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# SPECTRA OF EXPANSION GRAPHS*

SHMUEL FRIEDLAND† AND HANS SCHNEIDER‡

**Abstract.** Replace certain edges of a directed graph by chains and consider the effect on the spectrum of the graph. It is shown that the spectral radius decreases monotonically with the expansion and that, for a strongly connected graph that is not a single cycle, the spectral radius decreases strictly monotonically with the expansion. A limiting formula is given for the spectral radius of the expanded graph when the lengths of some chains replacing the original edges tend to infinity. The proofs depend on the construction of auxiliary nonnegative matrices of the same size and with the same support as the original adjacency matrix.

**Key words.** Graph expansion, graph spectrum.

**AMS subject classifications.** Primary 05C50, 15A18, 15A45; Secondary 05C20, 05C38.

**1. Introduction.** In [2] one of us considered the expansion graph of a (directed) graph, that is a (directed) graph obtained from a given graph by replacing certain edges by a chain. In this note we consider the effect of graph expansion on the spectrum (of the adjacency matrix) of the graph.

We show that the spectral radius decreases monotonically with the expansion and that, for a strongly connected graph that is not a single cycle, the spectral radius decreases strictly monotonically with the expansion. Moreover, if all edges are expanded to chains of the same length, there is a simple formula relating the spectral radius of the original and expanded graphs. The property that the spectral radius of the graph decreases with expansion may also be deduced from [1, Lemma 3], where an expansion of a different kind is considered for nonnegative matrices. The advantage of our approach lies in the construction of auxiliary nonnegative matrices of the same size and with the same support as the original $(0,1)$ adjacency matrix such that for each nonzero eigenvalue of the expanded graph there is an auxiliary matrix which has the same eigenvalue. We also give a limiting formula for the spectral radius of the expanded graph when the lengths of some chains replacing the original edges tend to infinity.

In this note we consider only directed graphs. Let , be a graph with the set of vertices $< n >:= \{1, ...., n\}$ and a set of edges $E \subset < n > \times < n >$. We denote the adjacency matrix of , by Adj(, ). As usual, the *spectrum* of a matrix is the set of its eigenvalues and the *spectral radius* of a matrix is the largest absolute value of an eigenvalue. The spectrum and spectral radius of a matrix $A \in \mathbb{C}^{nn}$ are denoted by spec($A$) and $\rho(A)$ respectively, and we write spec(Adj(, )) as spec(, ) and $\rho$(Adj(, )) as $\rho$(, ).

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†Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago Chicago, IL 60680, USA (friedlan@uic.edu).

‡Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA (hans@math.wisc.edu).

Let , be a graph. If , contains a cycle, $\mathrm{Adj}(,) \geq \mathrm{Adj}(\tilde{,})$ elementwise, where $\tilde{,}$ is the graph obtained from , by removing all arcs except those on the cycle, and hence, by a result of Perron-Frobenius theory which we use repeatedly, e.g., [3, Theorem 8.1.18], we have $\rho(,) \geq \rho(\tilde{,}) = 1$. If , doesn't contain a cycle, then $\mathrm{Adj}(,)$ is permutationally similar to a strictly triangular matrix and hence $\rho(,) = 0$. In this case, we call , an *acyclic* graph, otherwise the graph will be called *nonacyclic*. Expansion graphs of acyclic graphs are acyclic and our results are trivial for such graphs. Thus we confine our exposition to nonacyclic graphs , .

**2. Auxiliary Matrices.** Let $w$ be a function $w : E \to \mathbb{Z}_+$ of the edge set into the nonnegative integers. The *expansion graph* $,_w$ of , is obtained by replacing the edge $(i,j)$ by a chain from $i$ to $j$ with $w(i,j)+1$ edges by inserting $w(i,j)$ additional vertices. (If $w(i,j) = 0$ then the edge $(i,j)$ is not changed. In particular, $,_0 = ,$ .)

