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GROUP INVERSES OF MATRICES OF DIRECTED TREES*

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Abstract. A new class of directed trees is introduced. A formula for the group inverse of the matrices associated with any tree belonging to this class is obtained. This answers affirmatively, a conjecture of Catral et al., for this new class.

Key words. Group inverse, Moore–Penrose inverse, Directed tree, Maximum matching.

AMS subject classifications. 05C22, 05C50, 15A09.

1. Introduction. Let us start by recalling the definition of the group inverse of a matrix, the object of primary importance in this article. For a real $n \times n$ matrix A, the *group inverse*, if it exists, is the unique matrix X that satisfies the equations $AXA = A, XAX = X$, and $AX = XA$. Such an X is denoted by $A^{\#}$. Any matrix X that satisfies the first equation is called an *inner inverse* of A, while any matrix X that satisfies the second equation will be called an *outer inverse*. For a symmetric matrix A, always $A^{\#}$ exists and it is easy to show (and as was observed in [13]) that, if a symmetric matrix X satisfies the equations $AXA = A$ and $AX = XA$, then $X = A^{\#}$. Let us recall that for a real rectangular matrix A, the *Moore–Penrose inverse* of A is the unique matrix $A^{\dagger}$ that satisfies the equations $AA^{\dagger}A = A,\ A^{\dagger}AA^{\dagger} = A^{\dagger},\ (AA^{\dagger})^T = AA^{\dagger}$, and $(A^{\dagger}A)^T = A^{\dagger}A$. We refer the reader to [4] for more details on these notions of generalized inverses and Moore–Penrose inverses.

Let us recall some notation from [8]. Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries. The digraph $D(A) = (V, E)$ corresponding to A is the directed graph whose vertex set is $V = \{1, 2, \dots, n\}$ and whose edge set E is described by the requirement that, $(i, j) \in E$ iff $a_{ij} \neq 0$. For $m \geq 1$, a sequence $(i_1, i_2, \dots, i_m, i_{m+1})$ of distinct vertices with edges $(i_1, i_2), (i_2, i_3), \dots, (i_m, i_{m+1})$ in E is called a path of length m from i_1 to i_{m+1} in $D(A)$. For $m \geq 2$, a sequence $(i_1, i_2, \dots, i_m, i_1)$ with distinct vertices $i_1, i_2, \dots, i_m$, where $(i_1, i_2), (i_2, i_3), \dots, (i_m, i_1) \in E$, is called an *m-cycle* (a cycle of length m) in $D(A)$. Digraph $D(A)$ corresponding to a matrix A is called a *tree graph* if it is strongly connected, and all of its cycles have length 2. For r even, a set of $\frac{r}{2}$ disjoint 2-cycles in $D(A)$ given by $\{(i_1, i_2, i_1), (i_3, i_4, i_3), \dots, (i_{r-1}, i_r, i_{r-1})\}$ is called a *matching* of size r, and the product $a_{i_1,i_2}a_{i_2,i_1}a_{i_3,i_4}a_{i_4,i_3} \dots a_{i_{r-1},i_r}a_{i_r,i_{r-1}}$ is called a *matching product*. If this set of 2-cycles has a maximum cardinality, then the *matching* is referred to as a *maximum matching* and the matching product is then called a *maximum matching product*. The sum of all maximum matching products in $D(A)$ is denoted by Δ_A. Let us now recall a characterization for the existence of the group inverse.

THEOREM 1.1 ([8, Proposition 1.1]). *Let A be an $n \times n$ matrix with a tree graph $D(A)$. Then, the group inverse $A^{\#}$ exists if and only if $\Delta_A \neq 0$.*

This result is interesting from the perspective of determining if the group inverse of a matrix exists, purely based on the structure of the digraph $D(A)$. Further, a formula for the entries of the group inverse

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of a matrix A with a path graph $D(A)$ was derived, in terms of path length, sum of all maximal matchings, and the number Δ_A [8, Theorem 3.7].

Reverting back to the general discussion, a matching is said to be a *perfect matching* if it covers all the vertices of $D(A)$. For a path $(i_1, i_2, \ldots, i_m, i_{m+1})$, the product $a_{i_1,i_2}a_{i_2,i_3}a_{i_3,i_4}\ldots a_{i_{m-1},i_m}a_{i_m,i_{m+1}}$ is said to be the *path product,* denoted by $P(i_1, i_{m+1})$. For a cycle (i, j, i) in $D(A)$, the product $a_{ij}a_{ji}$ is called the *cycle product.* A sequence of m 2-cycles $((i_1, i_2, i_1), (i_2, i_3, i_2), \ldots, (i_m, i_{m+1}, i_m))$ with $m+1$ distinct vertices $i_1, i_2, \ldots, i_{m+1}$ in $D(A)$ is called a *cycle chain* from i_1 to i_{m+1} of length m and denoted by $C_m(i_1, i_{m+1})$. Suppose, $D(A)$ is a tree graph. For any two vertices i and j in $D(A)$, there is a unique cycle chain $C_l(i, j)$ for some nonnegative integer l. A cycle chain $C_l(i, j)$ is said to be an alternating cycle chain with respect to a maximum matching M if cycles of $C_l(i, j)$ alternatively belong to M and M^c, with the condition that both the first and the last cycles of $C_l(i, j)$ belong to M.

A cycle (i, j, i) is said to be incident to i as well as j in $D(A)$. A vertex i is called a pendant vertex if it is incident to only one 2-cycle and non-pendant vertex if it is incident to more than one 2-cycle in $D(A)$. A cycle (i, j, i) will be called a pendant cycle if at least one vertex i or j is pendant in $D(A)$, while a cycle which is not pendant will be called a non-pendant cycle. A pair of vertices i, j is said to be adjacent to each other if there is a cycle (i, j, i) in $D(A)$.

Before we define a new class of directed trees, we recall some more terminology for a tree graph $D(A)$. For arbitrary vertices, i and j in $D(A)$ denote $\mathbb{M}(i, j)$ to be the set of all maximum matchings M in $D(A)$ such that $C_m(i, j)$ is an alternating cycle chain with respect to M. Clearly, $\mathbb{M}(i, j) = \mathbb{M}(j, i)$. A necessary condition for the set $\mathbb{M}(i, j)$ to be non-empty is that the length of the path from i to j be odd. If (i, j, i) is a 2-cycle of some maximum matching, then $\mathbb{M}(i, j)$ is non-empty. Two distinct vertices i and j will be called *maximally matchable* if $\mathbb{M}(i, j) \neq \phi$.

Further, for any maximally matchable vertices i, j and a maximum matching $M \in \mathbb{M}(i, j)$, let $\beta_{\overline{i,j}}(M)$ denote the product of all cycle product, ranging over all the cycles of M that are not contained in the unique cycle chain $C_m(i, j)$ in $D(A)$ (product over an empty set is considered to be equal to 1). Since $\mathbb{M}(i, j) = \mathbb{M}(j, i)$, note that $\beta_{\overline{i,j}}(M) = \beta_{\overline{j,i}}(M)$. For a maximum matching M in $D(A)$, $\eta(M)$ denote the maximum matching product. Set

$$\beta_{ij} = \begin{cases} (-1)^{\frac{m-1}{2}} P(i, j) & \text{if } i, j \text{ are maximally matchable,} \\ 0 & \text{if } i, j \text{ are not maximally matchable.} \end{cases}$$

and

$$\mu_{ij} = \beta_{ij} \cdot \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M).$$

It follows $\mu_{ij} = 0$ if i, j are not maximally matchable. This includes the case $i = j$.

