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EXTENDING CSR DECOMPOSITION TO TROPICAL INHOMOGENEOUS MATRIX PRODUCTS*

ARTHUR KENNEDY-COCHRAN-PATRICK† AND SERGEĬ SERGEEV‡

Abstract. This article presents an attempt to extend the CSR decomposition, previously introduced for tropical matrix powers, to tropical inhomogeneous matrix products. The CSR terms for inhomogeneous matrix products are introduced, and then, a case is described where an inhomogeneous product admits such CSR decomposition after some length and a bound on this length is given. In the last part of the paper, a number of counterexamples are presented to show that inhomogeneous products do not admit CSR decomposition under more general conditions.

Key words. Max-plus algebra, Matrix product, Factor-rank, Walk, Matrix decompositions.

AMS subject classifications. 15A80, 68R99, 16Y60, 05C20, 05C22, 05C25.

1. Introduction.

Tropical (max-plus) linear algebra is the linear algebra developed over the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equipped with the additive operator $\oplus : a \oplus b = \max(a, b)$ and the multiplicative operator $\otimes : a \otimes b = a + b$. For brevity, we denote $\varepsilon = -\infty$: this element of the semiring is neutral with respect to addition, thus playing the role of semiring zero. In turn, the usual zero 0 plays the role of semiring unity, being neutral with respect to multiplication. Note that for any $a \in \mathbb{R}$, there is a multiplicative inverse: element $a^- = a$ such that $a^- \otimes a = a \otimes a^- = 0$.

We will be working with the max-plus multiplication of matrices $A \otimes B$ defined by the operation

$$(A \otimes B)_{i,j} = \bigoplus_{1 \leq k \leq n} a_{i,k} \otimes b_{k,j} = \max_{1 \leq k \leq n} (a_{i,k} + b_{k,j}),$$

using two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of appropriate sizes.

Consider the tropical dynamical system given by

$$\begin{aligned} x(0) &= x_0, \\ x(k) &= x(k-1) \otimes A_k \quad \text{for } k \geq 1, \\ \text{thus } x(k) &= x_0 \otimes A_1 \otimes \ldots \otimes A_k = x_0 \otimes \Gamma(k). \end{aligned}$$

Here, the matrices A_i are taken in some unspecified order from a possibly infinite set of matrices $\mathcal{X}$. In practical terms, this represents a dynamical system where some accidental changes may occur over time. This has useful applications in modelling scheduling systems that are subject to change.

Much work has been done for the case where the matrix A_i is the same at each step. Cohen et al. [8, 7] were the first to observe that, under some mild conditions, the tropical powers $\{A^t\}_{t \geq 1}$ become periodic

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†School of Mathematics, University of Birmingham, Birmingham, UK (arthurmkcphw@googlemail.com).

‡School of Mathematics, University of Birmingham, Birmingham, UK (s.sergeev@bham.ac.uk).

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after a big enough time. A number of bounds on the transient of such periodicity were then obtained, in particular, by Hartmann and Arguelles [9], Akian et al. [2], and Merlet et al. [17, 16]. In particular, Merlet et al. [17] offer an approach based on the CSR decompositions and CSR expansions of tropical matrix powers introduced by Sergeev and Schneider [20, 22]. Let us note that a preliminary version of such decompositions was introduced and studied before by Nachtigall [19] and Molnárová [18], and that similar decompositions appear in Akian et al. [2].

It is difficult to speak of ultimate periodicity in the case of inhomogeneous products. However, one can observe that CSR decompositions are an algebraic expression of turnpike phenomena occurring in tropical dynamical systems driven by one matrix. Namely, they express the fact that in such systems there are optimal trajectories (or walks) with a special structure: after a finite number of steps, they arrive to a well-defined group of nodes called critical nodes, then dwell within that group of nodes, and then use a finite number of steps to reach the destination. The same phenomena will likely occur in inhomogeneous products as well, but only under certain restrictive conditions. In particular, we can agree that all matrices constituting these inhomogeneous products have the same sets of critical nodes, and for a starter, we can consider the case where all these matrices have just one critical node. Under this and some other assumptions, Shue et al. [24] found that products $\Gamma(k)$ become tropical rank-1 matrices (i.e., tropical outer products) when k is sufficiently big. Kennedy-Cochran-Patrick et al. [13] improved this result by giving a lower bound for k to guarantee that $\Gamma(k)$ becomes a rank-1 matrix (i.e., a tropical outer product). In the present paper, we show that the above results of [13, 24] can be generalised further by introducing the *factor rank transient:* the length of the product after which the product is guaranteed to have a tropical factor rank not exceeding certain number. Rather than directly proving the factor rank property from an inhomogeneous product, a CSR analogue is used, which changes the aim to develop bounds on CSR transients rather than factor rank transients. Upon showing that the analogue definition of CSR exhibits similar properties to the original CSR (see the paper by Sergeev and Schneider [22]), then we can use similar proof methods and results from Merlet, Nowak, Schneider, and Sergeev [16] as well as Brualdi and Ryser [5] to develop the key result, which is Theorem 5.8, together with Corollary 5.9, which gives an explicit bound on the length of the product after which it becomes CSR. However, there are limitations to this approach, namely where it can be shown for other cases that no bound exists for the CSR transient, and then, we cannot guarantee a factor rank property. Three cases where CSR does not work are given along with the counterexamples that demonstrate this. In all these counterexamples, we present families of words of infinite length, in which the product made using such a word is not CSR.

Recall that tropical factor rank of a matrix A, studied together with many other concepts of rank in Akian et al. [1], can be defined as follows: for a matrix $A \in \mathbb{R}_{\max}^{n\times m}$, the *tropical factor rank* r of A is the smallest $r \in \mathbb{N}$ such that $A = U \otimes L$ where $U \in \mathbb{R}_{\max}^{n\times r}$ and $L \in \mathbb{R}_{\max}^{r\times m}$ for some $n, m \in \mathbb{N}$. Note that the factor rank of A is also equal to the minimum number of factor rank-1 matrices whose sum is equal to A, see [1][Definition 7.1].

For wider reading, Hook [11] shows that, by approximating the rank of the product in a min-plus setting, one can find and express the predominant structure in the associated digraph of the matrices forming the product. Hook has also looked at turnpike theory with respect to the max-plus linear systems in [12]. In this paper, he studies infinite length products and then uses a turnpike property to develop a factorisation of said matrix product. In terms of turnpikes, many results were obtained for them in the context of dynamic programming, in both discrete and continuous settings. Specifically, Kontorer and Yakovenko [15] used turnpike theory and Bellman equations to work with discrete optimal control problems. Following his work, Kolokoltsov and Maslov [14] developed turnpike theory for discrete optimal control problems in the context of idempotent analysis and tropical mathematics.

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The paper will proceed as follows. In Section 2, we will cover the necessary definitions and notation, and in Section 3, we will introduce the CSR decomposition for tropical inhomogeneous matrix products. In Section 4, we mostly generalise some important preliminary results obtained previously [13] for the case where the critical graph consists of just one loop. In Section 5, we describe the case in which the introduced CSR decomposition actually works and, for this case, obtain a bound on the factor rank threshold of inhomogeneous tropical matrix products. For Section 6, we look at the counterexamples that show the limitations of the proposed CSR approach.

2. Definitions and notation.

2.1. Weighted digraphs and tropical matrices. This subsection presents some concepts and notation expressing the connection between tropical matrices and weighted digraphs. Monographs [6, 10] are our basic references for such definitions.

Definition 2.1 (Weighted digraphs). *A* directed graph (digraph) *is a pair* (N, E) *where* N *is a finite set of nodes and* $E \subseteq N \times N = \{(i, j) \colon i, j \in N\}$ *is the set of edges, where* (i, j) *is a directed edge from node* i *to node* j.

A weighted digraph *is a digraph with associated weights* $w_{i,j} \in \mathbb{R}_{\max}$ *for each edge* (i, j) *in the digraph.*

A digraph associated with a square matrix A *is a weighted digraph* $\mathcal{D}(A) = (N_A, E_A)$ *where the set* N_A *has the same number of elements as the number of rows or columns in the matrix* A. *The set* $E_A \subseteq N_A \times N_A$ *is the set of edges in* $\mathcal{D}(A)$, *where* (i, j) *is an edge if and only if* $a_{i,j} \neq \varepsilon$, *and in this case, the weight of* (i, j) *equals the corresponding entry in the matrix* A, *i. e.* $w_{i,j} = a_{i,j} \in \mathbb{R}_{\max}$.

Definition 2.2 (Walks, paths and weights). *A sequence of nodes* $W = (i_0, \ldots, i_l)$ *is called a* walk on a weighted digraph $\mathcal{D} = (N, E)$ *if* $(i_{s-1}, i_s) \in E$ *for each* $s \colon 1 \leq s \leq l$. *This walk is a* cycle *if the start node* i_0 *and the end node* i_l *are the same. It is a* path *if no two nodes in* $i_0, \ldots, i_l$ *are the same. The* length *of* W *is* $l(W) = l$.
The weight *of* W *is defined as the max-plus product (i. e., the usual arithmetic sum) of the weights of each edge* (i_{s-1}, i_s) *traversed throughout the walk, and it is denoted by* $p_{\mathcal{D}}(W)$. *Note that a sequence* $W = (i_0)$ *is also a walk (without edges), and we assume that it has weight and length* 0.
The mean weight *of* W *is defined as the ratio* $p_{\mathcal{D}}(W)/l(W)$.

For a digraph, being strongly connected is a particularly useful property.

Definition 2.3 (Strongly connected, irreducible, completely reducible). *A digraph is* strongly connected, *if for any two nodes* i *and* j *there exists a walk connecting* i *to* j. *A square matrix is* irreducible *if the graph associated with it in the sense of Definition 2.1 is strongly connected.*

A digraph is called completely reducible, *if it consists of a number of strongly connected components, such that no two nodes of any two different components can be connected to each other by a walk.*

Note that, trivially, any strongly connected digraph is completely reducible.

The following more refined notions are crucial in the study of ultimate periodicity of tropical matrix powers, and also for the present paper.

Definition 2.4 (Cyclicity and cyclic classes). *Suppose that a digraph is completely reducible. Then, the* cyclicity *of that digraph is the lowest common multiple of the greatest common divisors of the lengths of cycles within each strongly connected component. It will be denoted by* γ.

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Suppose now that a digraph with set of nodes N and cyclicity γ is strongly connected. For two nodes $i, j \in N$ we say that i and j are in the same cyclic class *if there exists a walk of length modulo γ connecting i to j or j to i. This splits the set of nodes into γ cyclic classes: $\mathcal{C}_0, \ldots, \mathcal{C}_{\gamma-1}$. The notation $\mathcal{C}_l \to_k \mathcal{C}_m$ means that some (and hence all) walks connecting nodes of $\mathcal{C}_l$ to nodes of $\mathcal{C}_m$ have lengths congruent to k modulo γ. The cyclic class containing i will be also denoted by $[i]$.*

The correctness of the above definition of cyclic classes follows, for example, from [5, Lemma 3.4.1]: in fact, every walk from i to j on $\mathcal{D}$ has the same length modulo γ.

In tropical algebra, we often have to deal with two digraphs: 1) the digraph associated with A and 2) the critical digraph of A. The latter digraph (being a subdigraph of the first) is defined below.

Definition 2.5 (Maximum cycle mean and critical digraph). *For a square matrix A, the* maximum cycle mean *of $\mathcal{D}(A)$ denoted as $\lambda(A)$ (equivalently, the maximum cycle mean of A) is the biggest mean weight of all cycles of $\mathcal{D}(A)$.*

A cycle in $\mathcal{D}(A)$ is called critical *if its mean weight is equal to the maximum cycle mean (i.e., is maximal).*

The critical digraph *of A, denoted by $\mathbf{C}(A)$, is the subdigraph of $\mathcal{D}(A)$ whose node set $\mathcal{N}_c$ and edge set $\mathcal{E}_c$ consist of all nodes and edges that belong to the critical cycles (i.e., that are critical).*

Note that any critical digraph is completely reducible. As shown already in [8, 7], the cyclicity of critical digraph of A is the ultimate period of the tropical matrix powers sequence $\{A^t\}_{t\geq 1}$, provided that A is irreducible and $\lambda(A) = 0$. See also Butkovič [6] and Sergeev [20] for more detailed analysis of the ultimate periodicity of this sequence.

Below we will use notation for walk sets and their maximal weights that is similar to that of Merlet et al. [17].

Definition 2.6 (Sets of walks). *Let $\mathcal{D} = (N, E)$ be a weighted digraph and let $i, j \in N$. The three sets $\mathcal{W}_{\mathcal{D}}(i \to j)$, $\mathcal{W}^k_{\mathcal{D}}(i \to j)$ and $\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$, where $\mathcal{N} \subseteq N$ is a subset of nodes, are defined as follows:*

$\mathcal{W}_{\mathcal{D}}(i \to j)$ *is the set of walks over $\mathcal{D}$ connecting i to j;*
$\mathcal{W}^k_{\mathcal{D}}(i \to j)$ *is the set of walks over $\mathcal{D}$ of length k connecting i to j;*
$\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$ *is the set of walks over $\mathcal{D}$ connecting i to j that traverse at least one node of $\mathcal{N}$.*

The supremum of the weights of walks in these sets will be denoted by $p(\mathcal{W})$.

2.2. Main assumptions. In this subsection, we set out the main assumptions about $\mathcal{X}$ and the matrices A_α that are drawn from this set and give some relevant definitions.

