde{\phi}(I + P)\tilde{\phi}(T)\tilde{\phi}(I + P) =$$

$$\mathrm{tr}(I + P)\tilde{\phi}(T)(I + P) = \mathrm{tr} \tilde{\phi}(T) + 3 \mathrm{tr} \tilde{\phi}(T)P = \mathrm{tr} T + 3\alpha,$$

and on the other hand

$$\mathrm{tr}(I + P)T(I + P) = \mathrm{tr} T + 3 \mathrm{tr} TP = \mathrm{tr} T + 3\lambda.$$

This gives us that  $\alpha = \lambda$  and next that  $\beta = \mu$ . Consequently, we obtain that  $\tilde{\phi}(A) = A$  ( $A \in B(H)_{-1}^+$ ). Transforming back first to  $\phi$  and then to  $\Phi$  and having in mind how we have reached the assumption  $\Phi(I) = I$ , we deduce that our original transformation  $\Phi$  is necessarily of one of the forms appearing in the statement of Theorem 1.1.  $\square$

Turning to the proof of our second statement, just as in [3] we denote by  $\overline{A}$  the set of all positive scalar multiples of the invertible positive operator  $A \in B(H)$ . As noted in the introduction, we have

$$\overline{A} = \{B \in B(H)_{-1}^+ : d_H(A, B) = 0\} \quad (A \in B(H)_{-1}^+).$$

Now the proof of Theorem 1.2 is as follows.

*Proof.* We know from the introduction of [3] that the Hilbert projective metric can be calculated in the following way:

$$(2.13) \quad d_H(A, B) = \mathrm{diam} \log \sigma(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \quad (A, B \in B(H)_{-1}^+).$$

Similarly to the proof of our first result we see that the map

$$A \mapsto \Phi(I)^{-1/2}\Phi(A)\Phi(I)^{-1/2} \quad (A \in B(H)_{-1}^+)$$



is a surjective isometry of  $B(H)_{-1}^+$  corresponding to the Hilbert projective metric which has the additional property that it sends  $I$  to  $I$ . Therefore, we may and do assume that already  $\Phi$  has this property, i.e.,  $\Phi(I) = I$ . Then in the same way as in the proof of [3, Theorem 2] one can deduce that

$$(2.14) \quad \overline{\Phi(ABA)} = \overline{\Phi(A)\Phi(B)\Phi(A)} \quad (A, B \in B(H)_{-1}^+).$$

Again, as in the proof of [3, Theorem 2] this yields that  $\Phi$  preserves the commutativity, i.e., we have

$$AB = BA \iff \Phi(A)\Phi(B) = \Phi(B)\Phi(A) \quad (A, B \in B(H)_{-1}^+).$$

Let now  $P \in P_1(H)$  be fixed. In a similar way as in the proof of Theorem 1.1, we infer that there is a projection  $\tilde{P} \in P_1(H)$  such that for any numbers  $\lambda, \mu > 0$ , we have positive scalars  $\tilde{\lambda}, \tilde{\mu}$  for which

$$\Phi(\lambda P + \mu P^\perp) = \tilde{\lambda} \tilde{P} + \tilde{\mu} \tilde{P}^\perp.$$

Consequently, there exist functions  $f, g : ]0, \infty[ \rightarrow ]0, \infty[$  such that

$$(2.15) \quad \Phi(\lambda P + P^\perp) = f(\lambda) \tilde{P} + g(\lambda) \tilde{P}^\perp \quad (\lambda > 0).$$

Let  $\lambda, \mu > 0$ . Using the isometric property of  $\Phi$  for  $A = \lambda P + P^\perp$  and  $B = \mu P + P^\perp$ , and applying (2.13), (2.15) we obtain

$$\left| \log \frac{\mu}{\lambda} \right| = \left| \log \frac{f(\mu)}{f(\lambda)} - \log \frac{g(\mu)}{g(\lambda)} \right|.$$

Define  $h = f/g$ . We infer that

$$|\log \mu - \log \lambda| = |\log h(\mu) - \log h(\lambda)| \quad (\lambda, \mu > 0).$$