Let us now introduce a new class of graphs.

Definition 1.2. *Let $\mathbb{D}$ denote the set of all directed trees D such that each non-pendant vertex of D is adjacent to at least one pendant vertex of D.*

Example 1.3. *It is clear that the tree digraph $D_1 \in \mathbb{D}$, (Fig. 1), while $D_2 \notin \mathbb{D}$ (Fig. 2). The non-pendant vertex 3 (in D_2) is not adjacent to any pendant vertex.*

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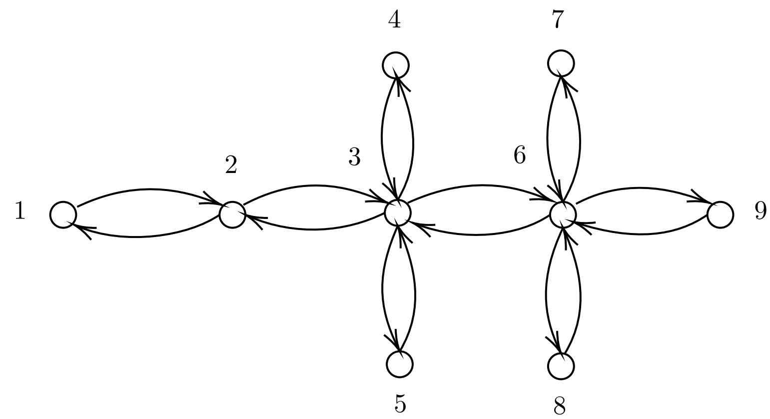

FIGURE 1. D_1.

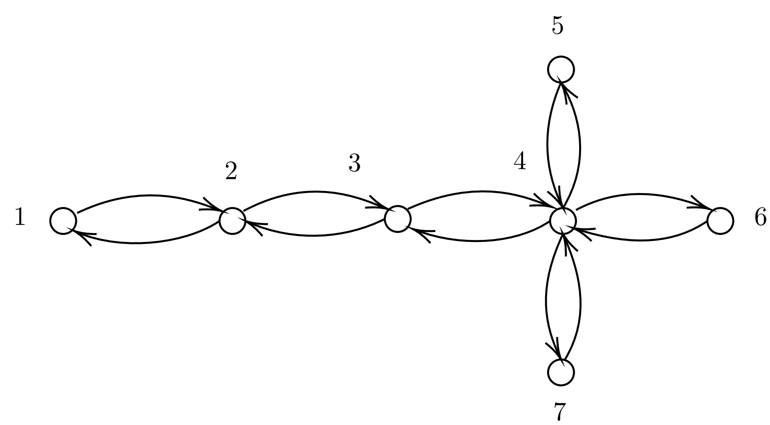

FIGURE 2. D_2.

Here is the main result of this article:

THEOREM 1.4. *Let A be an $n \times n$ real matrix with a tree graph $D(A) \in \mathbb{D}$ and assume that $\Delta_A \neq 0$. Let $A^{\#} = (\alpha_{ij})$ and let μ_{ij} be defined as above. Then, $\alpha_{ij} = \frac{\mu_{ij}}{\Delta_A}$.*

An interesting problem in matrix theory is to provide a formula for the inverse or the group inverse of a matrix, based on its graph structure. We refer the reader to the following articles, on determining the inverse [1, 2, 3, 12] and the group inverse [5, 7, 8, 9, 10, 13].

In [8], a formula for the group inverse of a 2×2 block matrix with bipartite digraph as well as a graphical description of the group inverse of a matrix A with path digraph $D(A)$ are presented. In the work [9], a necessary and sufficient condition for the existence of the group inverse of a special bipartite matrix is given and a formula is obtained for the group inverse in terms of block submatrices. A graphical description for the entries of the group inverse of a matrix A with directed broom tree $D(A)$ is presented.

In a recent work, the authors of [13] derived a formula for the entries of the group inverse of the adjacency matrix of an undirected weighted tree. The entries are given in terms of alternating paths and maximum

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matchings. A group inverse formula for the adjacency matrix of singular undirected cycle appeared in [11]; it can be obtained from [10, Theorem 5.5], too.

Let us present a brief overview of the main results of this article. In [8], the authors proposed a conjecture for the entries of the group inverse of a matrix with tree graph. We show that the conjecture is true for the class of trees $\mathbb{D}$, introduced here. This is presented in Theorem 2.11, achieved via a formula for the group inverse of matrices whose digraphs belong to $\mathbb{D}$ proved in Theorem 1.4. Extending another result of [8], we show the zero–nonzero pattern of the group inverse (when it exists) and the Moore–Penrose inverse of matrices A, for which $D(A) \in \mathbb{D}$.

2. Proof of the main result. Let us recall that a real square matrix $A = (a_{ij})$ is called *combinatorially symmetric* if $a_{ij} = 0$ iff $a_{ji} = 0$. Trivially, any symmetric matrix is combinatorially symmetric. Let A be an $n \times n$ matrix with real entries such that $D(A)$ is a directed tree. Let $a_{ij} \neq 0$. If $a_{ji} = 0$, then there is no path from j to i in $D(A)$, a contradiction, since $D(A)$ is strongly connected. Thus, A is a combinatorially symmetric matrix. It also follows that if (i, j) is an edge in $D(A)$, then there is a 2-cycle (i, j, i) in $D(A)$. Further, since $D(A)$ has only 2-cycles, the diagonals of A are zero.

The first result identifies a matrix that commutes with A (for which $D(A)$ is a tree); later, this is shown to satisfy further properties, under an additional assumption. It is pertinent to point to the fact that the proof of this result is a modification of the proof of [13, Proposition 2], which considers the case when A is symmetric.

THEOREM 2.1. *Let $A = (a_{ij})$ be an $n \times n$ real matrix such that $D(A)$ is a tree. Let $\Delta_A \neq 0$. Let $B = (b_{ij})$ be the matrix given by $b_{ij} = \frac{\mu_{ij}}{\Delta_A}, 1 \leq i, j \leq n$. Then, $AB = BA$.*

Proof. Let $A = (a_{ij})$. Then, $AB = BA$ has following equivalent form:

$$\sum_{k=1}^{n} a_{ik}\mu_{kj} = \sum_{l=1}^{n} \mu_{il} a_{lj} \quad \textit{for every} \ \ i, j \in \{1, 2, \ldots n\}. \tag{2.1}$$

Note that (2.1) is vacuously true if the length of the path from i to j is odd. Now, we discuss the case $i = j$. Let $\{i_1, i_2, \ldots, i_r\} \in \{1, 2, \ldots, n\}$ be such that for any $s \in \{1, 2, \ldots, r\}$, $a_{i,i_s} \neq 0$ and the cycle (i, i_s, i) belongs to some maximum matching in $D(A)$. Since $\mathbb{M}(i, j) = \mathbb{M}(j, i)$ and $\beta_{\overline{i,j}}(M) = \beta_{\overline{j,i}}(M)$, the expressions on both the sides of equation (2.1) are equal, and they equal the common value $\sum_{s=1}^{r} \left(a_{ii_s} a_{i_s i} \sum_{M \in \mathbb{M}(i_s, i)} \beta_{\overline{i_s,i}}(M)\right)$. Let $\mathbb{M}(i)$ be the set of all maximum matchings where i is matched. Then, this common value is equal to $\sum_{M \in \mathbb{M}(i)} \eta(M)$.

