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SUMS OF ORTHOGONAL, SYMMETRIC, AND SKEW-SYMMETRIC MATRICES*

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Abstract. An n-by-n matrix A is called symmetric, skew-symmetric, and orthogonal if $A^T = A$, $A^T = -A$, and $A^T = A^{-1}$, respectively. We give necessary and sufficient conditions on a complex matrix A so that it is a sum of type "orthogonal + symmetric" in terms of the Jordan form of $A - A^T$. We also give necessary and sufficient conditions on a complex matrix A so that it is a sum of type "orthogonal + skew-symmetric" in terms of the Jordan form of $A + A^T$.

Key words. Orthogonal, Symmetric, Skew-symmetric, Sums, Decompositions.

AMS subject classifications. 15A21, 15A23.

1. Introduction.

A matrix A is called

- *symmetric* if $A^T = A$,
- *skew-symmetric* if $A^T = -A$,
- *orthogonal* if A is nonsingular and $A^T = A^{-1}$.

Several mathematicians have studied matrix decompositions involving the above special matrices. Frobenius showed that every matrix over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a product of two symmetric matrices [3] (see also the work of Radjavi [12]). Gow and Laffey gave necessary and sufficient conditions for a matrix over an arbitrary field to be a product of two skew-symmetric matrices [5]. Laffey later on proved that if $n \equiv 0 \bmod 4$ and A is an n-by-n matrix over an algebraically closed field with characteristic not equal to 2, then A is a product of five skew-symmetric matrices [10]. Horn and Merino showed that a complex matrix A may be written as a product $A = QR$, where Q is orthogonal and R is symmetric if and only if AA^T is similar to $A^T A$ [9]. De la Cruz et al. gave necessary and sufficient conditions for a complex matrix A to be written as a product $A = QR$, where Q is orthogonal and R is skew-symmetric [1]. If $n > 1$, Merino showed that any matrix over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is a sum of a finite number of orthogonal matrices [11]. Granario et al. gave necessary and sufficient conditions for a complex matrix to be written as a sum of two orthogonal matrices [4].

The main result of this paper is the following theorem which gives necessary and sufficient conditions for a complex matrix A to be written as $A = Q + R$, where Q is orthogonal and R is either symmetric or skew-symmetric. For a complex number λ, we denote by $J_k(\lambda)$ the k-by-k upper triangular Jordan block with eigenvalue λ.

THEOREM 1.1. *Let $A \in \mathbb{C}^{n\times n}$ be given. Then*

(a) *$A = A_1 + A_2$, for some orthogonal A_1 and symmetric A_2 if and only if the Jordan blocks of $A - A^T$ with eigenvalue $2i$ of size greater than one come in pairs of $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$.*

(b) *$A = B_1 + B_2$, for some orthogonal B_1 and skew-symmetric B_2, if and only if $A + A^T$ is similar to a direct sum of matrices of the form*

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(1) $J_k(\lambda) \oplus J_k(\lambda)$, *where* $\lambda \neq \pm 2$,
(2) $J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda)$, *for any* $k > 1$ *and* $\lambda = \pm 2$,
(3) $J_{k+1}(\lambda) \oplus J_k(\lambda)$, *where* $\lambda = \pm 2$, *or*
(4) λI_k, *where* $\lambda = 2, -2$.

The authors have a similar study on symplectic, skew-Hamiltonian, and Hamiltonian matrices in [2].

If A has a decomposition as in Theorem 1.1(a), then

$$A - A^T = A_1 + A_2 - (A_1 + A_2)^T = A_1 + A_2 - A_1^T - A_2^T = A_1 - A_1^T. \tag{1.1}$$

Conversely, if $A - A^T = A_1 - A_1^T$ for some orthogonal A_1, then

$$A - A_1 = A^T - A_1^T = (A - A_1)^T, \tag{1.2}$$

is symmetric and $A = A_1 + (A - A_1)$ is a decomposition of A as in Theorem 1.1(a). Analogous arguments show that A has a decomposition from Theorem 1.1(b) if and only if $A + A^T = B_1 + B_1^T$ for some orthogonal B_1. The following theorem implies statements (a) and (b) of Theorem 1.1.

Theorem 1.2. *Let* $A \in \mathbb{C}^{n\times n}$ *be given.*

(a) *If* A *is skew-symmetric, then* $A = X - X^T$ *for some orthogonal* X *if and only if the Jordan blocks of* A *with eigenvalue* $2i$ *and size greater than one, if any, come in pairs of* $J_k(2i) \oplus J_k(2i)$ *or* $J_k(2i) \oplus J_{k+1}(2i)$.
(b) *If* A *is symmetric, then* $A = X + X^T$ *for some orthogonal* X *if and only if* A *is similar to a direct sum of matrices of the form*
 (i) $J_k(\lambda) \oplus J_k(\lambda)$, *where* $\lambda \neq \pm 2$,
 (ii) $J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda)$, *for any* $k > 1$ *and* $\lambda = \pm 2$,
 (iii) $J_{k+1}(\lambda) \oplus J_k(\lambda)$, *where* $\lambda = \pm 2$, *or*
 (iv) λI_k, *where* $\lambda = 2, -2$.

