SUMS OF ORTHOGONAL, SYMMETRIC, AND SKEW-SYMMETRIC MATRICES

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Abstract. An $n$-by-$n$ matrix $A$ is called symmetric, skew-symmetric, and orthogonal if $A^T = A$, $A^T = -A$, and $A^T = A^{-1}$, respectively. We give necessary and sufficient conditions on a complex matrix $A$ so that it is a sum of type “orthogonal + symmetric” in terms of the Jordan form of $A - A^T$. We also give necessary and sufficient conditions on a complex matrix $A$ so that it is a sum of type “orthogonal + skew-symmetric” in terms of the Jordan form of $A + A^T$.

Key words. Orthogonal, Symmetric, Skew-symmetric, Sums, Decompositions.

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1. Introduction. A matrix $A$ is called
- symmetric if $A^T = A$,
- skew-symmetric if $A^T = -A$,
- orthogonal if $A$ is nonsingular and $A^T = A^{-1}$.

Several mathematicians have studied matrix decompositions involving the above special matrices. Frobenius showed that every matrix over $F \in \{\mathbb{R}, \mathbb{C}\}$ is a product of two symmetric matrices [3] (see also the work of Radjavi [12]). Gow and Laffey gave necessary and sufficient conditions for a matrix over an arbitrary field to be a product of two skew-symmetric matrices [5]. Laffey later on proved that if $n \equiv 0 \mod 4$ and $A$ is an $n$-by-$n$ matrix over an algebraically closed field with characteristic not equal to 2, then $A$ is a product of five skew-symmetric matrices [10]. Horn and Merino showed that a complex matrix $A$ may be written as a product $A = QR$, where $Q$ is orthogonal and $R$ is symmetric if and only if $AA^T$ is similar to $A^T A$ [9]. De la Cruz et al. gave necessary and sufficient conditions for a complex matrix $A$ to be written as a product $A = QR$, where $Q$ is orthogonal and $R$ is skew-symmetric [1]. If $n > 1$, Merino showed that any matrix over $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is a sum of a finite number of orthogonal matrices [11]. Granario et al. gave necessary and sufficient conditions for a complex matrix to be written as a sum of two orthogonal matrices [4].

The main result of this paper is the following theorem which gives necessary and sufficient conditions for a complex matrix $A$ to be written as $A = Q + R$, where $Q$ is orthogonal and $R$ is either symmetric or skew-symmetric. For a complex number $\lambda$, we denote by $J_k(\lambda)$ the $k$-by-$k$ upper triangular Jordan block with eigenvalue $\lambda$.

**Theorem 1.1.** Let $A \in \mathbb{C}^{n \times n}$ be given. Then

(a) $A = A_1 + A_2$, for some orthogonal $A_1$ and symmetric $A_2$ if and only if the Jordan blocks of $A - A^T$ with eigenvalue $2i$ of size greater than one come in pairs of $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$.
(b) $A = B_1 + B_2$, for some orthogonal $B_1$ and skew-symmetric $B_2$, if and only if $A + A^T$ is similar to a direct sum of matrices of the form

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If \( A \) has a decomposition as in Theorem 1.1(a), then
\[
A - A^T = A_1 + A_2 - (A_1 + A_2)^T = A_1 + A_2 - A_1^T - A_2^T = A_1 - A_1^T.
\]

Conversely, if \( A - A^T = A_1 - A_1^T \) for some orthogonal \( A_1 \), then
\[
A - A_1 = A^T - A_1^T = (A - A_1)^T,
\]
is symmetric and \( A = A_1 + (A - A_1) \) is a decomposition of \( A \) as in Theorem 1.1(a). Analogous arguments show that \( A \) has a decomposition from Theorem 1.1(b) if and only if \( A + A^T = B_1 + B_1^T \) for some orthogonal \( B_1 \). The following theorem implies statements (a) and (b) of Theorem 1.1.