Since  $h(1) = 1$ , just as in the proof of Theorem 1.1 we can infer that either

$$h(\lambda) = \lambda \quad (\lambda > 0)$$

or

$$h(\lambda) = \frac{1}{\lambda} \quad (\lambda > 0).$$

In the first case we compute

$$\begin{aligned} \overline{\Phi(\lambda P + \mu P^\perp)} &= \overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\Phi((\lambda/\mu)P + P^\perp)} = \\ &= \overline{\Phi((\lambda/\mu)P + P^\perp)} = \overline{(\lambda/\mu)\tilde{P} + \tilde{P}^\perp} = \tilde{\lambda}\tilde{P} + \tilde{\mu}\tilde{P}^\perp \end{aligned}$$

while in the second case we have

$$\begin{aligned}\overline{\Phi(\lambda P + \mu P^\perp)} &= \overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\Phi((\lambda/\mu)P + P^\perp)} = \\ \overline{\Phi((\lambda/\mu)P + P^\perp)} &= (\mu/\lambda)\widetilde{P} + \widetilde{P}^\perp = \mu\widetilde{P} + \lambda\widetilde{P}^\perp\end{aligned}$$

for any scalars  $\lambda, \mu > 0$ . Therefore, in either case there is a projection  $\widehat{P} \in P_1(H)$  such that

$$(2.16) \quad \overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\lambda\widehat{P} + \mu\widehat{P}^\perp}$$

holds for all  $\lambda, \mu > 0$ . It now easily follows that

$$(2.17) \quad \frac{\max \sigma(\Phi(A))}{\min \sigma(\Phi(A))} = \frac{\max \sigma(A)}{\min \sigma(A)}$$

holds for every  $A \in B(H)_{-1}^+$ . We also infer that to the given rank-one projection  $P$  there corresponds a projection  $\psi(P) \in P_1(H)$  such that

$$\overline{\Phi(I + P)} = \overline{I + \psi(P)}.$$

It is clear that  $\psi(P)$  is uniquely determined by this equality.

Now pick two arbitrary rank-one projections  $P, Q \in P_1(H)$ . Define

$$S = (I + P)(I + Q)(I + P), \quad T = (I + \psi(P))(I + \psi(Q))(I + \psi(P)).$$

It follows from (2.14) that  $\overline{\Phi(S)} = \overline{T}$ . By (2.17) this implies that

$$(2.18) \quad \frac{\max \sigma(S)}{\min \sigma(S)} = \frac{\max \sigma(T)}{\min \sigma(T)}.$$

It is apparent that

$$(2.19) \quad \operatorname{tr} S = 6 + 3 \operatorname{tr} PQ, \quad \operatorname{tr} T = 6 + 3 \operatorname{tr} \psi(P)\psi(Q)$$

and

$$(2.20) \quad \det S = \det T = 8.$$

By (2.20) we have

$$\frac{\max \sigma(S)}{\min \sigma(S)} = \frac{(\max \sigma(S))^2}{8}, \quad \frac{\max \sigma(T)}{\min \sigma(T)} = \frac{(\max \sigma(T))^2}{8}$$

and these together with (2.18) yield that  $\max \sigma(S) = \max \sigma(T)$ . We next obtain that  $\min \sigma(S) = \min \sigma(T)$ , thus  $\operatorname{tr} S = \operatorname{tr} T$ . By (2.19) this implies that

$$\operatorname{tr} \psi(P)\psi(Q) = \operatorname{tr} PQ.$$

Since  $P$  and  $Q$  were arbitrary, this means that the map  $\psi : P_1(H) \rightarrow P_1(H)$  preserves the transition probability. It can easily be seen that  $\psi$  is in fact a bijection of  $P_1(H)$ , and by Wigner's theorem, we obtain that there is either a unitary or an antiunitary operator  $U$  on  $H$  such that

$$\psi(P) = UPU^* \quad (P \in P_1(H)).$$

We then infer

$$\overline{\Phi(2P + P^\perp)} = \overline{I + \psi(P)} = \overline{2UPU^* + (UPU^*)^\perp} \quad (P \in P_1(H)).$$

Taking (2.16) into consideration, it is easy to verify that  $\hat{P} = UPU^*$  and hence

$$\overline{\Phi(\lambda P + \mu P^\perp)} = \overline{\lambda UPU^* + \mu (UPU^*)^\perp}$$

holds for all scalars  $\lambda, \mu > 0$  and projection  $P \in P_1(H)$ . This clearly yields

$$\overline{\Phi(A)} = \overline{UAU^*} \quad (A \in B(H)_{-1}^+).$$

Having in mind the definition of the classes  $\overline{A}$  ( $A \in B(H)_{-1}^+$ ) and the reduction  $\Phi(I) = I$  which we have employed in the proof, we obtain the statement of Theorem 1.2.  $\square$

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