We give some preliminary observations in Section 2 and prove Theorem 1.2 in Section 3.

2. Preliminaries. The conditions for the decompositions in Theorem 1.2 can be stated in terms of the existence of symmetric or skew-symmetric square roots of a symmetric matrix. By $\mathcal{C}(A)$, we mean the *centralizer* of the square matrix A, that is,

$$\mathcal{C}(A) := \{X \in \mathbb{C}^{n\times n} \mid AX = XA\}. \tag{2.3}$$

Lemma 2.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be given.*

(a) *If* A *is skew-symmetric, then* $A = X - X^T$ *for some orthogonal* X *if and only if* $A^2 + 4I$ *has a symmetric square root in* $\mathcal{C}(A)$.
(b) *If* A *is symmetric, then* $A = X + X^T$ *for some orthogonal* X *if and only if* $A^2 - 4I$ *has a skew-symmetric square root in* $\mathcal{C}(A)$.

Proof. Let A be skew-symmetric. Suppose $A = X - X^T$ for some matrix X. If X is orthogonal, then $X^T = X^{-1}$, and so we consider orthogonal solutions to the matrix equation

$$A = X - X^{-1}. \tag{2.4}$$

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If X is a solution to (2.4), then $X \in \mathcal{C}(A)$. Now (2.4) is equivalent to

$$I = (X - A)X = X^2 - AX. \tag{2.5}$$

By completing the squares in (2.5), we get $(X - \frac{1}{2}A)^2 = \frac{1}{4}(A^2 + 4I)$. If, in addition, X is orthogonal and we set $Y := X - \frac{1}{2}A$, then $Y \in \mathcal{C}(A)$, $Y^2 = \frac{1}{4}(A^2 + 4I)$, and

$$Y^T = X^T - \frac{1}{2}A^T = X^{-1} + \frac{1}{2}A = (X - A) + \frac{1}{2}A = Y. \tag{2.6}$$

Thus, $Z := 2Y$ is a symmetric square root of $A^2 + 4I$ and $Z \in \mathcal{C}(A)$.

Conversely, suppose $Z \in \mathcal{C}(A)$ and Z is a symmetric square root of $A^2 + 4I$. Set $Y := \frac{1}{2}Z$ and set $X := Y + \frac{1}{2}A$. Then $Y \in \mathcal{C}(A)$, Y is symmetric,

$$XX^T = \left(Y + \frac{1}{2}A\right)\left(Y - \frac{1}{2}A\right) = Y^2 - \frac{1}{4}A^2 = \frac{1}{4}Z^2 - \frac{1}{4}A^2 = I, \tag{2.7}$$

and

$$X - X^T = Y + \frac{1}{2}A - \left(Y - \frac{1}{2}A\right) = A. \tag{2.8}$$

This proves (a). The proof of (b) is analogous. ∎

Note that the existence of a decomposition in Theorem 1.2 is invariant under orthogonal similarity, and so the following theorem is useful.

Lemma 2.2 ([8, Corollary 22]). *Two complex matrices which are both symmetric, both skew-symmetric, or both orthogonal are similar if and only if they are orthogonally similar.*

A matrix A has a square root if and only if the nilpotent part of A, if any, is similar to a direct sum of matrices of the form 0_m, $J_m(0) \oplus J_m(0)$, or $J_m(0) \oplus J_{m+1}(0)$ for any m [7, Theorem 6.4.12]. To prove Theorem 1.2 it helps to know the Jordan structure of a symmetric or skew-symmetric matrix. Any square complex matrix is similar to a symmetric matrix and so there are no restrictions on the Jordan form of a symmetric matrix [6, Theorem 4.4.9]. For a skew-symmetric matrix A, the Jordan form of A must be expressible as a direct sum of matrices of the form $J_k(\lambda) \oplus J_k(-\lambda)$ for $\lambda \neq 0$, $J_k(0) \oplus J_k(0)$ for even k, or $J_k(0)$ for odd k. Conversely, when A is similar to the direct sum of any of the preceding Jordan blocks, then A is similar to a skew-symmetric matrix [9]. Thus, by Lemma 2.2, we have the following.