**Theorem 1.2.** Let \( A \in \mathbb{C}^{n \times n} \) be given.

(a) If \( A \) is skew-symmetric, then \( A = X - X^T \) for some orthogonal \( X \) if and only if the Jordan blocks of \( A \) with eigenvalue \( 2i \) and size greater than one, if any, come in pairs of \( J_k(2i) \oplus J_k(2i) \) or 
\[
J_k(2i) \oplus J_{k+1}(2i).
\]

(b) If \( A \) is symmetric, then \( A = X + X^T \) for some orthogonal \( X \) if and only if \( A \) is similar to a direct sum of matrices of the form
\[
\begin{align*}
(1) & \ J_k(\lambda) \oplus J_k(\lambda), \text{ where } \lambda \neq \pm 2, \\
(2) & \ J_k(\lambda) \oplus J_k(\lambda) \oplus J_2(\lambda), \text{ for any } k > 1 \text{ and } \lambda = \pm 2, \\
(3) & \ J_{k+1}(\lambda) \oplus J_k(\lambda), \text{ where } \lambda = \pm 2, \text{ or} \\
(4) & \ \lambda I_k, \text{ where } \lambda = 2, -2.
\end{align*}
\]

We give some preliminary observations in Section 2 and prove Theorem 1.2 in Section 3.

**2. Preliminaries.** The conditions for the decompositions in Theorem 1.2 can be stated in terms of the existence of symmetric or skew-symmetric square roots of a symmetric matrix. By \( \mathcal{C}(A) \), we mean the centralizer of the square matrix \( A \), that is,
\[
\mathcal{C}(A) := \{ X \in \mathbb{C}^{n \times n} \mid AX =XA \}.
\]

**Lemma 2.1.** Let \( A \in \mathbb{C}^{n \times n} \) be given.

(a) If \( A \) is skew-symmetric, then \( A = X - X^T \) for some orthogonal \( X \) if and only if \( A^2 + 4I \) has a symmetric square root in \( \mathcal{C}(A) \).

(b) If \( A \) is symmetric, then \( A = X + X^T \) for some orthogonal \( X \) if and only if \( A^2 - 4I \) has a skew-symmetric square root in \( \mathcal{C}(A) \).

**Proof.** Let \( A \) be skew-symmetric. Suppose \( A = X - X^T \) for some matrix \( X \). If \( X \) is orthogonal, then \( X^T = X^{-1} \), and so we consider orthogonal solutions to the matrix equation
\[
A = X - X^{-1}.
\]
If $X$ is a solution to (2.4), then $X \in \mathcal{C}(A)$. Now (2.4) is equivalent to
\begin{equation}
I = (X - A)X = X^2 - AX.
\end{equation}

By completing the squares in (2.5), we get 
\begin{equation}
(X - \frac{1}{2}A)^2 = \frac{1}{4}(A^2 + 4I).
\end{equation}

If, in addition, $X$ is orthogonal and we set $Y := X - \frac{1}{2}A$, then $Y \in \mathcal{C}(A)$, $Y^2 = \frac{1}{4}(A^2 + 4I)$, and
\begin{equation}
Y^T = X^T - \frac{1}{2}A^T = X^{-1} + \frac{1}{2}A = (X - A) + \frac{1}{2}A = Y.
\end{equation}

Thus, $Z := 2Y$ is a symmetric square root of $A^2 + 4I$ and $Z \in \mathcal{C}(A)$.

Conversely, suppose $Z \in \mathcal{C}(A)$ and $Z$ is a symmetric square root of $A^2 + 4I$. Set $Y := \frac{1}{2}Z$ and set $X := Y + \frac{1}{2}A$. Then $Y \in \mathcal{C}(A)$, $Y$ is symmetric,
\begin{equation}
XX^T = \left(Y + \frac{1}{2}A\right)\left(Y - \frac{1}{2}A\right) = Y^2 - \frac{1}{4}A^2 = \frac{1}{4}Z^2 - \frac{1}{4}A^2 = I,
\end{equation}
and
\begin{equation}
X - X^T = Y + \frac{1}{2}A - \left(Y - \frac{1}{2}A\right) = A.
\end{equation}

This proves (a). The proof of (b) is analogous.