DEFINITION 2.1. Let , be a graph with the set of vertices $< n >$ and edge set $E \subset < n > \times < n >$. Let $w : E \to \mathbb{Z}_+$. Let $0 \neq t \in \mathbb{C}$. Then an *auxiliary matrix* $A_w(t) \in \mathbb{C}^{nn}$ is defined by

$$
\begin{aligned}
&a_{ij}(t) = t^{-w(i,j)}, \quad (i,j) \in E, \\
&a_{ij}(t) = 0, \quad (i,j) \in < n > \times < n > \setminus E.
\end{aligned} \tag{1}
$$

Note that for all functions $w$ of the type considered the matrix $A_w(1)$ is the adjacency matrix $\mathrm{Adj}(,)$. Further, for all $t \neq 0$, $A_0(t) = \mathrm{Adj}(,)$.

LEMMA 2.2. *Let , be a nonacyclic graph with set of vertices $< n >$ and edge set $E \subset < n > \times < n >$. Let $w : E \to \mathbb{Z}_+$. Then*

(i) *$\rho(A_w(t)/t)$ is a strictly monotonically decreasing function of $t$ in* $[1, \infty)$.

(ii) *There exists a unique $\tau \geq 1$ such that $\rho(A_w(\tau)/\tau) = 1$.*

*Proof.* (i) Let $1 \leq t' < t$. Since $,_w$ is nonacyclic and $A_w(t) \geq 0$ (elementwise), it follows that $0 \leq A_w(t) \leq A_w(t')$. By a well-known result (e.g., [3, Theorem 8.1.18]) it follows that $\rho(A_w(t)) \leq \rho(A_w(t'))$ and hence $\rho(A_w(t)/t) < \rho(A_w(t')/t')$ .

(ii) We note that $\rho(A_w(t)/t)$ is a continuous function of $t$ in $[1, \infty)$. Since $\rho(A_w(1)) = \rho(,) \geq 1$, and $\lim_{t\to\infty} A_w(t)/t = 0$, we deduce that $\lim_{t\to\infty} \rho(A_w(t)/t) = 0$. Thus (ii) now follows from (i). $\square$

**3. Spectra.** THEOREM 3.1. *Let , be a nonacyclic graph with set of vertices $< n >$ and edge set $E \subset < n > \times < n >$. Let $w : E \to \mathbb{Z}_+$. Let $0 \neq \tau \in \mathbb{C}$. Then $\tau$ is an eigenvalue of* $\mathrm{Adj}(,_w)$ *if and only if* $1$ *is an eigenvalue of* $A_w(\tau)/\tau$.

*Proof.* Let $V$ be the vertex set of $,_w$. Then the elements of $V$ will be indexed by the triples $(i,j,k), k = 0, \ldots, w(i,j)+1$, for $(i,j) \in E$. We identify all $(i,j,0) \in V$ with $i \in < n >$ and all $(i,j,w(i,j)+1) \in V$ with $j \in < n >$. Furthermore the chain from $i \in < n >$ to $j \in < n >$ in $,_w$ is given by

$$
(i,j,0) = i \to (i,j,1) \to \cdots \to (i,j,w(i,j)) \to (i,j,w(i,j)+1) = j, \quad (i,j) \in E.
$$

With each vertex $v \in V$ we associate a variable $x_v$. Thus for $i,j \in < n >$ the variable $x_i$ is identified with $x_{(i,j,0)}$ for all $(i,j) \in E$ and the variable $x_j$ is identified with $x_{(i,j,w(i,j)+1)}$ for all $(i,j) \in E$. Denote by $|V|$ the cardinality of $V$.

