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SPACES OF CONSTANT RANK MATRICES OVER $GF(2)$*

NIGEL BOSTON†

Abstract. For each n, we consider whether there exists an $(n+1)$-dimensional space of n by n matrices over $GF(2)$ in which each nonzero matrix has rank $n-1$. Examples are given for $n = 3, 4$, and 5, together with evidence for the conjecture that none exist for $n > 8$.

Key words. Constant rank, Matrices, Heuristics.

AMS subject classifications. 15A03, 15-04.

1. Introduction. There has been much interest [5], [7, Chapter 16D] in spaces of matrices in which every nonzero matrix has the same rank. We call this a space of matrices of constant rank. Often there is some algebraic construction behind the examples - for instance, taking a basis for $GF(q^n)$ over $GF(q)$ yields an n-dimensional space of n by n matrices over $GF(q)$ of constant rank n.

We focus on spaces of n by n matrices of constant rank $n-1$, and ask how large their dimensions can be. In [5], it was shown that for real matrices, the maximal dimension is $\max\{\rho(n-1), \rho(n), \rho(n+1)\}$, where ρ is the Hurwitz-Radon function, except for $n = 3$ and 7 when the maximal dimension is 3 and 7, respectively. As regards matrices over a general field F, it was shown in [2] that if $|F| \geq n$, then this maximal dimension is at most n. The question then arises as to whether for smaller fields F there can be such spaces of larger dimension, $n+1$.

As noted below, $GF(2)$ has the unusual property that there are about twice as many n by n matrices of rank $n-1$ over it as there are matrices of rank n, and so interest has focused on this case. By the above, if $n < 3$, then the maximal dimension is at most n. In [1], Beasley found a couple of spaces of n by n matrices of constant rank $n-1$ and dimension $n+1$ for $n = 3$. He conjectured that no examples exist for $n > 3$, but this author found, by search using the computer algebra system MAGMA [3], examples for $n = 4$ and $n = 5$. The temptation now is to conjecture that examples exist for all n, but as we shall see, heuristics do not support such a claim.

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†Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA (boston@math.wisc.edu). Supported by the Claude Shannon Institute, Science Foundation Ireland Grant 06/MI/006 and Stokes Professorship award, Science Foundation Ireland Grant 07/SK/I1252b.

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2. Low dimensional examples. This section exhibits spaces of n by n matrices of constant rank $n-1$ and dimension $n+1$ for $n = 3, 4$, and 5. For $n = 3$, Beasley [1] found some examples. An exhaustive MAGMA search shows that there are exactly 1176 such spaces. Under conjugation by $GL(3, 2)$, these fall into 12 orbits. A basis for a representative of each orbit is given:

$$\text{Orbit length 168:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\text{Orbit length 168:} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\text{Orbit length 168:} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 168:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 84:} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 84:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 84:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 84:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 56:} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 42:} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

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$$\text{Orbit length 42: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Orbit length 28: } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

An example of a 5-dimensional space of 4 by 4 matrices of constant rank 3 is given by the span of the following matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

An example of a 6-dimensional space of 5 by 5 matrices of constant rank 4 is given by the span of the following matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

These were discovered by careful search using the computer algebra system, MAGMA [3].

3. Heuristics. Let $C(n, r, q)$ denote the number of n by n matrices of rank r over $GF(q)$. Landsberg [6] (later refined by Buckheister [4] to count matrices with a given rank and trace) showed that

$$C(n, r, q) = q^{r(r-1)/2} \prod_{i=1}^{r} (q^{n-i+1} - 1)^2/(q^i - 1).$$

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As $n \to \infty$, the probability that an n by n matrix over $GF(q)$ has rank $n-r$, i.e., the ratio of $C(n, n-r, q)$ to the total number of matrices q^{n^2}, tends to a limit $K(r,q)$, where for instance $K(0,2) = 0.2888,\ K(1,2) = 0.5776$, (which is the basis for the statement above that an n by n matrix over $GF(2)$ is twice as likely to have rank $n-1$ as rank n), $K(2,2) = 0.1284,\ K(3,2) = 0.0052, \ldots$ Since we will make great use of $K(1,2)$ in this paper, note that to 20 decimal places $K(1,2) = 0.57757619017320484256$.