Note that the existence of a decomposition in Theorem 1.2 is invariant under orthogonal similarity, and so the following theorem is useful.

**Lemma 2.2** ([8, Corollary 22]). Two complex matrices which are both symmetric, both skew-symmetric, or both orthogonal are similar if and only if they are orthogonally similar.

A matrix $A$ has a square root if and only if the nilpotent part of $A$, if any, is similar to a direct sum of matrices of the form $0_m$, $J_m(0) \oplus J_m(0)$, or $J_m(0) \oplus J_{m+1}(0)$ for any $m$ [7, Theorem 6.4.12]. To prove Theorem 1.2 it helps to know the Jordan structure of a symmetric or skew-symmetric matrix. Any square complex matrix is similar to a symmetric matrix and so there are no restrictions on the Jordan form of a symmetric matrix [6, Theorem 4.4.9]. For a skew-symmetric matrix $A$, the Jordan form of $A$ must be expressible as a direct sum of matrices of the form $J_k(\lambda) \oplus J_k(-\lambda)$ for $\lambda \neq 0$, $J_k(0) \oplus J_k(0)$ for even $k$, or $J_k(0)$ for odd $k$. Conversely, when $A$ is similar to the direct sum of any of the preceding Jordan blocks, then $A$ is similar to a skew-symmetric matrix [9]. Thus, by Lemma 2.2, we have the following.

**Lemma 2.3.** Let $A \in \mathbb{C}^{n \times n}$ be given. Then

(a) $A$ is symmetric if and only if $A$ is orthogonal similar to $\oplus_i A_i$, where each $A_i$ is a symmetric matrix that is similar to a Jordan block.

(b) $A$ is skew-symmetric if and only if $A$ is orthogonal similar to $\oplus_i A_i$, where each $A_i$ is skew-symmetric and similar to one of the following:

1. $J_k(\lambda) \oplus J_k(-\lambda)$ for any $\lambda \neq 0$,
2. $J_k(0) \oplus J_k(0)$ for any even $k$, and
3. $J_k(0)$ for any odd $k$.

The following result reduces our problem to symmetric or skew-symmetric matrices having at most two eigenvalues. Let $\sigma(A)$ denote the spectrum of a matrix $A$. 
LEMMA 2.4. Let $A = \bigoplus_{i=1}^{n} A_i$ for some square complex matrices $A_i$ with pairwise disjoint spectra. Then
(a) $A$ is symmetric such that $A^2 - 4I$ has a skew-symmetric square root that commutes with $A$ if and only if each $A_i$ is symmetric and $A_i^2 - 4I$ has a skew-symmetric square root that commutes with $A_i$.
(b) $A$ is skew-symmetric such that $A^2 + 4I$ has a symmetric square root that commutes with $A$ if and only if each $A_i$ is skew-symmetric and $A_i^2 + 4I$ has a symmetric square root.

Proof. We only do (a). Sufficiency follows from the fact that a direct sum of skew-symmetric matrices is skew-symmetric. For necessity, let $B$ be a skew-symmetric square root of $A^2 - 4I$ that commutes with $A$. Since $\sigma(A_i) \cap \sigma(A_j) = \emptyset$ for $i \neq j$, Sylvester’s theorem [6, Theorem 2.4.4.1] implies that $B = \bigoplus_{i=1}^{n} B_i$ and partitioned conformal to $A$. Hence each $B_i$ is a skew-symmetric square root of $A_i^2 - 4I$ that commutes with $A_i$.

3. Proof of Theorem 1.2. For $A \in \mathbb{C}^{n \times n}$, we let $\mathbb{C}[A] := \{ p(A) | p(x) \in \mathbb{C}[x] \}$ denote the set of all polynomials in $A$.