Let $B = \mathrm{Adj}(\,_w)$. Suppose that $\tau \in \mathrm{spec}(B), \tau \neq 0$ and let $\tau x = Bx,\ x \neq 0$, where $x := (x_v)_{v \in V} \in \mathbb{C}^{|V|}$. If $w(i,j) \geq 1$, the equation $(\tau x)_v = (Bx)_v, v \in V$, yields

$$\tau x_{(i,j,k)} = x_{(i,j,k+1)}, \quad k = 1, ..., w(i,j), \quad (i,j) \in E. \tag{2}$$

Hence

$$x_{(i,j,k)} = \tau^{-w(i,j)+k-1} x_j,\ k = 1, \ldots, w(i,j), \tag{3}$$

and, in particular,

$$x_{(i,j,1)} = \tau^{-w(i,j)} x_j,\ (i,j) \in E. \tag{4}$$

Let $z \in \mathbb{C}^n$ be given by

$$z_j = x_j, \quad j = 1, \ldots, n. \tag{5}$$

Since $x \neq 0$, we deduce from (3), that $z \neq 0$.

We observe that the system $(\tau x)_i = (Bx)_i, i \in < n >$ may be written as

$$\tau x_i = \sum_{j,(i,j) \in E} x_{(i,j,1)}, \quad i = 1, ..., n, \tag{6}$$

and we can combine (4) and (6) to obtain

$$\tau x_i = \sum_{j,(i,j) \in E} \tau^{-w(i,j)} x_j, \quad i = 1, ..., n. \tag{7}$$

We now compare (7) and (1), and we deduce that

$$z = (A_w(\tau)/\tau) z. \tag{8}$$

Hence $1 \in \mathrm{spec}(A_w(\tau)/\tau)$.

Conversely, assume that $z \neq 0$ and that $z$ satisfies (8). Define $x \in \mathbb{C}^n$ by (5), and extend $x$ to be conformal with $B$ by (3). Then both (2) and (4) hold. We rewrite (8) in the equivalent form (7), and we use (4) to obtain (6). But (6) and (2) together imply that $Bx = \tau x$. $\square$

COROLLARY 3.2. *Let , be a nonacyclic graph with set of vertices $< n >$ and edge set $E \subset < n > \times < n >$. Let $w : E \to \mathbb{Z}_+$ satisfy $w(i,j) = m$, for all $(i,j) \in E$. Let $0 \neq \tau \in \mathbb{C}$. Then $\tau \in \mathrm{spec}(\,_w)$ if and only if $\tau^{m+1} \in \mathrm{spec}(\,)$. In particular,*

$$\rho(\,_w) = \rho(\,)^{\frac{1}{m+1}}.$$

*Proof.* Note that $A_w(\tau) = \tau^{-m} A_w(1) = \tau^{-m} \mathrm{Adj}(\,)$. If $\tau \in \mathrm{spec}(A_w(\tau))$ then $\tau^{m+1} \in \mathrm{spec}(\,)$, and conversely. Hence Theorem 3.1 implies the first part of the corollary and the second part follows immediately. $\square$

REMARK 3.3. *It is possible that* $0 \in \mathrm{spec}(,{}_w)$ *and* $0 \notin \mathrm{spec}(,)$. (See Example 3.8.)

THEOREM 3.4. *Let* , *be a nonacyclic graph with set of vertices* $< n >$ *and edge set* $E \subset < n > \times < n >$. *Let* $w : E \to \mathbb{Z}_+$. *Let* $\tau$ *be the unique solution in* $[1, \infty)$ *of* $\rho(A_w(\tau)/\tau) = 1$. *Then*

$$\rho(,{}_w) = \rho(A_w(\tau)).$$

*Proof.* By Perron-Frobenius $\rho(A_w(\tau)) \in \mathrm{spec}(A_w(\tau))$, and hence, by Theorem 3.1, $\tau \in \mathrm{spec}(,{}_w)$. It follows that $\tau' := \rho(,{}_w) \geq \tau$. By Lemma 2.2 we have $\rho(A_w(\tau')/\tau') \leq 1$. But, $\tau' \in \mathrm{spec}(,{}_w)$ also yields $\tau' \in \mathrm{spec}(A_w(\tau'))$ by Theorem 3.1. Using Perron-Frobenius again, we deduce that $\rho(A_w(\tau')/\tau') \geq 1$. It follows that $\rho(A_w(\tau')/\tau') = 1$, and we obtain $\tau' = \tau$ from Lemma 2.2. $\square$