Assume therefore, that the length of the path from i to j in $D(A)$ is even (say m). Let $(i, p, \ldots, q, j)$ be the unique path from i to j in $D(A)$. Let $\mathbb{M}(i, \tilde{j})$ be the set of maximum matchings $M \in \mathbb{M}(i, q)$ not containing j, so that

$$\mathbb{M}(i, q) = \cup_{t \in N(j) \setminus \{q\}} \mathbb{M}(i, t) \cup \mathbb{M}(i, \tilde{j}). \tag{2.2}$$

By using (2.1), (2.2), and the definition of μ_{ij}, we obtain

$$\begin{aligned} \mu_{iq} a_{qj} &= a_{qj} \left(\beta_{iq} \sum_{M \in \mathbb{M}(i,q)} \beta_{\overline{i,q}}(M) \right) \\ &= a_{qj} \beta_{iq} \left(\sum_{t \in N(j) \setminus \{q\}} a_{jt} a_{tj} \sum_{M \in \mathbb{M}(i,t)} \beta_{\overline{i,t}}(M) + \sum_{M \in \mathbb{M}(i,\tilde{j})} \beta_{\overline{i,q}}(M) \right) \end{aligned}$$

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$$= -\sum_{t\in N(j)\setminus\{q\}} a_{tj}\left(\beta_{it}\sum_{M\in\mathbb{M}(i,t)}\beta_{\overline{i,t}}(M)\right) + a_{qj}\beta_{iq}\sum_{M\in\mathbb{M}(i,\tilde{j})}\beta_{\overline{i,q}}(M)$$
$$= -\sum_{t\in N(j)\setminus\{q\}} a_{tj}\mu_{it} + a_{qj}\beta_{iq}\sum_{M\in\mathbb{M}(i,\tilde{j})}\beta_{\overline{i,q}}(M).$$

The above calculation implies that

$$\sum_{l=1}^{n}\mu_{il}a_{lj} = \mu_{iq}a_{qj} + \sum_{l\in N(j)\setminus\{q\}}\mu_{il}a_{lj} = a_{qj}\beta_{iq}\sum_{M\in\mathbb{M}(i,\tilde{j})}\beta_{\overline{i,q}}(M). \tag{2.3}$$

In an entirely similar manner, by interchanging the roles of i and j (as well as p and q), and letting $\mathbb{M}(j,\tilde{i})$ denote the set of all $M \in \mathbb{M}(j,p)$ not containing i, one obtains

$$\sum_{k=1}^{n} a_{ik}\mu_{kj} = a_{ip}\beta_{pj}\sum_{M\in\mathbb{M}(j,\tilde{i})}\beta_{\overline{j,p}}(M). \tag{2.4}$$

Since $a_{qj}\beta_{iq} = a_{ip}\beta_{pj}$, (2.3) and (2.4) imply that (2.1) holds if and only if

$$\sum_{M\in\mathbb{M}(i,\tilde{j})}\beta_{\overline{i,q}}(M) = \sum_{M\in\mathbb{M}(j,\tilde{i})}\beta_{\overline{j,p}}(M). \tag{2.5}$$

Note that, there is a bijection $f : \mathbb{M}(i,\tilde{j}) \to \mathbb{M}(j,\tilde{i})$ which transforms every maximum matching $M \in \mathbb{M}(i,\tilde{j})$ of $D(A)$ to a maximum matching $M^f \in \mathbb{M}(j,\tilde{i})$ by trading the matched cycles on the unique cycle chain $C_m(i,j)$ of $D(A)$ with the unmatched cycles. By its very definition, it is clear that this bijection satisfies $\beta_{\overline{i,q}}(M) = \beta_{\overline{j,p}}(M^f)$. This completes the proof of the validity of (2.1). □

Recall that an undirected *corona tree* is a tree obtained by attaching a new pendant vertex to each vertex of a given undirected tree. Let $\{i_1, i_2, \ldots, i_s\} \subseteq V(D(A))$. Then, $D(A)\setminus\{i_1, i_2, \ldots, i_s\}$ is the forest obtained from $D(A)$ by deleting the vertices $\{i_1, i_2, \ldots, i_s\}$ together with their incident 2-cycles.

In the next result, we identify a certain property that is satisfied by all the members of $\mathbb{D}$. This will be useful in further discussions.

PROPOSITION 2.2. *Let $D \in \mathbb{D}$. Then, no non-pendant cycle can belong to a maximum matching of D.*

Proof. If the underlying graph of D is a corona tree, then it has a perfect matching and each matching cycle is a pendant cycle. Now, we consider the case where the underlying graph of D is not a corona tree. In that case, there is at least one non-pendant vertex which is adjacent to at least two pendant vertices in D. We prove the assertion by induction on the number of vertices in D.

The smallest tree in $\mathbb{D}$ is directed star $K_{1,2}$, and every maximum matching has only pendant cycles. Let $D \in \mathbb{D}$ with n vertices. Let the statement be true for any $D \in \mathbb{D}$ having less than n vertices. Let i be a non-pendant vertex adjacent to s pendant vertices $\{i_1, i_2, \ldots, i_s\}$ in D. Let C be an arbitrary non-pendant cycle contained in a maximum matching M in D. Then, we show that this leads to a contradiction.

Case (i): C is incident to i. Then, none of 2-cycles (i, i_p, i), $p \in \{1, 2, \ldots, s\}$ belongs to M. So, M will also be a maximum matching of the tree $D\setminus\{i_1\} \in \mathbb{D}$. This contradicts the fact that a non-pendant cycle belongs to a maximum matching in $D\setminus\{i_1\}$.

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Case (ii): C is not incident to vertex i. Then, one of the 2-cycles (i, i_p, i), $p \in \{1, 2, \ldots, s\}$ belongs to M; otherwise, M will not be maximum. Let (i, i_p, i) belong to M for some $p \in \{1, 2, \ldots, s\}$. Then, M will also be a maximum matching of the tree $D \backslash \{i_q\}$ for some $q \neq p$, which again contradicts the fact that a non-pendant cycle belongs to a maximum matching in $D \backslash \{i_q\}$.