Definition 2.7 (Geometrical equivalence). *Let the matrices A and B have their respective digraphs $\mathcal{D}(A) = (N_A, E_A)$ and $\mathcal{D}(B) = (N_B, E_B)$. We say that A and B are* weakly geometrically equivalent *if $N_A = N_B$ and $E_A = E_B$, and they are* strongly geometrically equivalent *if they are weakly geometrically equivalent and $\mathbf{C}(A) = \mathbf{C}(B)$.*

We cannot assume that the maximum cycle mean of each $A_\alpha \in \mathcal{X}$ is zero therefore we normalise each matrix to give the new set of matrices $\mathcal{Y}$, where

$$\mathcal{Y} = \{A'_\alpha : A'_\alpha = \lambda^-(A_\alpha) \otimes A_\alpha \; \forall A_\alpha \in \mathcal{X}\}.$$

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Here, $\lambda^-(A_\alpha) = -\lambda(A_\alpha)$. From Assumption $\mathcal{A}$ stated below, it follows that $\lambda(A_\alpha) \in \mathbb{R}$; thus, the inverse $\lambda^-(A_\alpha)$ is well-defined.

NOTATION 2.8 (A^{sup} and A^{inf}).

> A^{sup}: *entrywise supremum of all matrices in* $\mathcal{Y}$. *In formula,* $A^{\text{sup}} = \bigoplus_{\alpha:\, A_\alpha \in \mathcal{Y}} A_\alpha$.
> A^{inf}: *entrywise infimum of all matrices in* $\mathcal{Y}$.

Note that the concept of A^{sup} has been used before for various purposes. In [4], Gursoy, Mason and Sergeev use the same definition to develop a common subeigenvector for the entire semigroup of matrices used to create A^{sup}, which is a technique we will use later on. In [3], Gursoy and Mason use A^{sup}, and $\lambda(A^{\text{sup}})$ to develop bounds for the max-eigenvalues over a set of matrices.

We now state the main assumptions to be used in the paper.

ASSUMPTION $\mathcal{A}$. *Any matrix* $A_\alpha \in \mathcal{X}$ *is irreducible.*

ASSUMPTION $\mathcal{B}$. *Any two matrices* $A_\alpha, A_\beta \in \mathcal{X}$ *are strongly geometrically equivalent to each other and to* A^{sup}, *which has all entries in* $\mathbb{R}_{\max}$.

The following notation is defined under assumptions $\mathcal{A}$ and $\mathcal{B}$.

NOTATION 2.9. *The common associated digraph of the matrices from* $\mathcal{X}$ *will be denoted by* $\mathcal{D}(\mathcal{X}) = (N, E)$, *and the common critical digraph by* $\mathbf{C}(\mathcal{X}) = (\mathcal{N}_c, \mathcal{E}_c)$. *In general, this critical digraph has* $m \geq 1$ *strongly connected components, denoted by* $\mathbf{C}_\nu$, *for* $\nu = 1, \ldots, m$.

ASSUMPTION $\mathcal{C}$. *Any matrix* $A_\alpha \in \mathcal{X}$ *is weakly geometrically equivalent to* A^{inf}. *In other words, for each* $(i,j) \in E$, *we have* $(A^{\text{inf}})_{ij} \neq -\infty$.

ASSUMPTION $\mathcal{D}1$. *For the matrix* A^{sup}, *we have* $\lambda(A^{\text{sup}}) = 0$.

The first three assumptions come from the previous works by Shue et al. [24] and Kennedy-Cochran-Patrick et al. [13]; however, we will no longer assume that the critical graph consists just of one loop.

The final assumption below is inspired by the visualisation scaling studied in Sergeev et al [23], see also [21] and references therein for more background on this scaling.

DEFINITION 2.10 (Visualisation). *Matrix* B *is called a* visualisation *of* A *if there exists a diagonal matrix* $X = \operatorname{diag}(x)$, *with entries* $X_{ii} = x_i$ *on the diagonal and* $X_{ij} = \varepsilon$ *off the diagonal (i.e., if* $i \neq j$*), such that* $B = X^{-1}AX$ *and* B *satisfies the following conditions:* $B_{ij} = \lambda(B)$ *for* $(i,j) \in \mathcal{E}_c(B)$ *and* $B_{ij} \leq \lambda(B)$ *for* $(i,j) \notin \mathcal{E}_c(B)$.

Once $\lambda(A) \neq \varepsilon$, a visualisation of A always exists, and, moreover, vectors x providing a visualisation by means of diagonal matrix scaling $A \mapsto X^{-1}AX$ are precisely the tropical subeigenvectors of A, that is, vectors satisfying $Ax \leq \lambda(A)x$. Using this information, we have the following lemma.

LEMMA 2.11. *Suppose that the vector* x *satisfies* $A^{\text{sup}}x \leq x$. *Then,* x *provides a simultaneous visualisation for all matrices of* $\mathcal{X}$ *(and* $\mathcal{Y}$*).*

Proof. Let x be the vector that satisfies $A^{\text{sup}}x \leq x$. By construction, A^{sup} is the supremum matrix of all the normalised generators in $\mathcal{X}$. Therefore for these normalised generators A_α, $A_\alpha \leq A^{\text{sup}}$. Hence, the vector x also satisfies $A_\alpha x \leq x$, and it can be used to visualise A_α. As this applies for all α, then they can

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be simultaneously visualised. As $\mathcal{Y}$ is the set of normalised matrices from $\mathcal{X}$, then the same applies to any matrix from $\mathcal{Y}$ as well. ∎

This is referred to as the set of matrices having a *common visualisation*, therefore, in what follows we assume that we have performed this common visualisation on all of the matrices in $\mathcal{X}$ (and $\mathcal{Y}$) to give the final core assumption.

Assumption $\mathcal{D}2$. *For all $A_\alpha \in \mathcal{Y}$, we have $(A_\alpha)_{ij} = 0$ and $(A^{\sup})_{ij} = 0$ for $(i,j) \in \mathcal{E}_c$, and $(A_\alpha)_{ij} \leq 0$ and $(A^{\sup})_{ij} \leq 0$ for $(i,j) \notin \mathcal{E}_c$.*

From now on, we will use Assumption $\mathcal{D}2$ instead of Assumption $\mathcal{D}1$. Note however, if the theory developed in this paper is applied to a set of matrices satisfying Assumption $\mathcal{D}1$, then the parameters appearing in the bounds are computed using the entries of their visualised counterparts.

2.3. Extension to inhomogeneous products. Recall now that we have a set of matrices $\mathcal{Y}$, from which we can select matrices in arbitrary sequence.

Definition 2.12. *The* word *associated with the matrix product $\Gamma(k)$ is the string of characters (subscript) i from $A_i \in \mathcal{Y}$ that make up said $\Gamma(k)$.*

Let us also introduce the trellis digraph associated with a matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ (as in [13], inspired by Viterbi algorithm).

Definition 2.13. *The* trellis digraph $\mathcal{T}(P) = (\mathcal{N}, \mathcal{E})$ *associated with the product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ made from the word P is the digraph with the set of nodes $\mathcal{N}$ and the set of edges $\mathcal{E}$, where:*

(1) *$\mathcal{N}$ consists of $k+1$ copies of N which are denoted $N_0, \ldots, N_k$, and the nodes in N_l for each $0 \leq l \leq k$ are denoted by $1 : l, \ldots, n : l$;*
(2) *$\mathcal{E}$ is defined by the following rules:*
   a) *there are edges only between N_l and N_{l+1} for each l,*
   b) *we have $(i : (l-1), j : l) \in \mathcal{E}$ if and only if (i,j) is an edge of $\mathcal{D}(\mathcal{Y})$, and the weight of that edge is $(A_l)_{i,j}$.*

The weight of a walk W on $\mathcal{T}(P)$ is denoted by $p_{\mathcal{T}}(W)$.

Below we will need to use (1) walks that start at one side of the trellis and end at an intermediate node, (2) walks that start at an intermediate node and end at the other side of the trellis, (3) walks that connect one side of the trellis to the other. More formally, we give the following definition.

Definition 2.14. *Consider a trellis digraph $\mathcal{T}(P)$.*

By an initial walk *connecting i to j on $\mathcal{T}(P)$, we mean a walk on $\mathcal{T}(P)$ connecting node $i : 0$ to $j : m$, where $0 \leq m \leq k$.*

By a final walk *connecting i to j on $\mathcal{T}(P)$, we mean a walk on $\mathcal{T}(P)$ connecting node $i : l$ to $j : k$, where $0 \leq l \leq k$.*

A full walk *connecting i to j on $\mathcal{T}(P)$ is a walk on $\mathcal{T}(P)$ connecting node $i : 0$ to $j : k$.*

We will mostly work with the following sets of walks on $\mathcal{T}$.

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NOTATION 2.15 (Walk sets on $\mathcal{T}(P)$).

$\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to j)$, $\mathcal{W}^l_{\mathcal{T},\text{init}}(i \to j)$ *and* $\mathcal{W}^l_{\mathcal{T},\text{final}}(i \to j)$: *set of full walks (of length k), and sets of initial and final walks of length l on $\mathcal{T}$ connecting i to j.*
$\mathcal{W}^k_{\mathcal{T},\text{full}}(i \xrightarrow{\mathcal{N}_c} j)$, $\mathcal{W}^l_{\mathcal{T},\text{init}}(i \xrightarrow{\mathcal{N}_c} j)$ *and* $\mathcal{W}^l_{\mathcal{T},\text{final}}(i \xrightarrow{\mathcal{N}_c} j)$: *set of full walks (of length k), and sets of initial and final walks of length l on $\mathcal{T}$ traversing a critical node and connecting i to j;*
$\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$*: set of initial walks connecting i to a node in $\mathcal{N}_c$ so that this node of $\mathcal{N}_c$ is the only node of $\mathcal{N}_c$ that is visited by the walk and it is visited only once;*
$\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$*: set of final walks connecting a node in $\mathcal{N}_c$ to j so that this node of $\mathcal{N}_c$ is the only node of $\mathcal{N}_c$ that is visited by the walk and it is visited only once.*
$i \to_{\mathcal{T}} j$: *this denotes the situation where $i:0$ can be connected to $j:k$ on $\mathcal{T}$ by a full walk.*

Recall that $p(\mathcal{W})$ denotes the optimal weight of a walk in a set of walks $\mathcal{W}$. The *optimal walk interpretation* of entries of $\Gamma(k)$ in terms of walks on $\mathcal{T} = \mathcal{T}(P)$ is now apparent:

$$\Gamma(k)_{i,j} = p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to j)\right). \tag{2.1}$$

We will also need special notation for the optimal weights of walks in the sets $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$ and $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$ introduced above.

NOTATION 2.16 (Optimal weights of walks on $\mathcal{T}(P)$).

$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|))$ *: the maximal weight of walks in* $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$,
$v^*_{\mathcal{N}_c,j} = p(\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j))$ *: the maximal weight of walks in* $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$.

The following notation is for optimal values of various optimisation problems involving paths and walks on $\mathcal{D}(A^{\text{sup}})$, $\mathcal{D}(A^{\text{inf}})$, which will be used in our factor rank bounds.

NOTATION 2.17 (Optimal weights of walks on $\mathcal{D}(A^{\text{sup}})$ and $\mathcal{D}(A^{\text{inf}})$).

$\alpha_{i,\mathcal{N}_c}$ *: the weight of an optimal path on $\mathcal{D}(A^{\text{sup}})$ connecting node i to a node in $\mathcal{N}_c$;*
$\beta_{\mathcal{N}_c,j}$ *: the weight of an optimal path on $\mathcal{D}(A^{\text{sup}})$ connecting a node in $\mathcal{N}_c$ to node j;*
$\gamma_{i,j}$ *: the weight of an optimal path on $\mathcal{D}(A^{\text{sup}})$ connecting node i to node j without traversing any node in $\mathcal{N}_c$.*
$w_{i,\mathcal{N}_c}$ *: the weight of an optimal path on $\mathcal{D}(A^{\text{inf}})$ connecting node i to a node in $\mathcal{N}_c$;*
$v_{\mathcal{N}_c,j}$ *: the weight of an optimal path on $\mathcal{D}(A^{\text{inf}})$ connecting a node in $\mathcal{N}_c$ to node j;*
$u^k_{i,j}$ *: the weight of an optimal walk on $\mathcal{D}(A^{\text{inf}})$ of length k connecting node i to node j.*

We remark by saying that the Kleene star, which is explored in [6] and is defined as $(A)^* = I \oplus A \oplus A^2 \oplus \ldots$, of A^{sup} can be used to find the values of $\alpha_{i,\mathcal{N}_c}$ and $\beta_{\mathcal{N}_c,j}$. Similarly, the Kleene star of A^{inf} can be used to find $w_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j}$. Let us end this section with the following observation, which follows from the geometric equivalence (Assumptions $\mathcal{B}$ and $\mathcal{C}$)

LEMMA 2.18. *The following are equivalent:* (i) $i \to_{\mathcal{T}} j$*;* (ii) $(\Gamma(k))_{i,j} > \varepsilon$*;* (iii) $u^k_{i,j} > \varepsilon$.

3. CSR products. In this section, we introduce CSR decomposition of inhomogeneous products and study its properties. It should be noted that in this section, we will use Assumptions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}2$ for every proof presented. We will give the two definitions of the CSR decomposition of $\Gamma(k)$ and prove their equivalence. However in order to do that we require another definition.