A graph , is *strongly connected* if there is a path in , from every vertex to every other vertex. Suppose that , is strongly connected. Then $\mathrm{Adj}(,)$ is irreducible. If , consists of a single loopless vertex then clearly $\mathrm{Adj}(,) = [0]$ and $\rho(,) = 0$; otherwise , is nonacyclic and $\rho(,) \geq 1$. Further, $\rho(,) = 1$ if and only if , consists of a single cycle, since for nonnegative matrices $A, B$ with $A$ irreducible, $A \geq B$ and $\rho(A) = \rho(B)$ imply that $A = B$, see [3, Theorem 8.4.5].

THEOREM 3.5. *Let* , *be a nonacyclic graph with vertex set* $< n >$ *and edge set* $E \subset < n > \times < n >$ *respectively. Let* $w : E \to \mathbb{Z}_+$. *Then*

$$1 \leq \rho(,{}_w) \leq \rho(,). \tag{9}$$

*Suppose also that* , *is strongly connected. Then* $\rho(,{}_w) = \rho(,)$ *if and only if either* $w = 0$ *or* , *is a cycle. Further,* $1 = \rho(,{}_w)$ *if and only if* , *is a cycle.*

*Proof.* Let $t \in [1, \infty)$. Since $A_w(t)/t \leq A_0(t)/t$, we also have $\rho(A_w(t)/t) \leq \rho(A_0(t)/t)$ and (9) follows from Theorem 3.4 and Lemma 2.2.

Now let , also be strongly connected. If $w = 0$, obviously $\rho(,{}_w) = \rho(,)$. If , is a cycle, then $\rho(,) = 1$, and it follows from (9) that also $\rho(,{}_w) = 1$.

Conversely, suppose that $w \neq 0$ and that , is not a cycle. Then $\rho(A_w(1)) = \rho(,) > 1$. Note that $\mathrm{Adj}(,)$ is irreducible and hence so is $A_w(t)$ for $t \geq 1$. Thus for $t > 1$ we have $\rho(A_w(t)/t) < \rho(A_0(t)/t)$ by [3, Theorem 8.4.5], since $A_w(t)/t \leq A_0(t)/t$ but $A_w(t)/t \neq A_0(t)/t$. It follows from Theorem 3.4 that $1 < \rho(,{}_w) < \rho(,)$.

The last part of the theorem follows from the remarks just preceding it and the fact that ${,}_w$ is a cycle if and only if , is a cycle. $\square$

The inequality (9) in Theorem 3.5 may be also derived from [1, Lemma 3, part (b)].

Let $E \subset < n > \times < n >$ and let $w, w' : E \to \mathbb{Z}_+$ be nonnegative integer valued functions on $E$. Then we write $w \geq w'$ if $w(i,j) \geq w'(i,j)$, for all $(i,j) \in E$. We have the following corollary to Theorem 3.5.

COROLLARY 3.6. *Let* , *be a nonacyclic graph with vertex set* $< n >$ *and edge set* $E \subset < n > \times < n >$ *respectively. Let* $w, w' : E \to \mathbb{Z}_+$. *Assume that* $w' \geq w$. *Then*

$$\rho(,{}_w) \geq \rho(,{}_{w'}).$$

*Suppose now that* , *is also strongly connected. If* , *is a cycle then* $\rho(,_w) = \rho(,_{w'})$*. If* , *is not a cycle and* $w \neq w'$ *then* $\rho(,_w) > \rho(,_{w'}) > 1$*.*