The proof is complete. ◻

Corollary 2.3. *Let $D \in \mathbb{D}$. Then, the length of any alternating cycle chain is at most three.*

Proof. Suppose D has an alternating cycle chain C of length at least five. Then, C must have at least one non-pendant maximum matching cycle, a contradiction to Proposition 2.2. ◻

Remark 2.4. *Let $D \in \mathbb{D}$ have k non-pendant vertices. Then, a maximum matching of D has a set of k pendant cycles incident to k non-pendant vertices. So, the number of edges in a maximum matching is always k. Note that, every non-pendant vertex is matched in any maximum matching of D.*

Remark 2.5. *Let $D \in \mathbb{D}$. Then, both the end points of a length three alternating cycle chain are pendant vertices and a length one alternating cycle chain is nothing but a pendant cycle.*

In the next result, we present a graph theoretic interpretation to the product AB, where A and B are as defined in Theorem 2.1, with $D(A) \in \mathbb{D}$.

Theorem 2.6. *Let A and B satisfy the hypotheses of Theorem 2.1. Let $D(A) \in \mathbb{D}$ and let $\mathbb{M}(i)$ be the set of all maximum matchings, where the vertex i is matched. Then,*

$$(AB)_{ii} = \begin{cases} 1 & \text{if } i \text{ is a non-pendant vertex,} \\ \frac{\sum_{M \in \mathbb{M}(i)} \eta(M)}{\Delta_A} & \text{if } i \text{ is a pendant vertex} \end{cases}$$

while for $i \neq j$,

$$(AB)_{ij} = \begin{cases} \frac{a_{qj}\mu_{iq}}{\Delta_A} & \text{if } i, j \text{ are pendant vertices and} \\ & \text{have a common neighbor } q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.1, it is clear that

$$(AB)_{ii} = \frac{1}{\Delta_A} \sum_{M \in \mathbb{M}(i)} \eta(M).$$

By Remark 2.4, a non-pendant vertex is matched in every maximum matching, and so for a non-pendant vertex i, $(AB)_{ii} = \frac{1}{\Delta_A} \cdot \Delta_A = 1$.

Now, let $i \neq j$. Let $(i, p, \ldots, q, j)$ be the unique path from i to j in $D(A)$. If the length of this path is odd, then $(AB)_{ij} = 0$. If the length is even, then, again from Theorem 2.1,

$$(AB)_{ij} = \frac{1}{\Delta_A} \cdot a_{qj}\beta_{iq} \sum_{M \in \mathbb{M}(i,\tilde{j})} \beta_{\overline{i,q}}(M),$$

where $\mathbb{M}(i, \tilde{j})$ is the set of all maximum matchings $M \in \mathbb{M}(i, q)$ not containing j. We consider four mutually exclusive and collective exhaustive cases.

Case (i): i is a non-pendant vertex. Then, q is a non-pendant vertex, irrespective of whether j is a pendant or a non-pendant vertex. By Remark 2.5, there is no alternating path between any two non-pendant vertex in $D(A)$ and so $\beta_{iq} = 0$. So, $(AB)_{ij} = 0$.

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Case (ii): i is pendant and j is non-pendant. Again, by Remark 2.4, since each non-pendant vertex is matched in every maximum matching of $D(A)$, $\mathbb{M}(i,\tilde{j}) = \phi$. So, $(AB)_{ij} = 0$.
Case (iii): i,j are pendant vertices having no common neighbor. Note that, to get a nonzero $(AB)_{ij}$, the length of the path from i to j should be at least 4. Since the last cycle of the cycle chain $C_m(i,q)$ for some odd $m \geq 3$ is always a non-pendant cycle, by Remark 2.5, $C_m(i,q)$ is not an alternating cycle chain with respect to any maximum matching in $D(A)$. So, $\beta_{iq} = 0$, which in turn, implies that $(AB)_{ij} = 0$.
Case (iv): i,j are pendant vertices having a common neighbor. Let q be such a common neighbor. Then, (i,q,i) and (j,q,j) cannot simultaneously be present in a maximum matching. So, $\mathbb{M}(i,\tilde{j}) = \mathbb{M}(i,q)$ and

$$(AB)_{ij} = \tfrac{1}{\Delta_A} \cdot a_{qj}\beta_{iq} \textstyle\sum_{M\in\mathbb{M}(i,q)} \beta_{\overline{i,q}}(M) = \tfrac{a_{qj}\mu_{iq}}{\Delta_A}. \qquad \square$$

Next, we show that B is an outer inverse of A.

THEOREM 2.7. *Let A and B satisfy the hypotheses of Theorem 2.6. Then, $BAB = B$.*

Proof. By Theorem 2.1, if we prove $ABB = B$, then we are done. This is equivalent to proving that,

$$\sum_{k=1}^{n}\left(\frac{1}{\Delta_A}\sum_{l=1}^{n} a_{il}\mu_{lk}\right)\mu_{kj} = \begin{cases} \mu_{ij}, & \text{if } i,j \text{ are maximally matchable,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Fix j and let b be the left hand side of (2.6). Then, b can be written in the form $b = b_i + \tilde{b_i}$, where

$$b_i = \left(\frac{1}{\Delta_A}\sum_{l=1}^{n} a_{il}\mu_{li}\right)\mu_{ij} \quad \text{and} \quad \tilde{b_i} = \sum_{\substack{k=1\\k\neq i}}^{n}\left(\frac{1}{\Delta_A}\sum_{l=1}^{n} a_{il}\mu_{lk}\right)\mu_{kj}.$$

First assume that i,j are maximally matchable. Then, by Corollary 2.3, the length of the path from i to j is at most three.
Case (i): The length of the path from i to j is one.
Subcase (i) : i is a non-pendant vertex. Then, by Corollary 2.6, the term in the parenthesis in b_i is 1 and the term in the parenthesis in $\tilde{b_i}$ is zero. So, $b = \mu_{ij}$.
Subcase (ii): i is a pendant vertex. Now, since the length of the path from i to j is 1, j must be a non-pendant vertex. Let $\mathbb{M}$ be the set of all maximum matchings and $\mathbb{M}(i)$ denote the set of all maximum matchings in which the vertex i is matched. Let $\{i_1, i_2, \ldots, i_s\}$ be the set of all pendant vertices other than i which have a common neighbor j. Then, $\mathbb{M} = \mathbb{M}(i) \cup_{m=1}^{s} \mathbb{M}(i_m)$, and they are mutually disjoint sets of maximum matchings. By Corollary 2.6, for $i \neq k$, $(AB)_{ik}$ can be nonzero only when i and k are pendant vertices and have a common neighbor. So, $b_i = \frac{\sum_{M\in\mathbb{M}(i)}\eta(M)}{\Delta_A}\mu_{ij}$ and

$$\begin{aligned}
\tilde{b_i} &= \sum_{m=1}^{s}\left(\frac{1}{\Delta_A}\sum_{l=1}^{n} a_{il}\mu_{li_m}\right)\mu_{i_m j} \\
&= \sum_{m=1}^{s}\left(\frac{a_{ji_m}\mu_{ij}}{\Delta_A}\right)\mu_{i_m j} \\
&= \frac{\mu_{ij}}{\Delta_A}\sum_{m=1}^{s} a_{ji_m}\left(\beta_{i_m j}\sum_{M\in\mathbb{M}(i_m,j)}\beta_{\overline{i_m,j}}(M)\right)
\end{aligned}$$

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$$= \frac{\mu_{ij}}{\Delta_A} \sum_{m=1}^{s} \left(a_{ji_m} a_{i_m j} \sum_{M \in \mathbb{M}(i_m, j)} \beta_{\overline{i_m, j}}(M) \right)$$

$$= \frac{\mu_{ij}}{\Delta_A} \sum_{m=1}^{s} \sum_{M \in \mathbb{M}(i_m)} \eta(M).$$

Thus, $b = \frac{\mu_{ij}}{\Delta_A} \left(\sum_{M \in \mathbb{M}(i)} \eta(M) + \sum_{m=1}^{s} \sum_{M \in \mathbb{M}(i_m)} \eta(M) \right) = \frac{\mu_{ij}}{\Delta_A} \cdot \Delta_A = \mu_{ij}$.