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Definition 3.1. *Let the matrix A have cyclicity γ. The* threshold of ultimate periodicity *of powers of A is a bound $T(A)$ such that $\forall k \geq T(A)$, $A^k = A^{k+\gamma}$.*

This threshold is required to develop the CSR decomposition for $\Gamma(k)$ as seen in the following definitions.

Definition 3.2 (CSR-1). *Let $\Gamma(k) = A_1 \otimes \ldots \otimes A_k$ be a matrix product of length k made using the word P. Define C, S and R as follows:*

S is the matrix associated with the critical graph, that is,

$$S = (s_{i,j}) = \begin{cases} 0 & \text{if } (i,j) \in \mathcal{E}_c \\ \varepsilon & \text{otherwise.} \end{cases} \tag{3.1}$$

Let γ be the cyclicity of critical graph and t be a big enough number, such that $t\gamma \geq T(S)$, where $T(S)$ is the threshold of ultimate periodicity of (the powers of) S.
C and R are defined by the following formulae:

$$C = \Gamma(k) \otimes S^{(t+1)\gamma - k (\operatorname{mod} \gamma)}, \quad R = S^{(t+1)\gamma - k (\operatorname{mod} \gamma)} \otimes \Gamma(k).$$

The product of C, $S^{k (\operatorname{mod} \gamma)}$ and R will be denoted by $CS^{k (\operatorname{mod} \gamma)} R[\Gamma(k)]$. We say that $\Gamma(k)$ is CSR *if $CS^{k (\operatorname{mod} \gamma)} R[\Gamma(k)]$ is equal to $\Gamma(k)$.*

For completeness, we must also state that for any matrix in $A \in \mathbb{R}_{\max}^{n\times n}$, $A^0 = I$, where I is the tropical identity matrix, that is, $I = \operatorname{diag}(0)$. In the next definition, we prefer to define CSR terms corresponding to the components of the critical graph.

Definition 3.3 (CSR-2). *Let $\Gamma(k) = A_1 \otimes \ldots \otimes A_k$ be a matrix product of length k, and let $\mathbf{C}_\nu$, for $\nu = 1, \ldots, m$ be the components of $\mathbf{C}(\mathcal{Y})$. For each $\nu = 1, \ldots, m$ define C_ν, S_ν and R_ν as follows:*

$S_\nu \in \mathbb{R}_{\max}^{n\times n}$ is the matrix associated with the s.c.c. $\mathbf{C}_\nu$ of the critical graph, that is,

$$S_\nu = (s_{i,j}) = \begin{cases} 0 & \text{if } (i,j) \in \mathbf{C}_\nu, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{3.2}$$

Let γ_ν be the cyclicity of critical component, and t_ν be a big enough number, such that $t_\nu \gamma_\nu \geq T(S_\nu)$, where $T(S_\nu)$ is the threshold of ultimate periodicity of (the powers of) S_ν.
C_ν and R_ν are defined by the following formulae:

$$C_\nu = \Gamma(k) \otimes S_\nu^{(t_\nu+1)\gamma_\nu - k (\operatorname{mod} \gamma_\nu)}, \quad R_\nu = S_\nu^{(t_\nu+1)\gamma_\nu - k (\operatorname{mod} \gamma_\nu)} \otimes \Gamma(k).$$

The product of C_ν, $S_\nu^{k (\operatorname{mod} \gamma_\nu)}$ and R_ν will be denoted by $C_\nu S_\nu^{k (\operatorname{mod} \gamma_\nu)} R_\nu[\Gamma(k)]$. We say that $\Gamma(k)$ is CSR *if*

$$\Gamma(k) = \bigoplus_{\nu=1}^{m} C_\nu S_\nu^{k (\operatorname{mod} \gamma_\nu)} R_\nu[\Gamma(k)].$$

Using the definitions given above, we can write out the CSR terms more explicitly:

$$\begin{aligned} CS^{k (\operatorname{mod} \gamma)} R[\Gamma(k)] &= \Gamma(k) \otimes S^{(t+1)\gamma - k (\operatorname{mod} \gamma)} \otimes S^{k (\operatorname{mod} \gamma)} \otimes S^{(t+1)\gamma - k (\operatorname{mod} \gamma)} \otimes \Gamma(k) \\ &= \Gamma(k) \otimes S^{2(t+1)\gamma - k (\operatorname{mod} \gamma)} \otimes \Gamma(k), \\ C_\nu S_\nu^{k (\operatorname{mod} \gamma_\nu)} R_\nu[\Gamma(k)] &= \Gamma(k) \otimes S_\nu^{2(t_\nu+1)\gamma_\nu - k (\operatorname{mod} \gamma_\nu)} \otimes \Gamma(k), \end{aligned}$$

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Since the powers of S are ultimately periodic with period γ and the powers of S_ν are ultimately periodic with period γ_ν, and since also we have $t\gamma \geq T(S)$ and $t_\nu\gamma_\nu \geq T(S_\nu)$, we can reduce the exponents of S and S_ν to $(t+1)\gamma - k(\operatorname{mod}\gamma)$ and $(t_\nu+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu)$, respectively, and thus

$$\begin{aligned} &CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)] = \Gamma(k)\otimes S^v\otimes\Gamma(k), \quad C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)}R_\nu[\Gamma(k)] = \Gamma(k)\otimes S_\nu^{v_\nu}\otimes\Gamma(k), \\ &\text{for } v=(t+1)\gamma-k(\operatorname{mod}\gamma),\ v_\nu=(t_\nu+1)\gamma_\nu-k(\operatorname{mod}\gamma_\nu),\ t\gamma\geq T(S),\ t_\nu\gamma_\nu\geq T(S_\nu). \end{aligned} \tag{3.3}$$

Below we will also need the following elementary observation.

LEMMA 3.4. *Let $v=(t+1)\gamma-k(\operatorname{mod}\gamma)$, where $t\gamma\geq T(S)$. Then, for any ν, we can find t_ν such that $v=(t_\nu+1)\gamma_\nu-k(\operatorname{mod}\gamma_\nu)$ and $t_\nu\gamma_\nu\geq T(S_\nu)$.*

Proof. The existence of t_ν such that $v=(t_\nu+1)\gamma_\nu-k(\operatorname{mod}\gamma_\nu)$ follows since γ is a multiple of γ_ν, and then we also have $t_\nu\gamma_\nu\geq t\gamma\geq T(S)\geq T(S_\nu)$. ∎

This lemma allows us to also write

$$C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)}R_\nu[\Gamma(k)] = \Gamma(k)\otimes S_\nu^v\otimes\Gamma(k), \tag{3.4}$$

with v as in (3.3).

PROPOSITION 3.5. *$\Gamma(k)$ is CSR by Definition 3.2 if and only if it is CSR by Definition 3.3.*

Proof. We need to show that

$$CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)] = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)}R_\nu[\Gamma(k)], \tag{3.5}$$

for arbitrary k. Using (3.3) and (3.4), we can rewrite this equivalently as

$$\Gamma(k)\otimes S^{(t+1)\gamma-k(\operatorname{mod}\gamma)}\otimes\Gamma(k) = \Gamma(k)\otimes\left(\bigoplus_{\nu=1}^m S_\nu^{(t+1)\gamma-k(\operatorname{mod}\gamma)}\right)\otimes\Gamma(k), \tag{3.6}$$

with $t\gamma\geq T(S)$. To obtain this equality, observe that $S=\bigoplus_{\nu=1}^m S_\nu$, and as $S_{\nu_1}\otimes S_{\nu_2}=-\infty$ for any ν_1 and ν_2 we can raise both sides to the same power to give us $S^t=\bigoplus_{\nu=1}^m S_\nu^t$ for any t. This shows (3.6), and the claim follows. ∎

For a similar reason, we also have the following identities:

$$\begin{aligned} &C=\bigoplus_{\nu=1}^m C_\nu, \qquad R=\bigoplus_{\nu=1}^m R_\nu, \\ &C\otimes S^{k(\operatorname{mod}\gamma)} = \bigoplus_{\nu=1}^m C_\nu\otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)}, \quad S^{k(\operatorname{mod}\gamma)}\otimes R = \bigoplus_{\nu=1}^m S_\nu^{k(\operatorname{mod}\gamma_\nu)}\otimes R_\nu. \end{aligned} \tag{3.7}$$

To give an optimal walk interpretation of CSR, we will need to define the trellis graph corresponding to these terms, by modifying Definition 2.13.

DEFINITION 3.6 (Symmetric extension of the trellis graph). *Let $v=(t+1)\gamma-k(\operatorname{mod}\gamma)$, where t is a large enough number such that $t\gamma\geq T(S)$.*
Define $\mathcal{T}'(\Gamma(k))$ as the digraph $\mathcal{T}'=(\mathcal{N}',\mathcal{E}')$ with the set of nodes $\mathcal{N}'$ and edges $\mathcal{E}'$, such that:

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(1) $\mathcal{N}'$ *consists of* $2k+v+1$ *copies of* N *which are denoted* $N_0, \ldots, N_{2k+v}$ *and the nodes for* N_l *for each* $0 \le l \le 2k+v$ *are denoted by* $1:l, \ldots, n:l$;

(2) $\mathcal{E}'$ *is defined by the following rules:*

a) *there are edges only between* N_l *and* N_{l+1},

b) *for* $1 \le l \le k$, *we have* $(i : l-1, j : l) \in \mathcal{E}'$ *if and only if* $(i,j) \in E(\mathcal{Y})$, *and the weight of the edge is* $(A_l)_{i,j}$,

c) *for* $k+v+1 \le l \le 2k+v$, *we have* $(i : l-1, j : l) \in \mathcal{E}'$ *if and only if* $(i,j) \in E(\mathcal{Y})$, *and the weight of the edge is* $(A_{l-k-v})_{i,j}$,

d) *for* $k < l < k+v+1$, *we have* $(i : l-1, j : l) \in \mathcal{E}'$ *if and only if* $(i,j) \in \mathbf{C}(\mathcal{Y})$, *and the weight of the edge is* 0.

The weight of a walk on $\mathcal{T}'(\Gamma(k))$ *is denoted by* $p_{\mathcal{T}'}(W)$.

If we consider the walks in $\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \to j)$ then, in the middle of the walk for l satisfying $k < l < k+v+1$, the walk is confined in one of the components of $\mathbf{C}(\mathcal{Y})$. The set of walks confined in the ν^{th} component of $\mathbf{C}(\mathcal{Y})$ in the middle of the walk for l satisfying $k < l < k+v+1$ is denoted by $\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{[\mathcal{N}_c^{\nu}]} j)$. The following optimal walk interpretation of CSR terms on $\mathcal{T}'$ is now obvious.

LEMMA 3.7 (CSR and optimal walks). *The following identities hold for all* i, j

$$(CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \to j)\right),$$

(3.8)

$$(C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)])_{i,j} = p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{[\mathcal{N}_c^{\nu}]} j)\right),$$

where $v = (t+1)\gamma - k(\mathrm{mod}\,\gamma)$, *with* $t\gamma \ge T(S)$.

Proof. With (3.3), the first identity follows from the optimal walk interpretation of $\Gamma(k) \otimes S^v \otimes \Gamma(k)$, and the second identity follows from (3.4) and the optimal walk interpretation of $\Gamma(k) \otimes S_\nu^v \otimes \Gamma(k)$. ∎

In what follows, we mostly work with Definition 3.3, but we can switch between the equivalent definitions if we find it convenient.

We now present a useful lemma that shows equality for columns of C_ν and rows of R_ν with indices in the same cyclic class.

LEMMA 3.8. *For any* i *and for any two nodes* x *and* y *in the same cyclic class of the critical component* $\mathbf{C}_\nu$ *we have*

$$(C_\nu)_{i,x} = (C_\nu)_{i,y} \quad \text{and} \quad (R_\nu)_{x,i} = (R_\nu)_{y,i}. \tag{3.9}$$

Proof. We prove the lemma for columns, as the case of the rows is similar.

For any i, x, denote $(C_\nu)_{i,x}$ by $c_{i,x}$. From the definition of C_ν, it follows that $c_{i,x}$ is the weight of an optimal walk in $\mathcal{W}^{k+(t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)}_{\mathcal{T}',\mathrm{init}}(i \xrightarrow{\mathcal{N}_c^{\nu}} x)$ where $t_\nu\gamma_\nu \ge T(S_\nu)$, and such walk consists of two parts. The first part is a full walk on $\mathcal{T}$ connecting i to the critical subgraph at some node s. The second part is a walk over the critical subgraph of length $(t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)$ connecting s to x with weight zero. As the length of the second walk is greater than $T(S_\nu)$, a walk connecting s to x exists if and only if $[s] \to_{-k(\mathrm{mod}\,\gamma_\nu)} [x]$. If a full walk connecting i to $[s]$ on $\mathcal{T}$ exists then, for arbitrary x, y in the same cyclic class, $c_{i,x}$ and $c_{i,y}$ are both equal to the optimal weight of all walks connecting i to $[s]$ on $\mathcal{T}$, where $[s] \to_{-k(\mathrm{mod}\,\gamma_\nu)} [x]$, otherwise both $c_{i,x}$ and $c_{i,y}$ are equal to $-\infty$. This shows that $c_{i,x} = c_{i,y}$.

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The case of rows of R_ν is considered similarly, but instead of initial walks one has to use final walks on $\mathcal{T}'$. ∎

We can use this to prove the same property for C and R of Definition 3.2.