*Proof.* The graph $,_{w'}$ may be obtained from $,_w$ by graph expansion. The corollary now follows from Theorem 3.5. □

COROLLARY 3.7. *Let* , *be a nonacyclic graph with vertex set* $< n >$ *and edge set* $E \subset < n > \times < n >$ *respectively. Let* $w : E \to \mathbb{Z}_+$ *and suppose that* $m' \leq w(i,j) \leq m$ *for all* $(i,j) \in E$*. Then*

$$\rho(,)^{\frac{1}{m+1}} \leq \rho(,_w) \leq \rho(,)^{\frac{1}{m'+1}}.$$

*Proof.* Immediate by Corollary 3.2 and Corollary 3.6. □

EXAMPLE 3.8. Let , be the graph with vertex set $\{1,2\}$ and edge set $E = \{(1,1),(1,2),(2,1)\}$. Let $w : E \to \mathbb{Z}_+$ be given by $w(1,1) = 1, w(1,2) = 0, w(2,1) = 0$, that is, we expand the arc $(1,1)$ to a chain of length 2 and we leave the other arcs unchanged. Then

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$A_w(t) = \begin{bmatrix} t^{-1} & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\sqrt{2}$ is an eigenvalue of $A_w(\sqrt{2})$, and $-\sqrt{2}$ is an eigenvalue of $A_w(-\sqrt{2})$. But these are precisely the nonzero eigenvalues of

$$\mathrm{Adj}(,_w) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{10}$$

in conformity with Theorem 3.1. Note also that $\rho(,_w) = \sqrt{2} < (1+\sqrt{5})/2 = \rho(,)$, as required by Theorem 3.5 since , is strongly connected.

**4. Limiting cases.** LEMMA 4.1. *Let* , *be a nonacyclic graph with the set of vertices* $< n >$ *and set of edges* $E$*. Let* $F' \subset E$ *and let* $,'$ *be the graph with the set of vertices* $< n >$ *and set of edges* $F'$*. Let* $w$ *be a mapping* $w : E \to \mathbb{Z}_+$ *such that* $w(i,j) = 0,\ (i,j) \in F'$*. Then*

$$\max(1, \rho(,')) \ \leq\ \rho(,_w) \ \leq\ \rho(,).$$

*Proof.* We have $1 \leq \rho(,_w)$ by Theorem 3.5. Since $\mathrm{Adj}(,') \leq A_w(t),\ t \geq 1$, we obtain $\rho(,') \leq \rho(A_w(t))$ by [3, Theorem 8.1.18], and $\rho(,') \leq \rho(,_w)$ follows by Theorem 3.4. □

DEFINITION 4.2. Let , be a graph on the set of vertices $< n >$ and the set of edges $E \subset < n > \times < n >$. Let $w_m,\ m = 1, 2, \ldots$ be an infinite sequence of mappings $w_m : E \to \mathbb{Z}_+$. Let $F$ be a subset of $E$ and let $F' = E \backslash F$.
(i) Let $\tilde{w}$ be a mapping: $E \to \mathbb{Z}_+$. We say that the sequence $w_m,\ m = 1, 2 \ldots,$ *coincides with* $\tilde{w}$ on $F'$ if

$$\omega_m(i,j) = \tilde{w}(i,j) \text{ for all } \quad (i,j) \in F' \text{ and } \quad m = 1, 2, \ldots \ .$$

(ii) We say that the sequence $w_m,\ m = 1, 2 \ldots,$ *tends to infinity on* $F$ if

$$\lim_{m\to\infty} w_m(i,j) = \infty \text{for all } \ (i,j) \in F.$$

In this terminology, the mapping $w$ considered in Lemma 4.1 coincides with 0 on $F'$.