Proof of Theorem 1.2(a). Let $A$ be skew-symmetric. Suppose that $A = Y - Y^T$ for some orthogonal $Y$. By Lemma 2.1, $A^2 + 4I$ has a symmetric square root $B$ which commutes with $A$. Lemma 2.3(b) implies that there is a nonsingular matrix $X$ such that

\[ XAX^{-1} = A_1 \oplus -A_1 \oplus A_2, \]

where each $A_i$ is symmetric, $\sigma(A_1) = \{2i\}$, and $2i, -2i \notin \sigma(A_2)$. Since $XBX^{-1} \in \mathbb{C}(XAX^{-1})$, the eigenvalue conditions above and Sylvester’s theorem imply that

\[ XBX^{-1} = B_1 \oplus B_2 \oplus B_3, \]

which is partitioned conformal to $XAX^{-1}$. It follows that $B_1^2 = A_1^2 + 4I$, that is, the nilpotent matrix $A_1^2 + 4I$ has a square root, and this gives the Jordan block restrictions stated in Theorem 1.2(a). This proves necessity.

Conversely, suppose the Jordan blocks of $A$ with eigenvalue $2i$ and size greater than 1, if any, come in pairs $J_k(2i) \oplus J_{k+1}(2i)$ or $J_k(-2i) \oplus J_{k+1}(-2i)$. By Lemma 2.3(b) and Lemma 2.4(b), we may assume that

1. $\sigma(A)$ does not contain $2i$ and $-2i$.
2. $A$ is similar to $J_k(2i) \oplus J_{k+1}(2i)$ for $k > 1$, or $A$ is similar to $2iI_n \oplus -2iI_n$.

If $\sigma(A) \cap \{2i, -2i\} = \emptyset$, then $A^2 + 4I$ is nonsingular, symmetric, and has a symmetric square root $B \in \mathbb{C}[A]$ [7, Theorem 6.4.12 (a)]. Since $B^2 = A^2 + 4I$, by Lemma 2.1(a), there exists an orthogonal $X$ such that $A = X - X^T$.

We consider the second case. Let $k > 1$. Set a symmetric $B$ similar to $J_{2k}(0)$ if $A$ is similar to $J_{k}(2i) \oplus J_{k+1}(2i)$, and $A$ is similar to $J_{2k}(0)$ if $A$ is similar to $J_{k}(2i) \oplus J_{k+1}(2i)$ or $J_{k}(2i) \oplus J_{k+1}(2i)$. Note that $B^2 - 4I$ is symmetric, nonsingular, and has a symmetric square root $S \in \mathbb{C}[B^2 - 4I]$ similar to $J_{k}(2i) \oplus J_{k}(2i)$ or $J_{k}(2i) \oplus J_{k+1}(2i)$. Set $D := \begin{bmatrix} 0 & iS \\ -iS & 0 \end{bmatrix}$. Notice that $D$ is skew-symmetric, $D^2 = S^2 \oplus S^2$, and $D$ is orthogonally similar to $A$. We also have that $D^2 + 4I = (B \oplus B)^2$, and $B \oplus B$ is symmetric and commutes with $D$, since $S$ is a polynomial in $B$. By Lemma 2.1(a), $D = Y - Y^T$. \[ \]
for some orthogonal $Y$, and so, since $A$ is orthogonally similar to $D$, we have that $A = X - X^T$ for some orthogonal $X$. This takes care of the case when the Jordan blocks of $A$ corresponding to $2i$ come in pairs of $J_k(2i) \oplus J_k(2i)$ or $J_k(2i) \oplus J_{k+1}(2i)$, for $k > 1$.

For the last case, let $A$ be similar to $2iI_n \oplus -2iI_n$. Set $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Observe that $2J$ is skew-symmetric and orthogonally similar to $A$. Since $J$ is orthogonal and $-J^T = J$, we have that $2J = J - J^T$. It follows that $A = X - X^T$ for some orthogonal $X$.