COROLLARY 3.9. *For any i and for any two nodes x and y in the same critical component and the same cyclic class of said critical component, we have*

$$C_{i,x} = C_{i,y} \quad \text{and} \quad R_{x,i} = R_{y,i}. \tag{3.10}$$

Proof. We will prove only the first identity, as the proof of the second identity is similar. Let x, y belong to the same component $\mathbf{C}_\mu$ of $\mathbf{C}(\mathcal{Y})$ and let them belong to the same cyclic class of that component. By Lemma 3.8, we have $(C_\mu)_{i,x} = (C_\mu)_{i,y}$, and we also have $(C_\nu)_{i,x} = (C_\nu)_{i,y} = \varepsilon$ for any $\nu \neq \mu$. Using these identities and (3.7), we have

$$C_{i,x} = \left(\bigoplus_{\nu=1}^{m} C_\nu\right)_{i,x} = (C_\mu)_{i,x} = (C_\mu)_{i,y} = \left(\bigoplus_{\nu=1}^{m} C_\nu\right)_{i,y} = C_{i,y}. \qquad ∎$$

The next theorem explains why CSR is useful for inhomogeneous products. Note that in the proof of it we use the CSR structure rather than the $\Gamma(k) \otimes S^v \otimes \Gamma(k)$ representation that was used above.

THEOREM 3.10. *The factor rank of each $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)]$ is no more than γ_ν, for $\nu = 1, \ldots, m$, and the factor rank of $CS^{k(\bmod \gamma)}R[\Gamma(k)]$ is no more than $\sum_{\nu=1}^{m} \gamma_\nu$.*

Proof. For each $\nu = 1, \ldots, m$, take all the nodes from $\mathcal{G}_\nu$ and order them into cyclic classes $\mathcal{C}_0^\nu, \ldots, \mathcal{C}_{\gamma_\nu - 1}^\nu$. Take two columns with indices $x, y \in \mathcal{C}_i^\nu$ from the matrix C_ν. As they are in the same cyclic class, by Lemma 3.8 the columns are equal to each other. This means that we can take a column representing a single node from each cyclic class, and since there are γ_ν distinct classes, then there will be γ_ν distinct columns of C_ν. The same also holds for any two rows of R_ν: if the row indices are in the same cyclic class, then the rows are equal, so that we have γ_ν distinct rows.

Let us now check that the same holds for $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$. By the construction of $S_\nu^{k(\bmod \gamma_\nu)},$ we know that if $(S_\nu^{k(\bmod \gamma_\nu)})_{ij} \neq 0$ then $[i] \to_{k(\bmod \gamma_\nu)} [j]$. Therefore,

$$(S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i,\cdot} = \bigoplus_{j \in N_c} (S_\nu^{k(\bmod \gamma_\nu)})_{ij} \otimes (R_\nu)_{j,\cdot} = \bigoplus_{j\colon [i] \to_{k(\bmod \gamma_\nu)} [j]} (S_\nu^{k(\bmod \gamma_\nu)})_{ij} \otimes (R_\nu)_{j,\cdot} = (R_\nu)_{j,\cdot}.$$

This means that for a row i such that $[i] \to_{k(\bmod \gamma_\nu)} [j]$, we have $(S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i,\cdot} = (R_\nu)_{j,\cdot}$ and all such rows of $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$ are equal to each other.

Our next aim is to define, for each ν, matrices C'_ν and R'_ν with γ_ν rows and γ_ν columns, such that $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$. To form matrix C'_ν, we select a node of $\mathbf{C}_\nu$ from each cyclic class $\mathcal{C}_0^\nu, \ldots, \mathcal{C}_{\gamma_\nu - 1}^\nu$ and define the column of C'_ν whose index is the number of this node to be the column of C_ν with the same index. The rest of the columns of C'_ν are set to $-\infty$. To form matrix R'_ν, we use the same selected nodes, but this time (instead of taking columns of C_ν and making them columns of C'_ν) we take the rows from $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$ whose indices are the numbers of selected nodes and make them rows of R'_ν. The rest of the rows of R'_ν are set to $-\infty$. Since the rows of C_ν with indices in the same cyclic class are equal to each other and the same is true about the rows of $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$, we have $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$; thus, the factor rank of any of these terms is no more than γ_ν.

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We next form the matrices $C' = \bigoplus_{\nu=1}^m C'_\nu$ and $R' = \bigoplus_{\nu=1}^m R'_\nu$. Obviously, $C'_{\nu_1} \otimes R'_{\nu_2} = -\infty$ for $\nu_1 \neq \nu_2$ and therefore

$$C' \otimes R' = \bigoplus_{\nu=1}^m C'_\nu \otimes R'_\nu = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)} R_\nu[\Gamma(k)] = CS^{k(\operatorname{mod}\gamma)} R[\Gamma(k)]. \qquad \square$$

Finally, as C' and, respectively, R' have $\sum_{\nu=1}^m \gamma_\nu$ columns with finite entries and, respectively, rows with finite entries with the same indices, $CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)] = C' \otimes R'$ has factor rank at most $\sum_{\nu=1}^m \gamma_\nu$.

Corollary 3.11. *If $\Gamma(k)$ is CSR, then its rank is no more than $\sum_{\nu=1}^m \gamma_\nu$.*

Let us also prove the following results that are similar to [22, Corollary 3.7].

Proposition 3.12. *For each $\nu = 1, \ldots, m$*

$$(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu)_{\cdot,j} = (C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)})_{\cdot,j} \quad \textit{for} \quad j \in \mathcal{N}_c^\nu$$
$$(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu)_{i,\cdot} = (S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu)_{i,\cdot} \quad \textit{for} \quad i \in \mathcal{N}_c^\nu.$$

Proof. As the proofs are very similar for both statements, we will only prove the first and omit the proof for the second statement. We begin by observing that

$$(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)})_{i,j} = p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right),$$

where we used the definitions of C_ν and S_ν and the identity $S_\nu^{(t_\nu+1)\gamma_\nu} = S_\nu^{t_\nu\gamma_\nu}$ (since $t_\nu\gamma_\nu \geq T(S_\nu)$). Here, it is convenient to choose t_ν that satisfies $(t_\nu+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu) = (t+1)\gamma - k(\operatorname{mod}\gamma)$, with t used in the definition of $\mathcal{T}'$. With this choice $t_\nu\gamma_\nu \leq t\gamma$.

Using (3.8), all we need to show is that $p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) = p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right)$, where $v = (t+1)\gamma - k(\operatorname{mod}\gamma)$. We will achieve this by proving these two inequalities:

$$\begin{aligned} &p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) \geq p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right), \\ &p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) \leq p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right). \end{aligned} \tag{3.11}$$

To prove the first inequality of (3.11), we first consider $\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j')$, where $j' \in [j]$. Optimal walk in any of these sets can be decomposed into 1) an optimal full walk on $\mathcal{T}$ connecting i to a node of $[j]$, and 2) a walk of weight 0 and length $t_\nu\gamma_\nu$ on $\mathbf{C}_\nu$ connecting that node of $[j]$ to j', whose existence follows since $t_\nu\gamma_\nu \geq T(S_\nu)$. This decomposition implies that the weights of all these optimal walks are equal. One of them, denote it by W_1 can be concatenated with a walk W_2 on $\mathbf{C}_\nu$ of length $k - k(\operatorname{mod}\gamma_\nu) + \gamma$ and ending in j. We see that $p(W_1W_2) = p(W_1)$ and $W_1W_2 \in \mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)$.

To prove the second inequality of (3.11), we take a walk in $\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)$ and decompose it into (1) a walk in $\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j')$, where $j' \in [j]$, (2) a walk in $\mathcal{W}_{\mathcal{T}',\mathrm{final}}^{k-k(\operatorname{mod}\gamma_\nu)+\gamma_\nu}(j' \to j)$. The weight of the first walk is bounded by $p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right)$, and the weight of the second walk is bounded by 0; thus, the second inequality also holds. $\square$

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COROLLARY 3.13. *For CSR as defined in Definition 3.2, we have,*

$$(C \otimes S^{k(\bmod \gamma)} \otimes R)_{\cdot,j} = (C \otimes S^{k(\bmod \gamma)})_{\cdot,j} \quad \textit{for} \quad j \in \mathcal{N}_c$$
$$(C \otimes S^{k(\bmod \gamma)} \otimes R)_{i,\cdot} = (S^{k(\bmod \gamma)} \otimes R)_{i,\cdot} \quad \textit{for} \quad i \in \mathcal{N}_c.$$

Proof. The proofs for both statements are similar so we will only prove the first one.

Let $j \in \mathcal{N}_c$. As all nodes from $\mathcal{N}_c$ can be sorted into $\mathcal{N}_c^{\nu}$ for some $\nu = 1, \ldots, m$, assume without loss of generality that $j \in \mathcal{N}_c^{\mu}$.

Taking the right-hand side of the first statement and using (3.7), we have

$$(C \otimes S^{k(\bmod \gamma)})_{\cdot,j} = \left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \right)_{\cdot,j}.$$

By Definition 3.3, if $j \in \mathcal{N}_c^{\mu}$ then for all $\nu \neq \mu$, $(C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)})_{\cdot,j} = -\infty$. Therefore, for every ν, $(C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)})_{\cdot,j}$ will be dominated by $(C_\mu \otimes S_\mu^{k(\bmod \gamma_\mu)})_{\cdot,j}$. Hence,

$$\left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \right)_{\cdot,j} = (C_\mu \otimes S_\mu^{k(\bmod \gamma_\mu)})_{\cdot,j}. \tag{3.12}$$

Turning our attention to the left-hand side of the first statement, by (3.7) we get

$$(C \otimes S^{k(\bmod \gamma)} \otimes R)_{\cdot,j} = \left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu \right)_{\cdot,j}.$$

Now we must show that, for $j \in \mathcal{N}_c^{\mu}$ and for all ν, $(C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{\cdot,j} \leq (C_\mu \otimes S_\mu^{k(\bmod \gamma_\mu)} \otimes R_\mu)_{\cdot,j}$. By (3.8), this is the same as saying

$$p\left(\mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^{\nu}]} j) \right) \leq p\left(\mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^{\mu}} j) \right),$$

for some arbitrary node i. Let W be the walk of length $2k + v$ connecting i to j that traverses $\mathcal{N}_c^{\nu}$, such that $p(W) = p\left(\mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^{\nu}]} j) \right)$. As $j \in \mathcal{N}_c^{\mu}$ then W is also a walk of length $2k + v$ connecting i to j that traverses $\mathcal{N}_c^{\mu}$; hence, $W \in \mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^{\mu}} j)$ and the inequality holds.

Therefore, as with the right-hand side, we have

$$\left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu \right)_{\cdot,j} = (C_\mu \otimes S_\mu^{k(\bmod \gamma_\mu)} \otimes R_\mu)_{\cdot,j}. \tag{3.13}$$

Finally, the first statement of Proposition 3.12 gives us equality between (3.12) and (3.13). As j was chosen arbitrarily, this holds for any $j \in \mathcal{N}_c$ and the result follows. ∎

4. General results. This section presents some results that hold for general inhomogeneous products satisfying Assumptions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}2$. Before we proceed, let us introduce the following piece of notation, inspired by the weak CSR expansion of Merlet et al. [17]:

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NOTATION 4.1 (B^{sup} and λ_*). *Denote*

$$(B^{\text{sup}})_{i,j} = \begin{cases} \varepsilon, & \text{if } i \in \mathcal{N}_c \text{ or } j \in \mathcal{N}_c, \\ (A^{\text{sup}})_{i,j}, & \text{otherwise} \end{cases}$$

and by λ_* *the maximum cycle mean of* B^{sup}.

We remark that the metric matrix, given in [6] and defined as $A^+ = A \oplus A^2 \oplus \ldots$, of B^{sup} is useful in calculating all the entries of $\gamma_{i,j}$ simultaneously.

NOTATION 4.2 (q). *We will denote by* q *the number of critical nodes, that is,* $q = |\mathcal{N}_c|$.

The following results generalise [13, Lemmas 3.1 and 3.2] for initial and final walks to the case of a general critical subgraph. Observe that, under Assumptions $\mathcal{B}$ and $\mathcal{D}2$, we have $\lambda_* < 0$, so that the bounds in the following lemmas make sense. Recall the sets of walks $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$ and $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$ introduced in Notation 2.15.