THEOREM 4.3. *Let , be a nonacyclic graph on the set of vertices $< n >$ and the set of edges $E \subset < n > \times < n >$. Let $F$ be a subset of $E$ and let $F' = E \backslash F$. Let $w_m,\ m = 1, 2, \ldots$ be an infinite sequence of mappings $w_m : E \to \mathbb{Z}_+$ which coincides with 0 on $F'$ and tends to infinity on $F$. Let $,'$ be the graph with vertex set $< n >$ and edge set $F'$.*
(i) *If $,'$ is acyclic then*

$$\lim_{m\to\infty} \rho(,_{\,w_m}) = 1. \tag{11}$$

(ii) *If $,'$ is nonacyclic then*

$$\lim_{m\to\infty} \rho(,_{\,w_m}) = \rho(,'). \tag{12}$$

*Proof.* In view of Lemma 4.1, we have

$$1 \le \limsup_{m\to\infty} \rho(,_{\,w_m}) \le \rho(,).$$

If

$$\limsup_{m\to\infty} \rho(,_{\,w_m}) = 1,$$

then it is immediate that equation (11) holds. Since a graph is either acyclic or nonacyclic, our result will follow if we prove the following claim:

Claim: If

$$\tau \ := \ \limsup_{m\to\infty} \rho(,_{\,w_m}) \ > \ 1, \tag{13}$$

then equation (12) holds and $,'$ is nonacyclic.

Thus we now assume that inequality (13) holds. Let $\tau_m = \rho(,_{\,w_m}),\ m = 1, 2, \ldots$. There exists an infinite increasing sequence of integers $m(k),\ k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} \tau_{m(k)} \ = \ \tau. \tag{14}$$

Let $(i,j) \in F$. Then $\lim_{k\to\infty} w_{m(k)}(i,j) = \infty$ and, since $\tau_{m(k)} \geq (1+\tau)/2 > 1$ for sufficiently large $k$, it follows that

$$\lim_{k\to\infty} \tau_{m(k)}^{-w_{m(k)}(i,j)} = 0.$$

Thus

$$\lim_{k\to\infty} A_{w_{m(k)}}(\tau_{m(k)}) = \mathrm{Adj}(,\ ').$$

By (14) and Theorem 3.4 we now have

$$\tau = \lim_{k\to\infty} \tau_{m(k)} = \rho(,\ '). \tag{15}$$

But, for $t \geq 1$, $A_{w_m}(t) \geq \mathrm{Adj}(,\ ')$ and hence $\tau_m = \rho(A_{w_m}(\tau_{w_m})) \geq \rho(\mathrm{Adj}(,\ '))$. Hence also

$$\liminf_{m\to\infty} \tau_m \geq \rho(,\ '). \tag{16}$$

We now combine (13), (15), and (16) to obtain (12). By (12) and the assumption that $\tau > 1$ we have $\rho(,\ ') > 1$. Hence $,\ '$ is nonacyclic. □

Applying Theorem 4.3 to an expanded graph we immediately obtain:

Corollary 4.4. *Let , be a nonacyclic graph on the set of vertices $< n >$ and the set of edges $E \subset < n > \times < n >$. Let $F$ be a subset of $E$ and let $F' = E \backslash F$. Let $w_m,\ m = 1, 2, \ldots$ be an infinite sequence of mappings $w_m : E \to \mathbb{Z}_+$ which coincides with a mapping $w : E \to \mathbb{Z}_+$ on $F'$ and tends to infinity on $F$. Let $,\ '$ be the graph with vertex set $< n >$ and edge set $F'$.* (i) *If $,\ '$ is acyclic then*

$$\lim_{m\to\infty} \rho(,\ _{w_m}) = 1.$$

(ii) *If $,\ '$ is nonacyclic then*

$$\lim_{m\to\infty} \rho(,\ _{w_m}) = \rho(,\ '_w).$$

Example 4.5. We use the notation of Corollary 4.4. We let , be the complete graph on 2 vertices. Thus $A := \mathrm{Adj}(,\ )$ is given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We write the mapping $w_m$ in the form of a matrix $W_m$ whose $(i,j)$-th entry is $w_m(i,j),\ i,j = 1,2$. Let