Our heuristic claims that, in the absence of any other algebraic structure, the probability that each matrix in a space of n by n matrices has rank $n-r$ should be independently approximated by $K(r,q)$. Let $N(n,r,q,d)$ denote the number of ordered d-tuples of n by n matrices over $GF(q)$ for which all nontrivial linear combinations have rank $n-r$. By the above heuristic, this should be about $K(r,q)^{q^d-1}$ multiplied by the total number of ordered d-tuples, namely q^{dn^2}, i.e.,

$$N(n,r,q,d) \approx K(r,q)^{q^d-1} q^{dn^2}.$$

To test our heuristic, let S_n be the set of all n by n matrices over $GF(2)$ of rank $n-1$. We seek the probability that, given $M_1, M_2 \in S_n$, $M_1 + M_2$ also lies in S_n. Exhaustive computation shows that it equals $(2/3)^2 = 0.4444, (85/147)^2 = 0.5782, (2722/4725)^2 = 0.5761, (174751/302715)^2 = 0.5773$ for $n = 2,3,4,5$, respectively. This is apparently approaching the limit $K(1,2)$, as proposed.

Likewise, we can test whether, given 3 matrices in S_n, the 4 nontrivial linear combinations of these matrices are all in S_n with probability approaching $K(1,2)^4 = 0.1113$ as the heuristic suggests. For example, $|S_3| = 294$ and of the 294^3 ordered triples, 2709504 or 10.66% satisfy this, which is close to the predicted 11.13%.

Finally, we consider some implications of the heuristic. Let $g(k)$ denote the order of $GL(k,2)$, i.e., $g(k) = C(k,k,2) = (2^k-1)(2^k-2)\cdots(2^k-2^{k-1})$. This counts the number of ordered bases of a k-dimensional vector space over $GF(2)$. If our heuristic holds true, then $N(n,1,2,n+1) \approx K(1,2)^{2^{n+1}-1}2^{(n+1)n^2}$ implies that the number of $(n+1)$-dimensional spaces of n by n matrices over $GF(2)$ of constant rank $n-1$ is $N(n,1,2,n+1)/g(n+1) \approx K(1,2)^{2^{n+1}-1}2^{(n+1)n^2}/g(n+1)$. Moreover, if conjugacy by $GL(n,2)$ acts faithfully on the set of such spaces, then the number of orbits under conjugacy $\approx K(1,2)^{2^{n+1}-1}2^{(n+1)n^2}/(g(n)g(n+1))$. If it is not faithful, then the number will be slightly larger (but not by orders of magnitude - see the examples for $n = 3$ in Section 2 where the stabilizers all have order ≤ 6).

For $n = 1, \ldots, 10$, this gives (to 4 significant figures) respectively $0.1285, 0.08713, 5.388, 244200, 6.783\times10^{12}, 1.162\times10^{21}, 1.868\times10^{24}, 1.006\times10^{9}, 3.562\times10^{-54}, 4.986\times10^{-223}$. It is easy to see that our estimate on the number of orbits is tending to zero very fast. The above data suggests the following:

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CONJECTURE 3.1. *There exists an* $(n+1)$*-dimensional space of* n *by* n *matrices over* $GF(2)$ *of constant rank* $n-1$ *if and only if* $3 \leq n \leq 8$*.*

Our results in Section 2 prove this for $n \leq 5$. Note also that for $n = 3$ the heuristic predicts about 5.388 orbits or equivalently about 905 spaces of dimension 4 and constant rank 2, whereas there are actually 1176 of them.

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