LEMMA 4.3. *Let* $W_{i,\mathcal{N}_c}$ *be an optimal walk in* $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$*, so that* $p(W_{i,\mathcal{N}_c}) = w^*_{i,\mathcal{N}_c}$. *Then, we have the following bound on the length of* $W_{i,\mathcal{N}_c}$*:*

$$l(W_{i,\mathcal{N}_c}) \leq \begin{cases} n-q, & \text{if } \lambda_* = \varepsilon, \\ \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n-q), & \text{if } \lambda_* > \varepsilon \end{cases} \tag{4.1}$$

Proof. If $\lambda_* = \varepsilon$, then any walk in $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$ has to be a path, and its length is bounded by $n-q$. Now let $\lambda_* > \varepsilon$. As $\lambda_* < 0$, the weight of the walk $W_{i,\mathcal{N}_c}$ connecting i to a node in $\mathcal{N}_c$ is less than or equal to that of a path $P_{i,\mathcal{N}_c}$ on $\mathcal{D}(A^{\text{sup}})$ connecting i to a node in $\mathcal{N}_c$ plus the remaining length multiplied by λ_*. The remaining length is bounded from above by $n-q$, since all intermediate nodes in $W_{i,\mathcal{N}_c}$ are non-critical. Hence

$$p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq p_{\text{sup}}(P_{i,\mathcal{N}_c}) + (l(W_{i,\mathcal{N}_c}) - (n-q))\lambda_*.$$

We can bound $p_{\text{sup}}(P_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c}$, so

$$p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (n-q))\lambda_*. \tag{4.2}$$

Now assuming for contradiction that $l(W_{i,\mathcal{N}_c}) > \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n-q)$. This is equivalent to

$$\alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (n-q))\lambda_* < w^*_{i,\mathcal{N}_c}. \tag{4.3}$$

In combining (4.2) and (4.3), we get $p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) < w^*_{i,\mathcal{N}_c}$ meaning that $W_{i,\mathcal{N}_c}$ is not optimal, a contradiction. So we know that for any $l \in \mathcal{N}_c$

$$l(W_{i,\mathcal{N}_c}) \leq \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n-q).$$

The proof is complete. ∎

LEMMA 4.4. *Let* $W_{\mathcal{N}_c,j}$ *be an optimal walk in* $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$*, so that* $p(W_{\mathcal{N}_c,j}) = v^*_{\mathcal{N}_c,j}$. *Then, we have the following bound on the length of* $W_{\mathcal{N}_c,j}$*:*

$$l(W_{\mathcal{N}_c,j}) \leq \begin{cases} n-q, & \text{if } \lambda_* = \varepsilon, \\ \frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (n-q), & \text{if } \lambda^* > \varepsilon. \end{cases} \tag{4.4}$$

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As the proof of this lemma is analogous to the proof of Lemma 4.3 it is omitted. Also, we can observe that $n-q$ is the limit of the expressions on the right-hand side of (4.1) and (4.4) as $\lambda_* \to \varepsilon$; hence, we will not consider this case separately in the rest of the paper.

The following result is a generalised form of [13, Lemma 3.4] which uses a nominal weight ω.

Lemma 4.5. *If $\gamma_{i,j} = \varepsilon$, then any full walk connecting i to j on $\mathcal{T}(P)$ traverses a node in $\mathcal{N}_c$. If $\gamma_{i,j} > \varepsilon$, let*

$$k > \frac{\omega - \gamma_{i,j}}{\lambda_*} + (n-q), \tag{4.5}$$

for some $\omega \in \mathbb{R}$. Then, any full walk W connecting i to j on $\mathcal{T}(P)$ that does not go through any node $l \in \mathcal{N}_c$ has weight smaller than ω.

Proof. In the case when $\gamma_{i,j} = \varepsilon$, the claim follows by the definition of $\gamma_{i,j}$ and by the geometric equivalence between $A^{\sup}$ and the matrices from $\mathcal{Y}$. So we assume that $\gamma_{i,j} > \varepsilon$. Any walk W that does not traverse any node in $\mathcal{N}_c$ can be decomposed into a path P connecting i to j avoiding $\mathcal{N}_c$ and a number of cycles. Hence we have the following bound:

$$p_{\mathcal{T}}(W) \leq p_{\sup}(P) + (k-(n-q))\lambda_*.$$

We can further bound $p_{\sup}(P) \leq \gamma_{i,j}$ so

$$p_{\mathcal{T}}(W) \leq \gamma_{i,j} + (k-(n-q))\lambda_*. \tag{4.6}$$

Now (4.5) can be rewritten as

$$\gamma_{i,j} + (k-(n-q))\lambda_* < \omega. \tag{4.7}$$

By combining (4.6) with (4.7), we have $p_{\mathcal{T}}(W) < \omega$, which completes the proof. □

Using this bound, we can obtain a condition under which the CSR term is (non-strictly) above $\Gamma(k)$.

Theorem 4.6. *If $\gamma_{i,j} = \varepsilon$ then $\Gamma(k) \leq CS^{k (\bmod \gamma)} R[\Gamma(k)]$. If $\gamma_{i,j} > \varepsilon$, let*

$$k > \max_{i,j\colon\, i \to_{\mathcal{T}} j, \gamma_{i,j} > \varepsilon} \left(\frac{\Gamma(k)_{i,j} - \gamma_{i,j}}{\lambda_*} + (n-q) \right). \tag{4.8}$$

Then ,$\Gamma(k) \leq CS^{k (\bmod \gamma)} R[\Gamma(k)]$.

Proof. If $i \not\to_{\mathcal{T}} j$, then $(\Gamma(k))_{i,j} = -\infty$. In this case, obviously, $\Gamma(k)_{i,j} \leq (CS^{k (\bmod \gamma)} R[\Gamma(k)])_{i,j}$.

If $i \to_{\mathcal{T}} j$, then $(\Gamma(k))_{i,j} \neq \varepsilon$. Let W^* be the optimal walk of length k on $\mathcal{T}(P)$ connecting i to j with weight $\Gamma(k)_{i,j}$. If k is greater than the bound (4.8) then, by Lemma 4.5, for the walk to have weight equal to $\Gamma(k)_{i,j}$, it must traverse at least one node in $\mathcal{N}_c$, and the same is true when $\gamma_{i,j} = \varepsilon$. Hence, this walk belongs to the set $\mathcal{W}^k_{\mathcal{T}}(i \xrightarrow{\mathcal{N}_c} j)$ and further $\Gamma(k)_{i,j} = p(W^*) \leq p\left(\mathcal{W}^k_{\mathcal{T}}(i \xrightarrow{\mathcal{N}_c} j)\right)$.

Let $f \in \mathcal{N}_c$ be the first critical node in the first critical s.c.c $\mathbf{C}_\nu$, with cyclicity γ_ν, that W^* traverses. We can split the walk into $W^* = W_1 W_3$ where W_1 is a walk connecting i to f of length r and W_3 is a walk connecting f to j of length $k-r$. We have $p(W^*) = p(W_1) + p(W_3)$.

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Let $\mathcal{T}'$ be the trellis extension for the matrix product $CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)]$ with length $2k+v$ where $v=(t+1)\gamma - k(\text{mod}\,\gamma)$ as described in Definition 3.6.

We now introduce the new walk $W'=W_1W_2W_3$ on $\mathcal{T}'$. Here, W_1 and W_3 are the subwalks from W^* introduced before, where W_1 is viewed as an initial walk on $\mathcal{T}'$ and W_3 as a final walk on $\mathcal{T}'$, and W_2 is a closed walk of length $k+v$ that starts and ends at f. Since $k+v\equiv 0(\text{mod}\,\gamma_\nu)$ and $k+v\geq T(S)\geq T(S_\nu)$, this closed walk exists and can be entirely made up of edges from $\mathbf{C}_\nu$. This means the walk W' is of length $2k+v$, and it traverses the set of nodes $\mathcal{N}_c^\nu$ therefore $W'\in\mathcal{W}^{2k+v}_{\mathcal{T}'}(i\xrightarrow{[\mathcal{N}_c^\nu]}j)$.

As W_2 is made entirely from critical edges, we have $p(W_2)=0$ and $p(W^*)=p(W')\leq p\left(\mathcal{W}^{2k+v}_{\mathcal{T}'}(i\xrightarrow{[\mathcal{N}_c^\nu]}j)\right)$, and using (5.8) gives us

$$\Gamma(k)_{i,j}=p(W^*)\leq (C_\nu S_\nu^{k(\text{mod}\,\gamma_\nu)}R_\nu[\Gamma(k)])_{i,j}\leq (CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)])_{i,j},$$

where the last inequality is due to Proposition 3.5. The claim follows. ◻

This condition looks like a bound for $\Gamma(k)$ to become equal to the corresponding CSR product, but it is implicit since it requires $\Gamma(k)$ to be calculated in order to generate the bound. However, we can develop a condition that does not depend on $\Gamma(k)$. This following result requires Assumption $\mathcal{C}$.

Corollary 4.7. *Let*

$$k>\max_{i,j\colon i\to_{\mathcal{T}}j,\gamma_{i,j}>\varepsilon}\left(\frac{u^k_{i,j}-\gamma_{i,j}}{\lambda_*}+(n-q)\right). \tag{4.9}$$

Then, $\Gamma(k)\leq CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)]$.

Proof. By Lemma 2.18, $i\to_{\mathcal{T}}j$ is equivalent to $u^k_{i,j}>\varepsilon$, so maximum in (4.9) is taken over i,j for which $u^k_{i,j}$ and $\gamma_{i,j}$ are finite. We also have $u^k_{i,j}\leq(\Gamma(k))_{i,j}$ by the definition of $A^{\inf}$.

Further, as $\lambda_*<0$, then any k that satisfies (4.9) will also satisfy (4.8). The claim now follows from Theorem 4.6. ◻

5. The case where CSR works. In the case, when $\mathbf{C}(\mathcal{X})$ is just one loop, Kennedy-Cochran-Patrick et al. [13] established a bound on the lengths of inhomogeneous products, after which these products are of tropical factor rank 1. In this section, we extend this result to the case when $\mathcal{D}(\mathcal{X})$ and $\mathbf{C}(\mathcal{X})$ satisfy the following assumption, in addition to Assumptions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}2$.

Assumption $\mathcal{P}0$. $\mathbf{C}(\mathcal{X})$ *is strongly connected and its cyclicity* γ *is equal to the cyclicity of* $\mathcal{D}(\mathcal{X})$.

The equality between cyclicities means that the associated digraph $\mathcal{D}(\mathcal{X})$ has the same number of cyclic classes γ as $\mathbf{C}(\mathcal{X})$.

Notation 5.1. *The cyclic classes of* $\mathcal{D}(\mathcal{X})$ *are denoted by* $\mathcal{C}'_0,\ldots,\mathcal{C}'_{\gamma-1}$.
For a node $i\in N$, *the cyclic class of this node with respect to* $\mathcal{D}(\mathcal{X})$ *will be denoted by* $[i]'$.

For a node $i\in\mathcal{N}_c$, we will use both $[i]$ (the cyclic class with respect to $\mathbf{C}(\mathcal{X})$) and $[i]'$ (the cyclic class with respect to $\mathcal{D}(\mathcal{X})$), and an obvious inclusion relation between them: $[i]\subseteq[i]'$.

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One of the ideas is to combine Lemmas 4.3 and 4.4 together with Schwarz's bound. To define this bound, following [17], we first introduce *Wielandt's number*

$$\mathrm{Wi}(n) = \begin{cases} (n-1)^2+1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and then *Schwarz's number*

$$\mathrm{Sch}(\gamma, n) = \gamma \,\mathrm{Wi}\left(\left\lfloor \frac{n}{\gamma} \right\rfloor\right) + n (\mathrm{mod}\, \gamma).$$

Let us now prove the following lemma.

Lemma 5.2. *Let*

$$k \geq \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_{q*}} + (n-q) + \mathrm{Sch}(\gamma, q) + \frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_{q*}} + (n-q). \tag{5.1}$$

Then

(i) If $[i]' \not\to_k [j]'$, then there are no full walks connecting i to j on $\mathcal{T}(P)$ (i.e., $i \not\to_{\mathcal{T}} j$).
(ii) If $[i]' \to_k [j]'$, then there is a full walk W connecting i to j on $\mathcal{T}(P)$ and going through a critical node, and we have $p_{\mathcal{T}}(W) = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$ if W is optimal.

Proof. The property $[i]' \not\to_k [j]'$ implies that there is no full walk W connecting i to j on $\mathcal{T}(P)$.

In the case $[i]' \to_k [j]'$, we construct a walk $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$ of length k, where $W_{i,\mathcal{N}_c}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ (see Lemma 4.3) , $W_{\mathcal{N}_c,j}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$ (see Lemma 4.4), and W_c is a walk that connects the end of $W_{i,\mathcal{N}_c}$ to the beginning of $W_{\mathcal{N}_c,j}$ and such that all edges of W_c are critical (the existence of such W_c is yet to be proved). Without loss of generality set $[i]' = \mathcal{C}'_0$ and $[j]' = \mathcal{C}'_{p_3}$: the cyclic classes of $\mathcal{D}(\mathcal{X})$ to which i and j belong. Let x be the final node of $W_{i,\mathcal{N}_c}$ and let y be the first node of $W_{\mathcal{N}_c,j}$. Set $[x]' = \mathcal{C}'_{p_1}$ and $[y]' = \mathcal{C}'_{p_2}$.

By [5, Lemma 3.4.1.iv] $l(W_{i,\mathcal{N}_c}) \equiv p_1 (\mathrm{mod}\, \gamma)$, $l(W_{\mathcal{N}_c,j}) \equiv (p_3 - p_2)(\mathrm{mod}\, \gamma)$. Hence, the congruence of the walk W_c to be inserted is $(p_3 - p_1 - (p_3 - p_2))(\mathrm{mod}\, \gamma) \equiv (p_2 - p_1)(\mathrm{mod}\, \gamma)$. As the cyclicity of the critical subgraph is the same as that of the digraph, the cyclic classes of the critical subgraph are $\mathcal{C}_0, \ldots, \mathcal{C}_{\gamma-1}$, and we can assume that the numbering is such that $\mathcal{C}_0 \subseteq \mathcal{C}'_0, \ldots, \mathcal{C}_{\gamma-1} \subseteq \mathcal{C}'_{\gamma-1}$. Then, $x \in \mathcal{C}_{p_1}$ and $y \in \mathcal{C}_{p_2}$ and by [5, Lemma 3.4.1.iv] there exists a walk on the critical subgraph of length congruent to $(p_2 - p_1)(\mathrm{mod}\, \gamma)$. Moreover, all walks connecting x to y have such length and by Schwarz's bound if $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j}) \geq \mathrm{Sch}(\gamma, q)$ then there is a walk of length equal to $l(W') - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$. According to Lemmas 4.3 and 4.4 $l(W_{i,\mathcal{N}_c}) \leq \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n-q)$, $l(W_{\mathcal{N}_c,j}) \leq \frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (n-q)$; therefore, k is a sufficient length for $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$ to satisfy Schwarz's bound, so a walk of the form $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$ exists and $p(W') = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$.