Case (ii): The length of the path from i to j is three. Since i and j are maximally matchable, i and j must be pendant vertices. Let q be the non-pendant vertex adjacent to i and $\{i_1, i_2, \ldots, i_s\}$ be the set of all pendant vertices adjacent to q, other than i. Since $\{i, i_1, i_2, \ldots, i_s\}$ have common neighbor q, for all $m \in \{1, 2, \ldots, s\}$,

$$\sum_{M \in \mathbb{M}(i,q)} \beta_{\overline{i,q}}(M) = \sum_{M \in \mathbb{M}(i_m,q)} \beta_{\overline{i_m,q}}(M) \tag{2.7}$$

and

$$\sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) = \sum_{M \in \mathbb{M}(i_m,j)} \beta_{\overline{i_m,j}}(M). \tag{2.8}$$

Let (i, q, z, j) be the unique path from i to j. Now, by (2.7), (2.8), Corollary 2.6 and using the mutual disjointness of the maximum matchings $\mathbb{M}(i), \mathbb{M}(i_1), \ldots, \mathbb{M}(i_s)$, we obtain $b_i = \frac{\sum_{M \in \mathbb{M}(i)} \eta(M)}{\Delta_A} \mu_{ij}$ and

$$\tilde{b_i} = \sum_{m=1}^{s} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il} \mu_{li_m} \right) \mu_{i_m j}$$

$$= \sum_{m=1}^{s} \left(\frac{a_{qi_m} \mu_{iq}}{\Delta_A} \right) \mu_{i_m j}$$

$$= \frac{1}{\Delta_A} \sum_{m=1}^{s} a_{qi_m} \left(\beta_{iq} \sum_{M \in \mathbb{M}(i,q)} \beta_{\overline{i,q}}(M) \right) \left(\beta_{i_m j} \sum_{M \in \mathbb{M}(i_m,j)} \beta_{\overline{i_m,j}}(M) \right)$$

$$= \frac{1}{\Delta_A} \sum_{m=1}^{s} a_{qi_m} \left(a_{iq} \sum_{M \in \mathbb{M}(i_m,q)} \beta_{\overline{i_m,q}}(M) \right) \left(-a_{i_m q} a_{qz} a_{zj} \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \right)$$

$$= \frac{1}{\Delta_A} \sum_{m=1}^{s} \left(a_{i_m q} a_{qi_m} \sum_{M \in \mathbb{M}(i_m,q)} \beta_{\overline{i_m,q}}(M) \right) \left(-a_{iq} a_{qz} a_{zj} \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \right)$$

$$= \frac{1}{\Delta_A} \sum_{m=1}^{s} \left(\sum_{M \in \mathbb{M}(i_m)} \eta(M) \right) \left(\beta_{ij} \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \right)$$

$$= \frac{\mu_{ij}}{\Delta_A} \sum_{m=1}^{s} \sum_{M \in \mathbb{M}(i_m)} \eta(M).$$

So, $b = b_i + \tilde{b_i} = \frac{\mu_{ij}}{\Delta_A} \cdot \Delta_A = \mu_{ij}$.

Next, we discuss the case when i and j are not maximally matchable. If the length of the path from i to j is even, then in (2.6), either the lengths of the paths from i to k as well as length of the path from k to

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j are both even or both odd. In the case of the former, μ_{kj} is zero and in the latter case, the term in the parenthesis is zero. So, (2.6) is vacuously true when the length of the path from i to j is even. Since i, j are not maximally matchable, $\mu_{ij} = 0$. So, $b = \tilde{b_i}$. Now, we consider the case when the length of the path from i to j is odd.

Case (i): i is a non-pendant vertex. By Corollary 2.6, the term in the parenthesis in $\tilde{b_i}$ is 0. Thus, $b = 0$.

Case (ii): i is a pendant vertex. Let q be the non-pendant vertex adjacent to i and $\{i_1, i_2, \ldots, i_s\}$ be the set of all pendant vertices that are adjacent to q other than i. As argued earlier,

$$\tilde{b_i} = \sum_{m=1}^{s} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il}\mu_{li_m} \right) \mu_{i_m j}.$$

Since i, j are not maximally matchable, for all $m \in \{1, 2, \ldots, s\}$, i_m and j are also not maximally matchable. Thus, $b = \tilde{b_i} = 0$, completing the proof. □

The next result shows that B is an inner inverse of A.

THEOREM 2.8. *Let A and B satisfy the hypotheses of Theorem 2.6. Then, $ABA = A$.*

Proof. To show that $ABA = A$, we show

$$\sum_{k=1}^{n} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il}\mu_{lk} \right) a_{kj} = \begin{cases} a_{ij}, & \text{when } (i, j) \text{ is an edge} \\ 0, & \text{when } (i, j) \text{ is not an edge.} \end{cases} \tag{2.9}$$

Let c be the left-hand side of (2.9). Then, c can be written in the form $c = c_i + \tilde{c_i}$, where

$$c_i = \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il}\mu_{li} \right) a_{ij} \quad \text{and} \quad \tilde{c_i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il}\mu_{lk} \right) a_{kj}.$$

First, we assume that (i, j) is an edge. Then, we are in two cases.

Case (i): i is a non-pendant vertex. Then, by Corollary 2.6, the terms in the parenthesis in c_i and $\tilde{c_i}$ are 1 and 0, respectively. So, $c = a_{ij}$.