Lemma 2.3. *Let $A \in \mathbb{C}^{n \times n}$ be given. Then*

(a) *A is symmetric if and only if A is orthogonal similar to $\oplus_i A_i$, where each A_i is a symmetric matrix that is similar to a Jordan block.*
(b) *A is skew-symmetric if and only if A is orthogonal similar to $\oplus_i A_i$, where each A_i is skew-symmetric and similar to one of the following:*
   (1) *$J_k(\lambda) \oplus J_k(-\lambda)$ for any $\lambda \neq 0$,*
   (2) *$J_k(0) \oplus J_k(0)$ for any even k, and*
   (3) *$J_k(0)$ for any odd k.*

The following result reduces our problem to symmetric or skew-symmetric matrices having at most two eigenvalues. Let $\sigma(A)$ denote the spectrum of a matrix A.

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LEMMA 2.4. *Let $A = \oplus_{i=1}^{m} A_i$ for some square complex matrices A_i with pairwise disjoint spectra. Then*

(a) *A is symmetric such that $A^2 - 4I$ has a skew-symmetric square root that commutes with A if and only if each A_i is symmetric and $A_i^2 - 4I$ has a skew-symmetric square root that commutes with A_i.*
(b) *A is skew-symmetric such that $A^2 + 4I$ has a symmetric square root that commutes with A if and only if each A_i is skew-symmetric and $A_i^2 + 4I$ has a symmetric square root .*

Proof. We only do (a). Sufficiency follows from the fact that a direct sum of skew-symmetric matrices is skew-symmetric. For necessity, let B be a skew-symmetric square root of $A^2 - 4I$ that commutes with A. Since $\sigma(A_i) \cap \sigma(A_j) = \emptyset$ for $i \neq j$, Sylvester's theorem [6, Theorem 2.4.4.1] implies that $B = \oplus_{i=1}^{m} B_i$ and partitioned conformal to A. Hence each B_i is a skew-symmetric square root of $A_i^2 - 4I$ that commutes with A_i. ∎

3. Proof of Theorem 1.2. For $A \in \mathbb{C}^{n \times n}$, we let $\mathbb{C}[A] := \{p(A) \,|\, p(x) \in \mathbb{C}[x]\}$ denote the set of all polynomials in A.

Proof of Theorem 1.2(a). Let A be skew-symmetric. Suppose that $A = Y - Y^T$ for some orthogonal Y. By Lemma 2.1, $A^2 + 4I$ has a symmetric square root B which commutes with A. Lemma 2.3(b) implies that there is a nonsingular matrix X such that

$$XAX^{-1} = A_1 \oplus -A_1 \oplus A_2, \tag{3.9}$$

where each A_i is symmetric, $\sigma(A_1) = \{2i\}$, and $2i, -2i \notin \sigma(A_2)$. Since $XBX^{-1} \in \mathcal{C}(XAX^{-1})$, the eigenvalue conditions above and Sylvester's theorem imply that

$$XBX^{-1} = B_1 \oplus B_2 \oplus B_3, \tag{3.10}$$

which is partitioned conformal to XAX^{-1}. It follows that $B_1^2 = A_1^2 + 4I$, that is, the nilpotent matrix $A_1^2 + 4I$ has a square root, and this gives the Jordan block restrictions stated in Theorem 1.2(a). This proves necessity.

Conversely, suppose the Jordan blocks of A with eigenvalue $2i$ and size greater than 1, if any, come in pairs $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$. By Lemma 2.3(b) and Lemma 2.4(b), we may assume that

(1) $\sigma(A)$ does not contain $2i$ and $-2i$, or
(2) A is similar to $J_k(2i) \oplus J_k(2i) \oplus J_k(-2i) \oplus J_k(-2i)$ or $J_k(2i) \oplus J_{k+1}(2i) \oplus J_k(-2i) \oplus J_{k+1}(-2i)$ for $k > 1$, or
(3) A is similar to $2iI_n \oplus -2iI_n$.

If $\sigma(A) \cap \{2i, -2i\} = \emptyset$, then $A^2 + 4I$ is nonsingular, symmetric, and has a symmetric square root $B \in \mathbb{C}[A]$ [7, Theorem 6.4.12 (a)]. Since $B^2 = A^2 + 4I$, by Lemma 2.1(a), there exists an orthogonal X such that $A = X - X^T$.