Let now W be an optimal full walk connecting i to j on $\mathcal{T}$ that passes through $\mathcal{N}_c$ at least once. As it passes through the critical nodes, then the walk can be decomposed into $W = \tilde{W}_{i,\mathcal{N}_c} \tilde{W}_c \tilde{W}_{\mathcal{N}_c,j}$ where $\tilde{W}_{i,\mathcal{N}_c}$ is a walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$, and $\tilde{W}_{\mathcal{N}_c,j}$ is a walk in $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$, and $\tilde{W}_c$ connects the end of $\tilde{W}_{i,\mathcal{N}_c}$ to the beginning of $\tilde{W}_{\mathcal{N}_c,j}$ on $\mathcal{T}(P)$. We then have $p_{\mathcal{T}}(\tilde{W}_{i,\mathcal{N}_c}) \leq p_{\mathcal{T}}(W_{i,\mathcal{N}_c})$ and $p_{\mathcal{T}}(\tilde{W}_{\mathcal{N}_c,j}) \leq p_{\mathcal{T}}(W_{\mathcal{N}_c,j})$ and also $p_{\mathcal{T}}(\tilde{W}_c) \leq p(W_c) = 0$. Since W is optimal, then all of these inequalities hold with equality, and $p_{\mathcal{T}}(W) = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, as claimed. ∎

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Remark 5.3. *It follows from the proof that, under the conditions of this lemma and in the case* $[i] \to_k [j]$, *there is an optimal full walk connecting* i *to* j *on* $\mathcal{T}_{\Gamma(k)}$ *and traversing a critical node that can be decomposed as* $W = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$, *where* $W_{i,\mathcal{N}_c}$ *is an optimal walk in* $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ *and* $W_{\mathcal{N}_c,j}$ *is an optimal walk in* $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$, *and* W_c *consists of edges solely in the critical subgraph. If the elements of* $\mathcal{Y}$ *are also strictly visualised in the sense of* [23], *then any such optimal full walk has to be of this form.*

Lemma 5.2 gives us the first part of the final bound for the case. In order to be able to use this lemma, we must ensure that the walk must traverse $\mathcal{N}_c$; hence, we can use Lemma 4.5 in conjunction with Lemma 5.2 to give us the following theorem.

Theorem 5.4. *Denote* $u^*_{i,\mathcal{N}_c,j} = w^*_{i\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. *Let*

$$k \geq \max\left(\frac{u^*_{i,\mathcal{N}_c,j} - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n-q) + \mathrm{Sch}(\gamma, q),\ \frac{u^*_{i,\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + (n-q+1)\right), \tag{5.2}$$

if $\gamma_{i,j} > \varepsilon$ *or just*

$$k \geq \frac{u^*_{i,\mathcal{N}_c,j} - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n-q) + \mathrm{Sch}(\gamma, q), \tag{5.3}$$

if $\gamma_{i,j} = \varepsilon$, *for some* $i, j \in N$. *Then*

(i) *If* $[i]' \not\to_k [j]'$ *then* $\Gamma(k)_{i,j} = -\infty$,
(ii) *If* $[i]' \to_k [j]'$ *then* $\Gamma(k)_{i,j} = u^*_{i,\mathcal{N}_c,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$.

Proof. We only need to prove the second part. By Lemma 4.5 and taking $\omega = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, if

$$k > \frac{w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_{q*}} + (n-q),$$

then any walk on $\mathcal{T}(P)$ that does not traverse the nodes in $\mathcal{N}_c$ will have weight smaller than $w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, or such walk will not exist if $\gamma_{i,j} = \varepsilon$. Using Lemma 5.2, if

$$k \geq \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_{q*}} + (n-q) + \mathrm{Sch}(\gamma, q) + \frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_{q*}} + (n-q),$$

and $[i]' \to_k [j]'$ then the weight of any optimal full walk on $\mathcal{T}(P)$ connecting i to j and traversing a critical node will be equal to $w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. If $\gamma_{i,j} = \varepsilon$, $[i]' \to_k [j]'$ and the above inequality holds, or if $\gamma_{i,j} > \varepsilon$, k satisfies both inequalities and $[i] \to_k [j]$, then any optimal full walk traverses nodes in $\mathcal{N}_c$ and has weight

$$\Gamma(k)_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}.$$

∎

Our next aim is to rewrite Theorem 5.4 in a CSR form, and we first want to look at the optimal walk representation of $w^*_{i,\mathcal{N}_c}$ and $v^*_{\mathcal{N}_c,j}$. This leads to the following lemma.

Lemma 5.5. *We have*

$$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T},\mathrm{full}}(i \to \mathcal{N}_c)), \quad v^*_{\mathcal{N}_c,j} = p(\mathcal{W}^k_{\mathcal{T},\mathrm{full}}(\mathcal{N}_c \to j)). \tag{5.4}$$

Proof. We will prove only the first of these two equalities, as the second one can be proved in a similar way.

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Let $W_{i,\mathcal{N}_c}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$, with weight $w^*_{i,\mathcal{N}_c}$. We are required to prove that

$$p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) = p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right), \tag{5.5}$$

where on the right we have the set of full walks connecting i to a critical node on $\mathcal{T}(P)$. We split (5.5) into two inequalities,

$$p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) \leq p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right), \quad p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) \geq p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right). \tag{5.6}$$

For the first inequality in (5.6), observe that we can concatenate $W_{i,\mathcal{N}_c}$ with a walk V on the critical graph which has length $l(V) = k - l(W_{i,\mathcal{N}_c})$. The resulting walk $W_{i,\mathcal{N}_c}V$ belongs to $\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)$ and has weight $w^*_{i,\mathcal{N}_c}$, which proves the first inequality. For the second inequality, take an optimal walk $W^* \in \mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)$, whose weight is $p(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c))$. By observing the first occurrence of a critical node in this walk, we represent $W^* = WV$, where $W \in \mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$. We then have $p(W^*) = p(W) + p(V) \leq p(W) \leq w^*_{i,\mathcal{N}_c}$ proving the second inequality. Combining both inequalities gives the equality (5.5) and finishes the proof of $w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c))$. The second part of the claim is proved similarly. ◻

REMARK 5.6. *In the previous lemma, the length of the walks on the right-hand side does not have to be restricted to k. We can obtain the following results:*

$$\begin{aligned} w^*_{i,\mathcal{N}_c} &= p(\mathcal{W}^l_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c)) \quad \textit{for any } l \geq \min\left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_{q*}} + (n-q), k\right) \\ v^*_{\mathcal{N}_c,j} &= p(\mathcal{W}^m_{\mathcal{T},\text{final}}(\mathcal{N}_c \to j)) \quad \textit{for any } m \geq \min\left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_{q*}} + (n-q), k\right). \end{aligned} \tag{5.7}$$

We now establish the connection between the previous Lemma and CSR.

LEMMA 5.7. *We have one of the following cases:*

(i) $(CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = \varepsilon$ *if* $[i]' \not\to_k [j]'$,
(ii) $(CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$ *if* $[i]' \to_k [j]'$.

Proof. By Lemma 3.7, we have $p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \to j)\right) = (CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)])_{i,j}$, where $v = (t+1)\gamma - k(\text{mod}\,\gamma)$ and $t\gamma \geq T(S)$, and let $W \in \mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \to j)$ be optimal. W can be decomposed as $W_1W_2W_3$ where W_1 is a full walk (of length k) connecting i to some $l \in \mathcal{N}_c$ on $\mathcal{T}$, W_3 is a (full) walk of length k connecting some $m \in \mathcal{N}_c$ to j and W_2 is a walk on the critical graph of length v connecting the end of W_1 to the beginning of W_3. In formula,

$$\begin{aligned} (CS^{k(\text{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = \max\{&p(W_1) + p(W_2) + p(W_3)\colon \\ &W_1 \in \mathcal{W}^k_{\mathcal{T},\text{full}}(i \to l),\ W_2 \in \mathcal{W}^v_{\mathbf{C}}(l \to m),\ W_3 \in \mathcal{W}^k_{\mathcal{T},\text{full}}(m \to j),\ l,m \in \mathcal{N}_c\}. \end{aligned} \tag{5.8}$$

If the weights of W_1, W_2, and W_3 in (5.8) are finite then $[i]' \to_k [l]'$, $[l]' \to_v [m]'$ and $[m]' \to_k [j]'$, hence $[i]' \to_k [j]'$. Thus, $(CS^tR[\Gamma(k)]_{i,j}) > \varepsilon$ implies $[i]' \to_k [j]'$ proving (i).

As the cyclicity of the associated graph is the same as the cyclicity of the critical graph, Lemma 5.5 implies that

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$$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T}}(i \to \mathcal{C}_{i,k})), \quad v^*_{\mathcal{N}_c,j} = p(\mathcal{W}^k_{\mathcal{T}}(\mathcal{C}_{k,j} \to j)), \tag{5.9}$$

where $\mathcal{C}_{i,k} = \mathcal{C}'_{i,k} \cap \mathcal{N}_c$ is the cyclic class of $\mathbf{C}(\mathcal{X})$ that can be found by intersecting with critical nodes $\mathcal{N}_c$ the cyclic class $\mathcal{C}'_{i,k}$ of $\mathcal{D}$ defined by $[i]' \to_k \mathcal{C}'_{i,k}$. Similarly, $\mathcal{C}_{k,j} = \mathcal{C}'_{k,j} \cap \mathcal{N}_c$ is the cyclic class of $\mathbf{C}(\mathcal{X})$ that can be found by intersecting with critical nodes $\mathcal{N}_c$ the cyclic class $\mathcal{C}'_{k,j}$ of $\mathcal{D}$ defined by $\mathcal{C}'_{k,j} \to_k [j]'$.

Now note that in (5.8), we can similarly restrict l to $\mathcal{C}_{i,k}$ and m to $\mathcal{C}_{k,j}$, which transforms it to

$$\begin{aligned}(CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)])_{i,j} = \max\{&p(W_1)+p(W_2)+p(W_3): \\ &W_1 \in \mathcal{W}^k_{\mathcal{T}}(i \to l),\ W_2 \in \mathcal{W}^v_{\mathbf{C}}(l \to m),\ W_3 \in \mathcal{W}^k_{\mathcal{T}}(m \to j),\ l \in \mathcal{C}_{i,k},\ m \in \mathcal{C}_{k,j}\}.\end{aligned} \tag{5.10}$$ □

Note that if a walk W_2 exists between any $l \in \mathcal{C}_{i,k}$ and $m \in \mathcal{C}_{k,j}$ then using (5.9) we immediately obtain $(CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)])_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. Thus, it remains to show existence of $W_2 \in \mathcal{W}^v_{\mathbf{C}}(l \to m)$ between any $l \in \mathcal{C}_{i,k}$ and $m \in \mathcal{C}_{k,j}$. For this, note that since $v = (t+1)\gamma - k(\operatorname{mod}\gamma) \geq T(S)$, either $\mathcal{C}_{i,k} \to_{(\gamma - k(\operatorname{mod}\gamma))} \mathcal{C}_{k,j}$ and a walk on $\mathbf{C}(\mathcal{X})$ of length v exists between each pair of nodes in $\mathcal{C}_{i,k}$ and $\mathcal{C}_{k,j}$, or $\mathcal{C}_{i,k} \not\to_{(\gamma - k(\operatorname{mod}\gamma))} \mathcal{C}_{k,j}$ and then no such walk exists. We thus have to check that $\mathcal{C}_{i,k} \to_{(\gamma - k(\operatorname{mod}\gamma))} \mathcal{C}_{k,j}$ on $\mathcal{D}$. But this follows since we have $[i]' \to_k [j]'$, and since in the sequence $[i]' \to_k \mathcal{C}'_{i,k} \to_l \mathcal{C}'_{k,j} \to_k [j]'$, we then must have $l \equiv_\gamma \gamma - k(\operatorname{mod}\gamma)$.

Combining Theorem 5.4 and Lemma 5.7 we obtain the following result.

THEOREM 5.8. *Denote $u^*_{i,\mathcal{N}_c,j} = w^*_{i\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. Let k be greater than or equal to*

$$\max\left(\max_{i,j} \frac{u^*_{i,\mathcal{N}_c,j} - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n-q) + \operatorname{Sch}(\gamma,q),\ \max_{i,j\,:\,\gamma_{i,j} > \varepsilon} \frac{u^*_{i,\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + n - q + 1\right).$$

Then $\Gamma(k) = CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)]$.

As with Theorem 4.6, this bound requires $\Gamma(k)$ in order to calculate the bound, which makes it implicit, but as with Corollary 4.7 we can use $w_{i,\mathcal{N}_c} \leq w^*_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j} \leq v^*_{\mathcal{N}_c,j}$ to give us an explicit bound. The following result requires Assumption $\mathcal{C}$ on $A^{\inf}$.

COROLLARY 5.9. *Denote $u_{i,\mathcal{N}_c,j} = w_{i\mathcal{N}_c} + v_{\mathcal{N}_c,j}$. Let k be greater than or equal to*

$$\max\left(\max_{i,j} \frac{u_{i,\mathcal{N}_c,j} - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n-q) + \operatorname{Sch}(\gamma,q),\ \max_{i,j\,:\,\gamma_{i,j} > \varepsilon} \frac{u_{i,\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + n - q + 1\right).$$

Then, $\Gamma(k) = CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)]$.

