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A NOTE ON BOUNDS FOR EIGENVALUES OF NONSINGULAR H-TENSORS*

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Abstract. A counterexample to a theorem in the paper ELA 29:3-16, (2015) is provided, and an upper bound on the H-spectral radius of H-tensors is given.

Key words. M-tensor; H-tensor; Z-spectral radius; Minimum H-eigenvalue.

AMS subject classifications. 15A18, 15A69, 65F15, 65F10

1. Introduction. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an mth order n-dimensional real square tensor, and let $x = (x_i)$. Then, we define the following real n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m} \right)_{1\le i\le n}, \ x^{[m-1]} = (x_i^{m-1})_{1\le i\le n}.$$

If there exists a real nonzero vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an H-eigenvalue of $\mathcal{A}$ and x is called an eigenvector of $\mathcal{A}$ associated with λ [1]. If there exists a real nonzero vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x, \quad x^T x = 1,$$

then λ is called a Z-eigenvalue of $\mathcal{A}$ and x is called an eigenvector of $\mathcal{A}$ associated with λ. Let $\sigma_Z(\mathcal{A})$ be the set of all Z-eigenvalues of $\mathcal{A}$ and $\sigma_H(\mathcal{A})$ be the set of all H-eigenvalues of $\mathcal{A}$. Assume that $\sigma_Z(\mathcal{A}) \neq \emptyset$, $\sigma_H(\mathcal{A}) \neq \emptyset$. Then, $\rho_Z(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma_Z(\mathcal{A})\}$ is called the Z-spectral radius of $\mathcal{A}$, $\rho_H(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma_H(\mathcal{A})\}$ the H-spectral radius of $\mathcal{A}$.

Recently, in [2], some bounds for the Z-spectral radius have been presented for the case when $\mathcal{A}$ is a nonsingular H-tensor. One of these upper bounds is given in the next theorem.

THEOREM 1.1 ([2, Theorem 3.3]). *Let $\mathcal{A}$ be an mth order and n-dimensional nonsingular H-tensor with $\sigma_Z(\mathcal{A}) \neq \emptyset$. Then,*

$$\rho_Z(\mathcal{A}) \le 2 \max_{1\le i\le n} |a_{ii\ldots i}|.$$

The following example is also given in [2] to illustrate Theorem 1.1.

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Example 1.2. Let $\mathcal{A} = (a_{ijk})$ be an third-order 2-dimension tensor with the form,

$$a_{111} = 1.1,\ a_{112} = -1,\ a_{121} = -1,\ a_{122} = 1,$$

$$a_{211} = -1,\ a_{221} = 1,\ a_{212} = 1,\ a_{222} = 1.1.$$

And from Theorem 1.1, it is further claimed in [2] that:

$$\rho_Z(\mathcal{A}) \leq 2 \max_{1\leq i\leq n} |a_{ii\dots i}| = 2.2.$$

But in fact, by the command "zeig" in the TenEig MATLAB toolbox[5], we obtain, $\rho_Z(\mathcal{A}) = 2.3667$ in this example. That is to say, the bound in Theorem 1.1 is not true in general.

2. Upper bound for the H-spectral radius of H-tensors. Let $\mathcal{A}$ and $\mathcal{B}$ be two mth order and n-dimensional tensors. Suppose that there exists an invertible diagonal matrix D such that $\mathcal{B} = D^{-(m-1)}\mathcal{A}D$, then $\mathcal{A}$ and $\mathcal{B}$ are similar. In [2], it is claimed that since $\mathcal{A}$ and $\mathcal{B}$ are similar, they have the same Z-spectrum. The similarity relation of tensors is also studied in [3, 4], and it is proved that $\mathcal{A}$ and $\mathcal{B}$ share the same H-spectrum if they are similar. It should be noted that $\mathcal{A}$ and $\mathcal{B}$ may have different Z-spectra even if they are similar, which can be seen in the following example.

Example 2.1. Let $\mathcal{A} = (a_{ijk})$ and $\mathcal{B} = (b_{ijk})$ be third-order 2-dimension tensors with the form,

$$a_{111} = 1, a_{222} = 1, a_{122} = 1, a_{211} = 1,$$

$$b_{111} = 1, b_{222} = 1, b_{122} = 4, b_{211} = 0.25.$$

If $D = diag(1, 2)$ is an invertible diagonal matrix, then we have $\mathcal{B} = D^{-2}\mathcal{A}D$. Thus, $\mathcal{A}$ and $\mathcal{B}$ are similar. But in fact, by the commands "zeig" and "heig" in the TenEig MATLAB toolbox[5], we obtain $\sigma_Z(\mathcal{A}) = \{-1.4142,\ 1.4142\}$, $\sigma_Z(\mathcal{B}) = \{-1.2127,\ 1.2127\}$, $\sigma_H(\mathcal{A}) = \sigma_H(\mathcal{B}) = \{2\}$.

In the proof of Theorem 1.1([2, Theorem 3.3]), let X be an invertible diagonal matrix, then, the authors claim, $\rho_Z(\mathcal{A}) = \rho_Z(X^{-(m-1)}\mathcal{A}X)$, from the analysis above, we find that the result $\rho_Z(\mathcal{A}) = \rho_Z(X^{-(m-1)}\mathcal{A}X)$ does not hold in general. Therefore, the result of Theorem 1.1([2, Theorem 3.3]) is not true in general. Since we have $\rho_H(\mathcal{A}) = \rho_H(X^{-(m-1)}\mathcal{A}X)$ by similar arguments, as in the proof of Theorem 1.1, we can get the following upper bounds for the H-spectral radius of H-tensors.

THEOREM 2.2. *Let $\mathcal{A}$ be an mth order and n-dimensional nonsingular H-tensor with $\sigma_H(\mathcal{A}) \neq \emptyset$. Then,*

$$\rho_H(\mathcal{A}) \leq 2 \max_{1\leq i\leq n} |a_{ii\dots i}|.$$

COROLLARY 2.3. *Let $\mathcal{A}$ be an mth order and n-dimensional nonsingular H-tensor with $\sigma_H(\mathcal{A}) \neq \emptyset$. Then,*

$$\rho_H(\mathcal{A}) \leq 2 \min\left\{R_i(\mathcal{A}), \max_{1\leq i\leq n} |a_{ii\dots i}|\right\}.$$

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