Case (ii): i is a pendant vertex. Since (i, j) is an edge and $D(A) \in \mathbb{D}$, j must be a non-pendant vertex. Let $\mathbb{M}$ and $\mathbb{M}(i)$ be the set of all maximum matchings in $D(A)$, and the set of all maximum matchings in which vertex i is matched, respectively. Let $\{i_1, i_2, \ldots, i_s\}$ be the set of all pendant vertices other than i, which have a common neighbor j. Then, $\mathbb{M} = \mathbb{M}(i) \cup_{m=1}^{s} \mathbb{M}(i_m)$, a mutually disjoint union. Again, by Corollary 2.6 and (2.7), $c_i = \frac{\sum_{M \in \mathbb{M}(i)} \eta(M)}{\Delta_A} a_{ij}$ and

$$\begin{aligned}
\tilde{c_i} &= \sum_{m=1}^{s} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il}\mu_{li_m} \right) a_{i_m j} \\
&= \sum_{m=1}^{s} \left(\frac{a_{ji_m}\mu_{ij}}{\Delta_A} \right) a_{i_m j} \\
&= \frac{1}{\Delta_A} \sum_{m=1}^{s} a_{ji_m} a_{i_m j} \left(\beta_{ij} \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \right) \\
&= \frac{1}{\Delta_A} \sum_{m=1}^{s} a_{ji_m} a_{i_m j} \left(a_{ij} \sum_{M \in \mathbb{M}(i_m,j)} \beta_{\overline{i_m,j}}(M) \right)
\end{aligned}$$

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Volume 38, pp. 617-631, September 2022.

$$= \frac{a_{ij}}{\Delta_A} \sum_{m=1}^{s} \left(a_{ji_m} a_{i_m j} \sum_{M \in \mathbb{M}(i_m,j)} \beta_{\overline{i_m,j}}(M) \right)$$
$$= \frac{a_{ij}}{\Delta_A} \sum_{m=1}^{s} \sum_{M \in \mathbb{M}(i_m)} \eta(M).$$

Thus, $c = \frac{a_{ij}}{\Delta_A}(\sum_{M\in\mathbb{M}(i)} \eta(M) + \sum_{m=1}^{s} \sum_{M\in\mathbb{M}(i_m)} \eta(M)) = \frac{a_{ij}}{\Delta_A} \cdot \Delta_A = a_{ij}$.

Next, let (i, j) be not an edge. Then, $a_{ij} = 0$. One has $(AB)_{ik} = 0$ when the length of the path from i to k is odd and so, (2.9) is vacuously true when the length of the path from i to j is even. So, $c = \tilde{c_i}$.

Case (i): i is a non-pendant vertex. Once again, by Corollary 2.6, the term in the parenthesis in $\tilde{c_i}$ is 0. Thus, $c = 0$.

Case (ii): i is a pendant vertex. Let q be the non-pendant vertex adjacent to i and $\{i_1, i_2, \ldots, i_s\}$ be the set of all pendant vertices that are adjacent to q other than the vertex i. Using Corollary 2.6, we obtain

$$\tilde{c_i} = \sum_{m=1}^{s} \left(\frac{1}{\Delta_A} \sum_{l=1}^{n} a_{il} \mu_{l i_m} \right) a_{i_m j}.$$

Since (i, j) is not an edge, $q \neq j$. So, for all $m \in \{1, 2, \ldots, s\}$, $a_{i_m j} = 0$. Thus, $c = \tilde{c_i} = 0$, completing the proof. ∎

Proof of Theorem 1.4:
Follows from the conclusions of Theorems 2.1, 2.7, and 2.8.

Here is an illustration.

Example 2.9. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $D(A)$ is the digraph D_1 in Fig. 1. Here, all the maximum matchings of $D(A)$ are given by:

$$M_1 = \{(1,2,1),(3,4,3),(6,7,6)\}$$
$$M_2 = \{(1,2,1),(3,4,3),(6,9,6)\}$$
$$M_3 = \{(1,2,1),(3,4,3),(6,8,6)\}$$
$$M_4 = \{(1,2,1),(3,5,3),(6,7,6)\}$$
$$M_5 = \{(1,2,1),(3,5,3),(6,7,9)\}$$
$$M_6 = \{(1,2,1),(3,5,3),(6,7,8)\}.$$

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So, $\Delta_A = 1+2+(-2)+4+8+(-8) = 5$*. Let* $A^{\#} = (\alpha_{ij})$*. Let us compute* α_{15}*. First,* $P(1,5) = 1\cdot 2\cdot 2 = 4$*. Note that* $C_3(1,5)$ *cycle chain is alternating with respect to the maximum matchings* M_4*,* M_5*, and* M_6*. Thus,* $\beta_{15} = (-1)\cdot 4 = -4$*,* $\beta_{\overline{1,5}}(M_4) = 1\cdot(-1) = -1$*,* $\beta_{\overline{1,5}}(M_5) = 2\cdot(-1) = -2$ *and* $\beta_{\overline{1,5}}(M_6) = (-1)\cdot(-2) = 2$*. So,*

$$\mu_{15} = (-4)\cdot(-1-2+2) = 4.$$

Therefore, $\alpha_{15} = \frac{4}{5}$*.*

An $n\times n$ real matrix $A = (a_{ij})$ is said to be an *irreducible* matrix if the corresponding directed graph $D(A)$ is strongly connected. An irreducible matrix is *nearly reducible* if it is reducible whenever any nonzero entry is set to zero [6, Section 3.3].

Consider a tree graph $D(A)$, for an $n\times n$ real matrix $A = (a_{ij})$. Then, the *term rank* of A is twice of the number of 2-cycles in a maximum matching in $D(A)$. For a tree graph $D(A)$, the matrix A is nearly reducible, so the term rank of A is equal to the rank of A [5, Theorem 4.5].

A *path graph*, denoted by $p(i_1, i_n)$ (which is nothing but a cycle chain from i_1 to i_n) on n vertices $i_1, i_2, \cdots, i_n$ which consists of the path $p = (i_1, i_2, \cdots, i_n)$ from i_1 to i_n and its reversal (i.e., the path obtained by reversing all of the arcs in p). Let $\gamma(i_1, i_n)$ denote the sum of all maximum matchings not on the path subgraph $p(i_1, i_n)$ of $D(A)$. Consider the following conjecture, stated in [8]. This was proposed after proving that it holds for the special class of matrices A with the property that $D(A)$ are path graphs.

Conjecture 2.10. *[8, Conjecture 5.1] Let* A *be a singular matrix with a tree graph* $D(A)$*. Let* r *be the term rank of* A *and* $\Delta_A \neq 0$*. Then,* $A^{\#} = (\alpha_{ij})$ *exists and*

$$\alpha_{ij} = \begin{cases} \frac{1}{\Delta_A}(-1)^s P(i,j)\gamma(i,j) & \text{if the path in } D(A) \text{ from } i \text{ to } j \text{ is of length } 2s+1, s\geq 0 \\ & \text{and the matrix associated with } D(A)\backslash p(i,j) \text{ has} \\ & \text{term rank } r-2(s+1), \\ 0 & \text{otherwise.} \end{cases}$$

The second main result of this article shows that the above conjecture holds for matrices A for which $D(A)\in\mathbb{D}$ with $\Delta_A \neq 0$. Thus, the given conjecture is true for more classes of graphs than path graphs.

Theorem 2.11. *Let* A *be a real square matrix of order* n *such that* $D(A)\in\mathbb{D}$ *with* $\Delta_A\neq 0$*. Then,* $A^{\#} = (\alpha_{ij})$ *exists, where* α_{ij} *is as given above.*

Proof. By Theorem 1.4,

$$\alpha_{ij} = \begin{cases} \frac{1}{\Delta_A}(-1)^s P(i,j)\sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M) & \text{if } i,j \text{ are maximally matchable and } 2s+1 \\ & \text{is the length of the path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

We show that the description for α_{ij} as in Conjecture 2.10, and the one given above are equivalent. The proof of the theorem then follows.