We will now present an example of this bound in action.

Let $\mathcal{D}(G)$ be the eight node digraph with the following structure:

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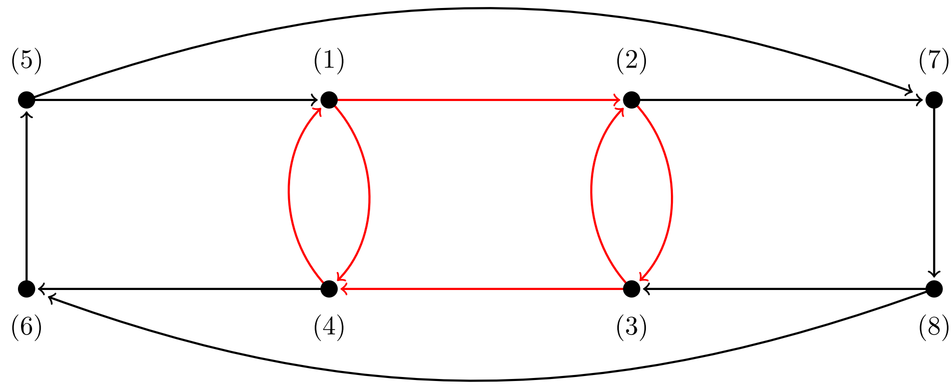

along with the associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & A_{2,7} & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{4,6} & \varepsilon & \varepsilon \\ A_{5,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,7} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{6,5} & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{7,8} \\ \varepsilon & \varepsilon & A_{8,3} & \varepsilon & \varepsilon & A_{8,6} & \varepsilon & \varepsilon \end{pmatrix}.$$

There are three critical cycles in this digraph, one cycle of length 4 traversing $1 \to 2 \to 3 \to 4$, and two cycles of length 2 traversing $1 \to 4 \to 1$ and $2 \to 3 \to 2$, respectively. There are also cycles of length 4, 6, and 8 which means that the cyclicity of the whole digraph is 2, which is the same cyclicity of the critical subgraph. Therefore, Assumption $\mathcal{P}0$ is satisfied, and we can continue.

The semigroup of matrices $\mathcal{X}$ used by this example will be generated by these five matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -14 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -20 \\ \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon & -3 & \varepsilon & \varepsilon \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -17 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -17 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -5 \\ \varepsilon & \varepsilon & -19 & \varepsilon & \varepsilon & -7 & \varepsilon & \varepsilon \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -13 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -17 \\ \varepsilon & \varepsilon & -12 & \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon \end{pmatrix}, A_4 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -19 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -16 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -12 \\ \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon \end{pmatrix},$$

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$$A_5 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon & \varepsilon \\ -19 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -12 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 \\ \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon & -7 & \varepsilon & \varepsilon \end{pmatrix}.$$

Using these matrices, we can calculate A^{sup} and A^{inf},

$$A^{\text{sup}} = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -5 \\ \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon \end{pmatrix}, A^{\text{inf}} = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -19 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon & \varepsilon \\ -19 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -20 \\ \varepsilon & \varepsilon & -19 & \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon \end{pmatrix},$$

as well as $\alpha_{i,\mathcal{N}_c}$, $\beta_{\mathcal{N}_c,j}$, $\gamma_{i,j}$, $w_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j}$:

$$\alpha_{i,\mathcal{N}_c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -9 \\ -17 \\ -6 \\ -1 \end{pmatrix}, \qquad \beta^T_{\mathcal{N}_c,j} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -14 \\ -6 \\ -3 \\ -8 \end{pmatrix}, \qquad \gamma_{i,j} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & -10 & -3 & -8 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & -10 & -3 & -8 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -15 & -7 & -18 & -5 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 & -2 & -13 & -18 \end{pmatrix}$$

$$w^T_{i,\mathcal{N}_c} = \begin{pmatrix} 0 & 0 & 0 & 0 & -19 & -37 & -39 & -19 \end{pmatrix}, v_{\mathcal{N}_c,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & -34 & -16 & -19 & -39 \end{pmatrix}.$$

With all the pieces ready, we can now form the bound of Corollary 5.9,

$$k \geq \max\left(\begin{pmatrix} 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 14.2 & 14.2 & 14.2 & 14.2 & 18.7 & 16.4 & 17.8 & 21.1 \\ 16.4 & 16.4 & 16.4 & 16.4 & 20.9 & 18.7 & 20 & 23.3 \\ 19.3 & 19.3 & 19.3 & 19.3 & 23.8 & 21.6 & 22.9 & 26.2 \\ 16 & 16 & 16 & 16 & 20.4 & 18.22 & 19.6 & 22.9 \end{pmatrix}, \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 12.8 & 10.6 & 12.8 & 16.1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 19 & 12.8 & 15 & 18.3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 17.9 & 15.7 & 13.9 & 21.2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 14.6 & 12.3 & 10.6 & 13.9 \end{pmatrix}\right)$$

$$\Rightarrow k \geq 23.8.$$

Therefore, by Corollary 5.9 if the length of a product using the matrices from $\mathcal{X}$ is greater than or equal to 24 then the resulting product will be CSR. We will show such a product. Let $\Gamma(24)$ be the inhomogeneous matrix product made using the word $P = 551541235515535135454155$ which gives us:

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$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ \varepsilon & -19 & \varepsilon & -19 & -47 & \varepsilon & \varepsilon & -40 \\ -31 & \varepsilon & -31 & \varepsilon & \varepsilon & -47 & -42 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon & \varepsilon & -27 & -22 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 & -29 & \varepsilon & \varepsilon & -22 \end{pmatrix}.$$

This matrix product is indeed CSR, and by Definition 3.2 we have,

$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & -19 & \varepsilon & -19 \\ -31 & \varepsilon & -31 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}$$

$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \\ 0 & \varepsilon \\ \varepsilon & 0 \\ \varepsilon & -19 \\ -31 & \varepsilon \\ -11 & \varepsilon \\ \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}.$$

We can see that, for the C matrix, columns 3 and 4 are copies of columns 1 and 2, respectively. The same is also true for the rows of the R matrix so they can be deleted. As $24 (\mathrm{mod}\, 2) = 0$, we replace the S matrix with the tropical identity matrix which shows us that the matrix product $\Gamma(24)$ using the word P is indeed CSR, and it has factor rank-2.

6. Counterexamples.

Here we present a number of counterexamples for the different cases of digraph structure. These counterexamples present families of products which are not CSR, and we construct them in such a way that they have no upper bound on their length.

6.1. The ambient graph is primitive but the critical graph is not.

We will now look at two cases where we are unable to create a bound for matrix products to become CSR. For the first case, we will be looking at digraphs that are primitive but have a critical subgraph with a non-trivial cyclicity. Therefore, we have the following assumption:

Assumption $\mathcal{P}1$. *$\mathcal{D}(\mathcal{X})$ is primitive (i.e., $\gamma(\mathcal{D}(\mathcal{X})) = 1$) and the critical subgraph $\mathbf{C}(\mathcal{X})$, which is a single strongly connected component, has cyclicity $\gamma(\mathbf{C}(\mathcal{X})) = \gamma > 1$.*

We now present a counterexample which shows that under this assumption, in general, no bound for k in terms of $A^{\sup}$ and $A^{\inf}$ can exist that ensures that $\Gamma(k)$ is equal to the corresponding CSR product.

Let $\mathcal{D}(G)$ be the five node digraph with the following structure:

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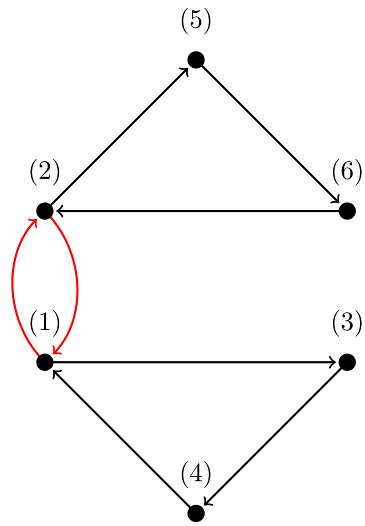

This digraph will have the following associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & A_{1,3} & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & A_{2,5} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & A_{3,4} & \varepsilon & A_{3,6} \\ A_{4,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,6} \\ \varepsilon & A_{6,2} & A_{6,3} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

There is a critical subgraph consisting of the cycle between nodes 1 and 2. There also exist two cycles, $1 \to 3 \to 4 \to 1$ and $2 \to 5 \to 6 \to 2$, both of length 3 which makes $\mathcal{D}(A)$ primitive. We aim to present a family of words with infinite length such that the products made up using these words are not CSR. Since the cyclicity of the critical subgraph is 2, then we will have to create two classes of words, one of even length and one of odd length to define the family.

The semigroup of matrices we will use is generated by the two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

Let us first consider the class of words $(1)^{2t}2$ where $t \geq 2$, and let $U = (A_1)^{2t}A_2$ for arbitrary such t. We will first examine entries $U_{6,1}$, $U_{2,5}$, $U_{6,2}$ and $U_{1,5}$.

The entry $U_{6,1}$ can be obtained as the weight of the walk $6\underbrace{(21)(21)\ldots(21)}_{t-1}341$, which is -301. For this observe that the walk 621 has an even length and therefore we need to use one of the three-cycles to make it odd, and using the southern three-cycle in the end of the walk is the most profitable way to do so. The entry U_{25} is equal to -1, as there is a walk that mostly rests on the critical cycle and only in the end jumps to node 5. We also have $U_{6,2} = -100$ (go to node 2 and remain on the critical cycle) and $U_{1,5} = -301$ (use the

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southern triangle once and then dwell on the critical cycle and in the end jump to node 5). Note that in the case of $U_{1,5}$ we again need to use one of the triangles to create a walk of an odd length.

We then compute

$$(CSR)[U]_{6,5} = (US^3U)_{6,5} = \max(U_{6,1} + U_{2,5},\ U_{6,2} + U_{1,5}) = -301 - 1 = -302.$$

However, $U_{6,5}$ results from the walk $6\underbrace{(21)(21)\dots(21)}_{t-1}2562$, with weight -401, needing to use the northern triangle to make a walk of odd length.

The following an example of U and $CS^{2t+1}R[U]$ for $t = 10$:

$$U = \begin{pmatrix} -201 & 0 & -100 & -500 & -301 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \end{pmatrix}$$

$$CS^{21(\mathrm{mod}\,2)}R[U] = \begin{pmatrix} -201 & 0 & -100 & -401 & -202 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \end{pmatrix}.$$

We now consider the class of words $(1)^{2t+1}2$ where $t \geq 1$, and let $V = (A_1)^{2t+1}A_2$ for arbitrary such t. We will first examine entries $V_{2,1}$, $V_{1,5}$, $V_{2,2}$ and $V_{2,5}$.

The entry $V_{2,1} = -201$ is obtained as the weight of the walk $2\underbrace{(12)(12)\dots(12)}_{t-1}341$: it is necessary to use one of the triangles to create a walk of even length, and using the southern triangle once in the end of the walk is the most profitable way to do so. The walk 125 already has an even length, and we only have to augment it with enough copies of the critical cycle and use the arc $2 \to 5$ in the end of the walk, thus getting $V_{1,5} = -1$. Obviously, $V_{2,2} = 0$: we just stay on the critical cycle. The entry $V_{2,5} = -301$ is obtained as the weight of the walk $\underbrace{(21)(21)\dots(21)}_{t-1}5625$, where we have to use the northern triangle in the end of the walk to create a walk of even walk and minimise the loss.

We then find

$$(CS^2R[V])_{2,5} = (VS^2V)_{2,5} = \max(V_{2,1} + V_{1,5}, V_{2,2} + V_{2,5}) = V_{2,1} + V_{1,5} = -202,$$

which is bigger than $V_{2,5} = -301$.

The case for $V_{2,5}$ is one for connecting a critical node to a non-critical node. For completeness, we should also look at a walk connecting two non-critical nodes, namely the walk representing $V_{4,5}$. To do this, we will need to also look at the entries $V_{4,1}$ and $V_{4,2}$. For $V_{4,1} = -301$, the entry is obtained as the weight of the walk

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$4\underbrace{(12)(12)\dots(12)}_{t-1}341$. As the walk 41 has odd length, one of the triangles is required to make the walk even so choosing the southern triangle is the most profitable way to achieve an even length walk. The walk 412 already has an even length so we can augment it with enough copies of the critical cycle to give us the desired length for the walk representing the entry $V_{4,2} = -100$. Using $V_{1,5}$ and $V_{2,5}$ discussed earlier we calculate

$$(CS^2R[V])_{4,5} = (VS^2V)_{4,5} = \max(V_{4,1} + V_{1,5}, V_{4,2} + V_{2,5}) = V_{4,1} + V_{1,5} = -302,$$

which is bigger than $V_{4,5} = -401$.