**Proof of Theorem 1.2(b).** Let $A$ be symmetric. Suppose that $A^2 - 4I$ has a skew-symmetric root $B$ in $\mathbb{C}(A)$. Lemma 2.3(a) implies that there is an orthogonal matrix $T$ such that

\[(3.11) \quad XAX^{-1} = A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k} \oplus A_2 \oplus A_{-2},\]

where $\sigma(A_{\mu}) = \{\mu\}$ and $\lambda_1, \ldots, \lambda_k, 2, -2$ are the $k + 2$ distinct eigenvalues of $A$. Since $XBX^{-1}$ commutes with $XAX^{-1}$,

\[(3.12) \quad XBX^{-1} = B_{\lambda_1} \oplus \cdots \oplus B_{\lambda_k} \oplus B_2 \oplus B_{-2},\]

where each $B_{\mu}$ is skew-symmetric and $B_{\mu}^2 = A_{\mu}^2 - 4I$. If $\mu = \pm 2$, then $B_{\mu}^2$ is nilpotent and the Jordan form of a nilpotent skew-symmetric $B_{\mu}$ given by Lemma 2.3(b) yields the Jordan block restrictions (ii), (iii), and (iv) for $A_{\mu}$. If $\mu \neq \pm 2$, then $B_{\mu}$ is nonsingular. By Lemma 2.3(b)(i), $\sigma(B_{\mu}) = \{\sqrt{\mu^2 - 4}, -\sqrt{\mu^2 - 4}\}$ and the Jordan form of $B_{\mu}$ is a direct sum of matrices of the form $J_k(\sqrt{\mu^2 - 4}) \oplus J_k(-\sqrt{\mu^2 - 4})$. Hence, the Jordan form of $A_{\mu}$ is a direct sum of matrices of the form $J_k(\mu) \oplus J_k(\mu)$. This proves necessity. For the converse, we may assume, by Lemma 2.3(a) and Lemma 2.4(a), that $A$ is similar to

1. $J_k(\lambda) \oplus J_k(\lambda)$, where $\lambda \neq \pm 2, 0$,
2. $J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda) \oplus J_k(\lambda)$, for any $k > 1$ and $\lambda = \pm 2$,
3. $J_{k+1}(\lambda) \oplus J_k(\lambda)$, where $\lambda = \pm 2$,
4. $\lambda I_k$, where $\lambda = \pm 2$, or
5. $A_0 := A_1 \oplus A_1$, where $A_1$ is symmetric and similar to $J_k(0)$.

We show for each case that $A = X + X^T$ for some orthogonal $X$. For each of the cases (1) - (4), we respectively set a skew-symmetric $B$ similar to

1. $J_k(\sqrt{\mu^2 - 4}) \oplus J_k(-\sqrt{\mu^2 - 4})$
2. $J_{2k}(0) \oplus J_{2k}(0)$
3. $J_{2k+1}(0)$
4. $0_k$

Observe that $B^2 + 4I$ is nonsingular, symmetric, and has a symmetric square root $R \in \mathbb{C}[B]$ that is similar to $A$. By Lemma 2.1, $R = Y + Y^T$ for some orthogonal matrix $Y$, and since $R$ is orthogonally similar to $A$, we have $A = X + X^T$ for some orthogonal $X$.

For the last case, we observe that $A_1^2 - 4I$ is nonsingular and symmetric, and thus has a symmetric square root $T \in \mathbb{C}[A_1]$. Set $B := \begin{bmatrix} 0 & iT \\ -iT & 0 \end{bmatrix}$. Note that $B$ is skew-symmetric, commutes with $A_0$, and $B^2 = A_0^2 - 4I$. By Lemmas 2.1 and 2.4, we have that $A = X + X^T$ for some orthogonal $X$. \qed
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