We consider the second case. Let $k > 1$. Set a symmetric B similar to $J_{2k}(0)$ if A is similar to $J_k(2i) \oplus J_k(2i) \oplus J_k(-2i) \oplus J_k(-2i)$; and set a symmetric B similar to $J_{2k+1}(0)$ if A is similar to $J_k(2i) \oplus J_{k+1}(2i) \oplus J_k(-2i) \oplus J_{k+1}(-2i)$. Note that $B^2 - 4I$ is symmetric, nonsingular, and has a symmetric square root $S \in \mathbb{C}[B^2 - 4I]$ similar to $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$. Set $D := \begin{bmatrix} 0 & iS \\ -iS & 0 \end{bmatrix}$. Notice that D is skew-symmetric, $D^2 = S^2 \oplus S^2$, and D is orthogonally similar to A. We also have that $D^2 + 4I = (B \oplus B)^2$, and $B \oplus B$ is symmetric and commutes with D, since S is a polynomial in B. By Lemma 2.1(a), $D = Y - Y^T$

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for some orthogonal Y, and so, since A is orthogonally similar to D, we have that $A = X - X^T$ for some orthogonal X. This takes care of the case when the Jordan blocks of A corresponding to $2i$ come in pairs of $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$, for $k > 1$.

For the last case, let A be similar to $2iI_n \oplus -2iI_n$. Set $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Observe that $2J$ is skew-symmetric and orthogonally similar to A. Since J is orthogonal and $-J^T = J$, we have that $2J = J - J^T$. It follows that $A = X - X^T$ for some orthogonal X. □

Proof of Theorem 1.2(b). Let A be symmetric. Suppose that $A^2 - 4I$ has a skew-symmetric root B in $\mathcal{C}(A)$. Lemma 2.3(a) implies that there is an orthogonal matrix X such that

$$XAX^{-1} = A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k} \oplus A_2 \oplus A_{-2}, \tag{3.11}$$

where $\sigma(A_\mu) = \{\mu\}$ and $\lambda_1, \ldots, \lambda_k, 2, -2$ are the $k+2$ distinct eigenvalues of A. Since XBX^{-1} commutes with XAX^{-1},

$$XBX^{-1} = B_{\lambda_1} \oplus \cdots \oplus B_{\lambda_k} \oplus B_2 \oplus B_{-2}, \tag{3.12}$$

where each B_μ is skew-symmetric and $B_\mu^2 = A_\mu^2 - 4I$. If $\mu = \pm 2$, then B_μ^2 is nilpotent and the Jordan form of a nilpotent skew-symmetric B_μ given by Lemma 2.3(b) yields the Jordan block restrictions (ii), (iii), and (iv) for A_μ. If $\mu \neq \pm 2$, then B_μ is nonsingular. By Lemma 2.3(b)(i), $\sigma(B_\mu) = \{\sqrt{\mu^2 - 4}, -\sqrt{\mu^2 - 4}\}$ and the Jordan form of B_μ is a direct sum of matrices of the form $J_k(\sqrt{\mu^2 - 4}) \oplus J_k(-\sqrt{\mu^2 - 4})$. Hence, the Jordan form of A_μ is a direct sum of matrices of the form $J_k(\mu) \oplus J_k(\mu)$. This proves necessity. For the converse, we may assume, by Lemma 2.3(a) and Lemma 2.4(a), that A is similar to

(1) $J_k(\lambda) \oplus J_k(\lambda)$, where $\lambda \neq \pm 2, 0$,
(2) $J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda)$, for any $k > 1$ and $\lambda = \pm 2$,
(3) $J_{k+1}(\lambda) \oplus J_k(\lambda)$, where $\lambda = \pm 2$,
(4) λI_k, where $\lambda = \pm 2$, or
(5) $A_0 := A_1 \oplus A_1$, where A_1 is symmetric and similar to $J_k(0)$.

We show for each case that $A = X + X^T$ for some orthogonal X. For each of the cases (1) - (4), we respectively set a skew-symmetric B similar to

(1) $J_k(\sqrt{\lambda^2 - 4}) \oplus J_k(-\sqrt{\lambda^2 - 4})$
(2) $J_{2k}(0) \oplus J_{2k}(0)$
(3) $J_{2k+1}(0)$
(4) 0_k

Observe that $B^2 + 4I$ is nonsingular, symmetric, and has a symmetric square root $R \in \mathbb{C}[B]$ that is similar to A. By Lemma 2.1, $R = Y + Y^T$ for some orthogonal matrix Y, and since R is orthogonally similar to A, we have $A = X + X^T$ for some orthogonal X.

For the last case, we observe that $A_1^2 - 4I$ is nonsingular and symmetric, and thus has a symmetric square root $T \in \mathbb{C}[A_1]$. Set $B := \begin{bmatrix} 0 & iT \\ -iT & 0 \end{bmatrix}$. Note that B is skew-symmetric, commutes with A_0, and $B^2 = A_0^2 - 4I$. By Lemmas 2.1 and 2.4, we have that $A = X + X^T$ for some orthogonal X. □

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