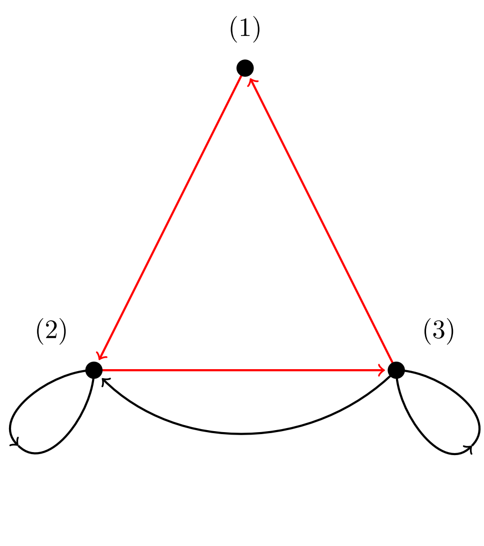
We now show an example of V for $t = 10$:

$$V = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -500 & -301 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}$$

$$CS^{22 (\mathrm{mod}\, 2)}R[V] = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -401 & -202 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}.$$

Combining both classes, we have a family of words covering all lengths greater than 29 such that any product made using these words will not be equal to the corresponding CSR product. Therefore, there cannot be a transient for this case as there is no upper limit to the lengths of these words.

We now also construct a counterexample where all nodes of $\mathcal{D}(G)$ are critical. Let $\mathcal{D}(G)$ be the three node digraph with the following structure:



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The digraph has the following associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & A_{2,2} & 0 \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix}.$$

For this example, there is a single critical cycle of length 3 traversing all of the nodes. There also exists two loops $2 \to 2$ and $3 \to 3$ and a cycle $2 \to 3 \to 2$ of length 2. Like the previous example, this digraph is primitive but the critical subgraph has cyclicity 3. As the cyclicity is greater than one we need to present three different classes of words making up a family of words such that any product $\Gamma(k)$ made using these words will not be CSR.

The semigroup of matrices that we will use is again generated only by two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -100 & 0 \\ 0 & -100 & -100 \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix}.$$

Let the first class of words be $(1)^{3t+2}2$ for $t \geq 0$, and let $M = (A_1)^{3t+2}A_2$ for any arbitrary t. We will now examine the entries $M_{1,1}$, $M_{1,2}$, $M_{2,2}$ $M_{1,3}$ and $M_{3,2}$.

Since all the walks are of length 0 modulo 3, then any walk connecting i to i will have weight zero as we can simply use the critical cycle. This gives $M_{1,1} = M_{2,2} = 0$. The entry $M_{1,2}$ can be obtained as the weight of the walk $(123)^{t+1}2$ which is -100. In this entry, observe that the walk 12 is of length 1 modulo 3; therefore, we need to use the two cycle $2 \to 3 \to 2$ to give us a walk of the desired length. The entry $M_{1,3}$ is equal to the weight of the walk $(123)^{t+1}3$ and the entry $M_{3,2}$ is equal to the weight of the walk $(312)^{t+1}2$. For these entries, observe that the walks 123 and 312 are both of length 2 modulo 3; therefore, we require a loop for both walks to give us the required length. The most profitable time to use these loops are right at the end of the walk.

We then compute

$$(CSR)[M]_{1,2} = (MS^3M)_{1,2} = \max(M_{1,1} + M_{1,2}, M_{1,2} + M_{2,2}, M_{1,3} + M_{3,2}) = -1 - 1 = -2.$$

However, as seen earlier the entry M_{12} has weight -100 which is less than the CSR suggestion.

The following is an example of M and $CS^{3t+3}R[M]$ for $t = 10$:

$$M = \begin{pmatrix} 0 & -100 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix} \quad CS^{33 (\bmod 3)}R[M] = \begin{pmatrix} 0 & -2 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix}.$$

For efficiency, we will simply present the final two classes and omit the in-depth analysis of them:

For walks of length 1 modulo 3, we have the class of words $(1)^{3t+3}2$ for $t \geq 0$.
For walks of length 2 modulo 3, we have the class of words $(1)^{3t+4}2$ for $t \geq 0$.

We will also present examples of products and their CSR counterparts made using these words for $t = 10$ where $N = (A_1)^{3t+3}A_2$ and $P = (A_1)^{3t+4}A_2$.

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$$N = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix} \quad CS^{34 (\mathrm{mod}\, 3)}R[N] = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -100 & -1 \\ -100 & 0 & -100 \end{pmatrix} \quad CS^{35 (\mathrm{mod}\, 3)}R[P] = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -2 & -1 \\ -100 & 0 & -100 \end{pmatrix}.$$

The combination of these three classes create a family of words such that any product $\Gamma(k)$ made using these words is not equal to the corresponding CSR product.

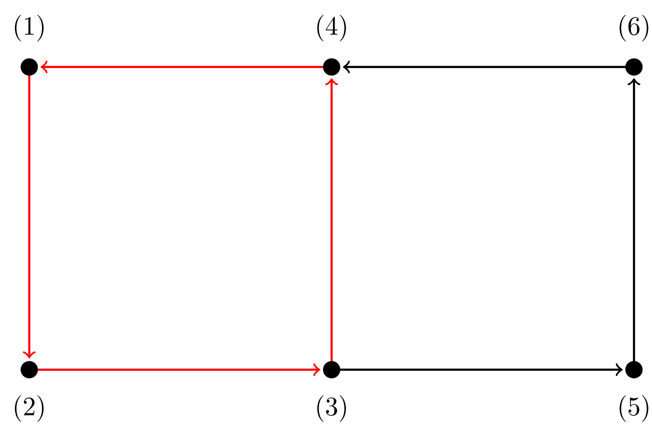
We now extend these counterexamples to a more general form where we consider digraphs with non-trivial cyclicity r along with critical subgraphs with cyclicity γ which is greater than r. This leads to the following assumptions.

6.2. More general case.

Assumption $\mathcal{P}2$. *$\mathcal{D}(\mathcal{X})$ has cyclicity r and the critical subgraph $\mathbf{C}(\mathcal{X})$, which is strongly connected, has cyclicity $\gamma > r$.*

In a similar method to the primitive example above, using the new assumptions, we can now describe a counterexample that shows that no bound for k in terms of A^{sup} and A^{inf} can exist that ensures $\Gamma(k)$ is equal to the corresponding CSR product.

Let $\mathcal{D}(\mathcal{X})$ be a six node digraph with the following structure:



along with the following associated weight matrix,

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & A_{3,5} & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,6} \\ \varepsilon & \varepsilon & \varepsilon & A_{6,4} & \varepsilon & \varepsilon \end{pmatrix}.$$

Here, the critical cycle traverses nodes $1 \to 2 \to 3 \to 4 \to 1$; however, there also exists another non-critical cycle of length six traversing $1 \to 2 \to 3 \to 5 \to 6 \to 4 \to 1$. This means that while the cyclicity of the critical

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subgraph is 4 the cyclicity of $\mathcal{D}(G)$ is 2. Therefore, the digraph structure satisfies the assumptions, and we can develop a family of words with infinite length such that any $\Gamma(k)$ made using these words will not be equal to the corresponding CSR product. As the cyclicity of the critical subgraph is 4, then we will require four classes of words to fully define the family.

The semigroup of matrices that will be used is generated by two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -100 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon \end{pmatrix}.$$

Let us begin with the first class of words $(1)^{4t}2$ where $t \geq 2$, and let $L = (A_1)^{4t}A_2$ for arbitrary such t. We will begin by examining the entries $L_{1,2}$, $L_{1,5}$, $L_{1,4}$ and $L_{3,5}$.

The entry $L_{1,2}$ can be obtained as the weight of the walk $\underbrace{(1234)}_{t}12$, which is 0. As the walk 12 has length congruent to $1(\bmod 4)$, then a walk exists on the critical cycle connecting these nodes. The entry $L_{1,5}$ is obtained from the weight of the walk $\underbrace{(1234)}_{t-2}1235641235$, which is -301. As the walk 1235 has length congruent to $3(\bmod 4)$, then we need to add on the six cycle with weight -300 to give us a walk of length congruent to $1(\bmod 4)$ and finally the last step of the walk is to go from 3 to 5 with weight -1. For the entry, $L_{1,4} = -201$ which is the weight of the walk $\underbrace{(1234)}_{t-1}123564$ and the entry $L_{35} = -1$ comes from the weight of the walk $\underbrace{(3412)}_{t}35$. Note that in the case of $L_{1,4,}$ we used the six cycle to give us the desired length of walk.

We then compute

$$(CSR)[L]_{1,5} = (L \otimes S^3 \otimes L)_{1,5} = \max(L_{1,2} + L_{1,5},\ L_{1,4} + L_{3,5}) = -201 - 1 = -202.$$

However L_{15}, as explained earlier, results from a walk with weight -301.

The following is an example of L and $CS^{4t+1}R[L]$ for $t = 10$

$$L = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}$$

$$CS^{41(\bmod 4)}R[L] = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}.$$

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The other classes behave in a similar way so we omit the in-depth explanation of them. We present the words used for each class:

For walks of length congruent to $2 (\text{mod } 4)$, we have the words $(1)^{4t+1}2$ for $t \geq 2$;
For walks of length congruent to $3 (\text{mod } 4)$, we have the words $(1)^{4t+2}2$ for $t \geq 2$;
For walks of length congruent to $0 (\text{mod } 4)$, we have the words $(1)^{4t+3}2$ for $t \geq 2$.

For example, if $t = 10$ then for the first of these classes

$$F = (A_1)^{41} \otimes A_2 = \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix},$$

$$CS^{42 (\text{mod } 4)}R[F] = \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix}.$$

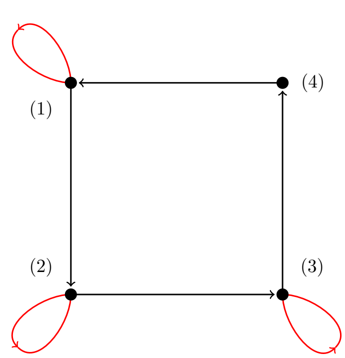
Combining all classes gives us a family of words covering all lengths greater than 9 such that any product made using these words will not be equal to the corresponding CSR product.

6.3. Critical graph is not connected. For this counterexample, we now consider a digraph with multiple critical components $\mathbf{C}_1, \ldots, \mathbf{C}_m$ which are each strongly connected components with respective cyclicities $\gamma_1, \ldots, \gamma_m$.

ASSUMPTION $\mathcal{P}3$. *$\mathbf{C}(\mathcal{X})$ is composed of multiple strongly connected components $\mathbf{C}_1, \ldots, \mathbf{C}_m$ where the component $\mathbf{C}_i$ has cyclicity γ_i. The cyclicity of $\mathcal{D}(\mathcal{X})$ is $\text{lcm}_i(\gamma_i)$, which is the same as the cyclicity of $\mathbf{C}(\mathcal{X})$.*

Let us now show a counterexample, which demonstrates that, for the case of several critical components, we cannot have any bounds after which the product becomes CSR in terms of A^{sup} and A^{inf}. The reason is that the non-critical parts of optimal walks whose weights are the entries of C and R cannot be separated in time: in general, they will use the same letters, and such walks on the symmetric extension of $\mathcal{T}(P)$ cannot be transformed back to the walks on $\mathcal{T}(P)$.

Let $\mathcal{D}(\mathcal{X})$ be the four node digraph with the following structure:



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along with the following associated weight matrix

$$A = \begin{pmatrix} 0 & A_{12} & \varepsilon & \varepsilon \\ \varepsilon & 0 & A_{23} & \varepsilon \\ \varepsilon & \varepsilon & 0 & A_{34} \\ A_{41} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

For this digraph, we have a the critical subgraph comprised of three separate loops at nodes 1,2 and 3. There is also a cycle of length 4 which means the cyclicity of the digraph is 1. We are going to present a class of words of infinite length such that the matrix generated by this class of words is not CSR.

We introduce a semigroup of tropical matrices with two generators $\mathcal{X} = \{A_1, A_2\}$ where A_1 to A_2 are

$$A_1 = \begin{pmatrix} 0 & -100 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -100 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -1 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix},$$

and the class of the words that we will consider is $(1)^t 2$, where $t \geq 2$. In other words, we will consider a set of matrices of the form $U = (A_1)^t A_2$ (the actual value of $t \geq 2$ will not matter to us).

We have: $U_{1,2} = -1$ (as the weight of the walk $\underbrace{11\ldots1}_{t+1}2$), $U_{2,3} = -1$ (as the weight of the walk $\underbrace{22\ldots2}_{t+1}3$),and therefore $(CS^{t+1}R[U])_{1,3} = U^2_{1,3} = U_{1,2} \otimes U_{2,3} = -2$, but $U_{1,3} = -101$ (as the weight of the walk $1\underbrace{22\ldots2}_{t}3$).

Similarly, we can also look at the entry $U_{4,3}$. Then, we have $U_{4,2} = -101$ (as the weight of the walk $4\underbrace{11\ldots1}_{t}2$), $U_{2,3} = -1$ and hence $(CS^{t+1}R)_{4,3} = (USU)_{4,3} = U_{4,2} \otimes U_{2,3} = -102$, but $U_{4,3} = -201$ (as the weight of the walk $41\underbrace{22\ldots2}_{t-1}3$).

Here is an example of the word from the class for $t = 10$ and the corresponding CSR

$$W = \begin{pmatrix} 0 & -1 & -101 & -300 \\ -300 & 0 & -1 & -200 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -201 & -400 \end{pmatrix}, \qquad CS^{11 \pmod 1} R[W] = \begin{pmatrix} 0 & -1 & -2 & -201 \\ -201 & 0 & -1 & -101 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -102 & -301 \end{pmatrix}.$$

Therefore, any matrix product of length greater than 3 which has been made following this word will not be CSR. Hence, there can be no upper bound to guarantee the CSR decomposition in this case.

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