$$W_m = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}.$$

Thus we leave unchanged the arcs $(1,2)$ and $(2,1)$, replace the arc $(1,1)$ by a chain of length 2 and replace the arc $(2,2)$ by a sequence of chains of length $m+1,\ m=1,2,\ldots$. The matrix $A_{w_m}(t)$ is given by

$$A_{w_m}(t)=\begin{bmatrix} t^{-1} & 1 \\ 1 & t^{-m}\end{bmatrix}.$$

The characteristic polynomial of $A_{w_m}(t)/t$ is given by

$$\lambda^2-(t^{-2}+t^{-(m+1)})\lambda-t^{-2}+t^{-(m+3)} \tag{17}$$

and we shall now prove that

$$\rho(A_{w_m}(\tau_m)/\tau_m)=1 \tag{18}$$

for a unique $\tau_m$ in $[1,\infty)$ and that

$$\lim_{m\to\infty}\tau_m=\sqrt{2}. \tag{19}$$

It follows from (17) and (18) that $\tau_m$ is the largest positive solution of

$$f_m(t):=2t^{-2}+t^{-(m+1)}-t^{-(m+3)}=1. \tag{20}$$

For $t\geq 1$,

$$\begin{aligned} f_m'(t)=-4t^{-3}-(m+1)t^{-(m+2)}+(m+3)t^{-(m+4)} &< \\ -2t^{-3}-(m+1)t^{-(m+2)}+(m+3)t^{-(m+4)} &\leq \\ -2t^{-(m+2)}-(m+1)t^{-(m+2)}+(m+3)t^{-(m+4)} &= \\ -(m+3)t^{-(m+2)}(1-t^{-2}) &\leq 0. \end{aligned}$$

Since $f_m(\sqrt{2})>1$ while $f_m(t)<1$ for large positive $t$ it follows that $\tau_m$ is the unique solution of (20) in $(\sqrt{2},\infty)$. Clearly $f_{m+1}(t)<f_m(t)$ in $(1,\infty)$, and hence $\{\tau_m\}$ is a decreasing sequence in $(\sqrt{2},\infty)$. It follows that $\tau:=\lim_{m\to\infty}\tau_m$ exists. But by (20), and since $\tau_m>\sqrt{2}$ for all $m$ we have

$$1=\lim_{m\to\infty}f_m(\tau_m)=2\tau^{-2}$$

and (19) now follows.

Note that $\Gamma'_w$ is the graph considered in Example 3.8 and that $\sqrt{2}$ is also the spectral radius of $\mathrm{Adj}(\Gamma'_w)$, the matrix displayed in (10), as required by Corollary 4.4(ii).

Example 4.6. We choose $\Gamma$ as in Example 4.5 and define the expansion mapping $w_m$ by

$$W_m=\begin{bmatrix} m & 0 \\ 2m & m\end{bmatrix},$$

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that is we leave the arc $(1,2)$ unchanged and replace the arcs $(1,1)$, $(2,2)$ and $(2,1)$ by chains of length $m+1$, $m+1$ and $2m+1$ respectively. This time we have

$$A_{w_m}(t) = \begin{bmatrix} t^{-m} & 1 \\ t^{-2m} & t^{-m} \end{bmatrix}.$$

The characteristic polynomial of $A_{w_m}(t)/t$ is $\lambda^2 - 2t^{-(m+1)}\lambda$. Hence $\rho(A_{w_m}(\tau_m)/\tau_m)$ $= 1$ if and only if $\tau_m = 2^{1/(m+1)}$ and it follows that

$$\lim_{m\to\infty} \tau_m = 1,$$

as required by Corollary 4.4(i). But note that

$$\mathrm{Adj}(, ') = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and that therefore $\rho(, ') = 0$.

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