Suppose that i and j are maximally matchable and $2s+1$ is the length of the path from i to j. Then, by definition, $\gamma(i,j) = \sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M)$. Suppose A has term rank r, i.e., the number of cycles in a maximum matching of $D(A)$ is $\frac{r}{2}$. The term rank of a matrix associated with the cycle chain $C_{2s+1}(i,j)$ is $2s+2$ due to the fact that an alternating cycle chain $C_{2s+1}(i,j)$ contains exactly $s+1$ non-pendant vertices. Now,

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$D(A)\backslash C_m(i,j)$ is a forest, wherein each nontrivial component either belongs to $\mathbb{D}$ or is just a 2-cycle. So, the number of cycles in a maximum matching of $D(A)\backslash C_{2s+1}(i,j)$ is $\frac{r}{2}-(s+1)$. Thus, the term rank of a matrix associated with $D(A)\backslash C_{2s+1}(i,j)$ is $r-2(s+1)$.

Next, for the converse part let i and j be not maximally matchable. Then, $C_{2s+1}(i,j)$ contains at least $s+2$ non-pendant vertices. Again, since $D(A)\backslash C_{2s+1}(i,j)$ is a forest wherein each nontrivial component either belongs to $\mathbb{D}$ or is just a 2-cycle, the number of cycles in a maximum matching in $D(A)\backslash C_{2s+1}(i,j)$ is at most $\frac{r}{2}-\frac{2s+4}{2}$. Thus, the term rank of a matrix associated with $D(A)\backslash C_{2s+1}(i,j)$ will be at most $r-(2s+4)$ which is less than $r-(2s+2)$.

Observe that if the matrix associated with $D(A)\backslash C_{2s+1}(i,j)$ does not have rank $r-2(s+1)$, then the two odd distance vertices i and j are not maximally matchable. ∎

Corollary 2.12. *Let $A=(a_{ij})$ be an $n\times n$ real matrix with tree graph $D(A)\in\mathbb{D}$ and assume that $\Delta_A\neq 0$. If i and j are maximally matchable and $A^{\#}=(\alpha_{ij})$, then $\sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M)\neq 0$, so that $\alpha_{ij}\neq 0$.*

Proof. Since $D(A)\in\mathbb{D}$, the length of the alternating cycle chain from i to j is at most three. Let $\mathbb{M}$ denote the set of all maximum matchings in $D(A)$. Since the length of a cycle is either 1 or 3, we have the following two cases:
Case (i): The length of the alternating cycle chain is 1, so that (i,j,i) is a pendant cycle. Without loss of generality let i be the pendant vertex. Let $\{i_1(=i),i_2,\ldots,i_s\}$ be the set of pendant vertices which have j, the non-pendant vertex, as a common neighbor. Then, $\mathbb{M}=\cup_{x=1}^{s}\mathbb{M}(i_x,j)$, a disjoint union. Now, using (2.7),

$$\begin{aligned}\Delta_A &= \sum_{x=1}^{s}\sum_{M\in\mathbb{M}(i_x,j)}\eta(M)\\ &= \sum_{x=1}^{s}a_{i_xj}a_{ji_x}\sum_{M\in\mathbb{M}(i_x,j)}\beta_{\overline{i_x,j}}(M)\\ &= \sum_{x=1}^{s}a_{i_xj}a_{ji_x}\sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M)\\ &= \left(\sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M)\right)\sum_{x=1}^{s}a_{i_xj}a_{ji_x}.\end{aligned}$$

Since $\Delta_A\neq 0$, $\sum_{M\in\mathbb{M}(i,j)}\beta_{\overline{i,j}}(M)\neq 0$ and so, $\alpha_{ij}\neq 0$.
Case (ii): The length of the alternating cycle chain is 3. Let (i,q,p,j) be the path from i to j. So, i and j are pendant, while p and q are non-pendant vertices. Let $\{i_1(=i),i_2,\ldots,i_s\}$ and $\{j_1(=j),j_2,\ldots,j_t\}$ be the set of all pendant vertices having q and p as a common neighbor, respectively. Then, $\mathbb{M}=\cup_{x=1}^{s}\cup_{y=1}^{t}\mathbb{M}(i_x,j_y)$, again a mutually disjoint union. Now, using (2.8), we obtain

$$\begin{aligned}\Delta_A &= \sum_{x=1}^{s}\sum_{y=1}^{t}\sum_{M\in\mathbb{M}(i_x,j_y)}\eta(M)\\ &= \sum_{x=1}^{s}\sum_{y=1}^{t}(a_{i_xq}a_{qi_x})(a_{j_yp}a_{pj_y})\sum_{M\in\mathbb{M}(i_x,j_y)}\beta_{\overline{i_x,j_y}}(M)\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 617-631, September 2022.

$$= \sum_{x=1}^{s}\sum_{y=1}^{t}(a_{i_xq}a_{qi_x})(a_{j_yp}a_{pj_y}) \sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M)$$
$$= \left(\sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M)\right)\sum_{x=1}^{s}\sum_{y=1}^{t}(a_{i_xq}a_{qi_x})(a_{j_yp}a_{pj_y}).$$

Again, since $\Delta_A \neq 0$, $\sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \neq 0$ and so, $\alpha_{ij} \neq 0$. ∎

3. Relation between $A^\#$ and $A^\dagger$ for a matrix A with $D(A) \in \mathbb{D}$. Now, we will use some notation from [5]. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be disjoint sets. For an $n \times n$ matrix $A = (a_{ij})$, $B(A)$ is the bipartite graph with vertices $U \cup V$ and edges $\{\{u_i, v_j\} : u_i \in U, v_j \in V, a_{ij} \neq 0\}$. For $k \geq 1$ and any bipartite graph B, let $\mathbb{M}_k(B)$ denotes the family of subsets of k distinct edges of B such that no two of which are adjacent.

Let A be a matrix with acyclic bipartite graph $B(A)$ and $rank(A) \geq 2$, and let $A^\dagger = (\alpha_{ij})$. Then, the following is shown [5, Proposition 2.8]. Let $\{u_i, v_j\}$ be an edge in $B(A)$. Then, $\{u_j, v_i\}$ is an edge in $B(A^\dagger)$ if and only if $\{u_i, v_j\}$ belongs to some member in $\mathbb{M}_r(B(A))$. Further, if $\{u_i, v_j\}$ is contained in every member in $\mathbb{M}_r(B(A))$, then $\alpha_{ji} = \frac{1}{a_{ij}}$.

Interestingly, we have the following analog of the second part of [5, Proposition 2.8], for the class of trees $\mathbb{D}$.

Corollary 3.1. *Let $A = (a_{ij})$ be an $n \times n$ real matrix with tree graph $D(A) \in \mathbb{D}$ and assume that $\Delta_A \neq 0$. If (i, j, i) is a cycle that belongs to each maximum matching of $D(A)$ and $A^\# = (\alpha_{ij})$, then $\alpha_{ij} = \frac{1}{a_{ji}}$ and $\alpha_{ji} = \frac{1}{a_{ij}}$.*

Proof. Let $\mathbb{M}$ be the set of all maximum matching of $D(A)$. Since the cycle (i, j, i) belongs to each maximum matching of $D(A)$, (i, j, i) is an alternating cycle chain of length one and $\mathbb{M} = \mathbb{M}(i, j) = \mathbb{M}(j, i)$. Now,

$$\alpha_{ij} = \frac{\beta_{ij} \cdot \sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M)}{\Delta_A} = \frac{\sum_{M\in\mathbb{M}(i,j)} a_{ji}a_{ij}\beta_{\overline{i,j}}(M)}{a_{ji}\Delta_A} = \frac{\sum_{M\in\mathbb{M}} \eta(M)}{a_{ji}\Delta_A} = \frac{1}{a_{ji}}$$

and

$$\alpha_{ji} = \frac{\beta_{ji} \cdot \sum_{M\in\mathbb{M}(j,i)} \beta_{\overline{j,i}}(M)}{\Delta_A} = \frac{\sum_{M\in\mathbb{M}(j,i)} a_{ij}a_{ji}\beta_{\overline{j,i}}(M)}{a_{ij}\Delta_A} = \frac{\sum_{M\in\mathbb{M}} \eta(M)}{a_{ij}\Delta_A} = \frac{1}{a_{ij}}.$$

Let A be a square matrix with path graph $D(A)$ and $\Delta_A \neq 0$. Let $\gamma(i, j)$ be defined as in the paragraph just before Conjecture 2.10. It is shown in [8, Theorem 4.1 (iii)] that if $\gamma(i, j) \neq 0$, then the zero–nonzero sign patterns of $A^\#$ and $A^\dagger$ are the same. Observe that, for a maximally matchable pair i, j in $D(A) \in \mathbb{D}$, it follows that $\gamma(i, j) = \sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M)$. Further, as is shown in Corollary 2.12, $\sum_{M\in\mathbb{M}(i,j)} \beta_{\overline{i,j}}(M) \neq 0$. Thus, the next result is an extension of the corresponding result of [8], stated earlier, for any matrix A satisfying the property that $D(A) \in \mathbb{D}$.

Theorem 3.2. *Let $A = (a_{ij})$ be an $n \times n$ real matrix with tree graph $D(A) \in \mathbb{D}$ and assume that $\Delta_A \neq 0$. Let $A^\# = (\alpha_{ij})$ and $A^\dagger = (\lambda_{ij})$. Then, $\alpha_{ij} \neq 0$ if and only if $\lambda_{ij} \neq 0$.*

Proof. Let $G(A)$ be the underlying graph of $D(A)$. Then, $B(A)$ is a forest with two components and each component is isomorphic to $G(A)$. Note that, i is a pendant vertex in $D(A)$ iff u_i and v_i are pendant vertices in $B(A)$; a similar statement holds for non-pendant vertices. Suppose the term rank of A is r. Then,

Electronic Journal of Linear Algebra, ISSN 1081-3810
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$\mathbb{M}_r(B(A))$ is non-empty and $\mathbb{M}_l(B(A)) = \phi$, for all $l > r$. For two distinct vertices i, j let the cycle chain $C_q(i,j)$ in $D(A)$ be $((i, k_2, i), (k_2, k_3, k_2), \ldots, (k_q, j, k_q))$. By Corollary 2.12, for $i \neq j$, $\alpha_{ij} \neq 0$ and

$$\alpha_{ij} = \frac{1}{\Delta_A}(-1)^{\frac{q-1}{2}} a_{i,k_2} a_{k_2,k_3} a_{k_3,k_4} \cdots a_{k_{q-1},k_q} a_{k_q,j} \sum_{M \in \mathbb{M}(i,j)} \beta_{\overline{i,j}}(M), \tag{3.1}$$

if and only if i and j are maximally matchable in $D(A)$. From [5, Corollary 2.7], $\lambda_{ji} \neq 0$ if and only if $B(A)$ contains a path P from u_i to v_j

$$u_i \to v_{k_2} \to u_{k_3} \to v_{k_4} \to \cdots \to v_{k_{q-1}} \to u_{k_q} \to v_j$$

of odd length q and $\mathbb{M}_{r-\frac{q+1}{2}}(B(A))$ has at least one element with $r - \frac{q+1}{2}$ edges none of which is adjacent to P. Furthermore, if such a path exists, then λ_{ji} has the same sign as

$$(-1)^{\frac{q-1}{2}} a_{i,k_2} a_{k_3,k_2} a_{k_3,k_4} \cdots a_{k_q,k_{q-1}} a_{k_q,j}. \tag{3.2}$$

Since A is a matrix with a tree graph $D(A)$, it is combinatorially symmetric. Thus, there is a cycle chain $C_q(i,j)$ of odd length in $D(A)$ if and only if there is a path from u_j to v_i of odd length in $B(A)$. Let G_1 and G_2 be the two components of $B(A)$ and assume, without loss of generality, that the path $Q : u_j \to v_{k_q} \to u_{k_{q-1}} \to v_{k_{q-2}} \to \cdots \to v_{k_3} \to u_{k_2} \to v_i$ belongs to G_1. Using the fact that G_1 is isomorphic to $G(A)$, it follows that $G_1 \backslash Q$ is isomorphic to the underlying graph of $D(A) \backslash C_q(i,j)$. So, by the discussion in Theorem 2.11, for an odd q, when i and j are maximally matchable in $D(A)$, $\mathbb{M}_{r-\frac{q+1}{2}}(B(A) \backslash Q) \neq \phi$ and when i and j are not maximally matchable in $D(A)$, $\mathbb{M}_{r-\frac{q+1}{2}}(B(A) \backslash Q) = \phi$.

By Theorem 1.4 and [5, Corollary 2.7], $\alpha_{ii} = 0 = \lambda_{ii}$. Also, when q is even, $\alpha_{ij} = \lambda_{ij} = 0$. Now, suppose q is odd and i and j are not maximally matchable so that, $\alpha_{ij} = 0$. Then, since $\mathbb{M}_{r-\frac{q+1}{2}}(B(A) \backslash Q) = \phi$, $\lambda_{ij} = 0$. If possible, suppose, $\alpha_{ij} \neq 0$. Then, by Corollary 2.12, i and j should be maximally matchable. In this case, there exists a path Q from u_j to v_i of length q and $\mathbb{M}_{r-\frac{q+1}{2}}(B(A) \backslash Q) \neq \phi$. Now, using (3.1) and (3.2) and by combinatorial symmetry, $\lambda_{ij} \neq 0$. ∎

The above result is not true for a matrix A with tree graph $D(A) \notin \mathbb{D}$. We show this by using the same example as in [8, Example 4.2]. Consider the 5×5 matrix A for which $D(A)$ is a path digraph on five vertices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, $\lambda_{45} = 2$, whereas $\alpha_{45} = 0$.

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