Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

ELA

EVENTUALLY CYCLIC MATRICES AND A TEST FOR STRONG EVENTUAL NONNEGATIVITY*

LESLIE HOGBEN†

Abstract. Eventually r-cyclic matrices are defined, and it is shown that if A is an eventually r-cyclic matrix A having rank A^2 = rank A, then A is r-cyclic with the same cyclic structure. This result and known Perron-Frobenius theory of eventually nonnegative matrices are used to establish an algorithm to determine whether a matrix is strongly eventually nonnegative (i.e., is an eventually nonnegative matrix having a power that is both irreducible and nonnegative).

Key words. Eventually nonnegative matrix, Eventually r-cyclic matrix, Strongly eventually nonnegative matrix, Perron-Frobenius.

AMS subject classifications. 15B48, 05C50, 15A18.

1. Introduction. A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually nonnegative* (respectively, *eventually positive*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k \geq 0$ (respectively, $A^k > 0$), and the least such k_0 is called the *power index* of A. A matrix $A \in \mathbb{R}^{n\times n}$ is *strongly eventually nonnegative* if A is eventually nonnegative and there is a positive integer k such that $A^k \geq 0$ and A^k is irreducible [4].

For a fixed n, the power index of an eventually positive or eventually nonnegative $n \times n$ matrix may be arbitrarily large, so it is not possible to show a matrix is not eventually positive or eventually nonnegative by computing powers. Eventual positivity is characterized by Perron-Frobenius properties, which provide necessary and sufficient conditions to determine whether a matrix is eventually positive. Unfortunately, nilpotent matrices, which have no Perron-Frobenius structure, are eventually nonnegative, and there is no known "if and only if" test using Perron-Frobenius-type properties for eventual nonnegativity. Strongly eventually nonnegative matrices are a subset of the eventually nonnegative matrices having weaker connections with Perron-Frobenius theory than eventually positive matrices, but still allowing an "if and only if" test, presented here in Algorithm 3.1, which provides a way to show a matrix is not strongly eventually nonnegative. The proof of the algorithm is based on results from the literature and the result that if rank A^2 = rank A and A is eventually r-cyclic, then A is r-cyclic (Corollary 2.8 below).

*Received by the editors on May 3, 2010. Accepted for publication on October 16, 2010. Handling Editor: Judith McDonald.

†Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu) & American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

ELA

Throughout this paper all matrices are real. An eigenvalue λ of A is a *dominant* eigenvalue if $|\lambda| = \rho(A)$ (where $\rho(A)$ denotes the spectral radius). A matrix is eventually positive if and only if $\rho(A)$ is a simple eigenvalue having positive right and left eigenvectors and A has no other dominant eigenvalue [6].

Just as digraphs are central to the Perron-Frobenius theory of nonnegative matrices, they are central to our analysis of strongly eventually nonnegative matrices, and we need additional notation and terminology. A *digraph* $\Gamma = (V, E)$ consists of a finite, nonempty set V of vertices, together with a set $E \subseteq V \times V$ of arcs. Note that a digraph allows loops (arcs of the form (v, v)) and may have both arcs (v, w) and (w, v) but not multiple copies of the same arc.

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. The *digraph of* A, denoted $\Gamma(A)$, has vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : a_{ij} \neq 0\}$. If $R, C \subseteq \{1, 2, \ldots, n\}$, then $A[R|C]$ denotes the *submatrix* of A whose rows and columns are indexed by R and C, respectively. If $C = R$, then $A[R|R]$ can be abbreviated to $A[R]$. For a digraph $\Gamma = (V, E)$ and $W \subseteq V$, the *induced subdigraph* $\Gamma[W]$ is the digraph with vertex set W and arc set $\{(v, w) \in E : v, w \in W\}$. For a square matrix A, $\Gamma(A[W])$ is identified with $\Gamma(A)[W]$ by a slight abuse of notation.

A square matrix A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are nonempty square matrices and 0 is a (possibly rectangular) block consisting entirely of zero entries, or A is the 1×1 zero matrix. If A is not reducible, then A is called *irreducible*. A digraph Γ is *strongly connected* (or *strong*) if for any two distinct vertices v and w of Γ, there is a walk in Γ from v to w. It is well known that for $n \geq 2$, A is irreducible if and only if $\Gamma(A)$ is strong. For a strong digraph Γ, the *index of imprimitivity* is the greatest common divisor of the the lengths of the closed walks in Γ. A strong digraph is *primitive* if its index of imprimitivity is one; otherwise it is *imprimitive*. The *strong components* of Γ are the maximal strongly connected subdigraphs of Γ.

For $r \geq 2$, a digraph $\Gamma = (V, E)$ is *cyclically r-partite* if there exists an ordered partition $(V_1, \ldots, V_r)$ of V into r nonempty sets such that for each arc $(i, j) \in E$, there exists $\ell \in \{1, \ldots, r\}$ with $i \in V_\ell$ and $j \in V_{\ell+1}$ (where we adopt the convention that index $r + 1$ is interpreted as 1). For $r \geq 2$, a strong digraph Γ is cyclically r-partite if and only if r divides the index of imprimitivity (see, for example, [2, p. 70]). For $r \geq 2$, a matrix $A \in \mathbb{R}^{n\times n}$ is called *r-cyclic* if $\Gamma(A)$ is cyclically r-partite. If $\Gamma(A)$ is cyclically r-partite with ordered partition Π, then we say A is *r-cyclic with partition* Π, or Π *describes* the r-cyclic structure of A. The ordered partition $\Pi = (V_1, \ldots, V_r)$ is *consecutive* if $V_1 = \{1, \ldots, i_1\}, V_2 = \{i_1 + 1, \ldots, i_2\}, \ldots, V_r = \{i_{r-1} + 1, \ldots, n\}$. If

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

A is r-cyclic with consecutive ordered partition Π, then A has the block form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r} \\ A_{r1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{1.1}$$

where $A_{i,i+1} = A[V_i|V_{i+1}]$. For any r-cyclic matrix A, there exists a permutation matrix P such that PAP^T is r-cyclic with consecutive ordered partition. The *cyclic index* of A is the largest r for which A is r-cyclic.

An irreducible nonnegative matrix B is *primitive* if $\Gamma(B)$ is primitive, and the *index of imprimitivity* of B is the index of imprimitivity of $\Gamma(B)$. It is well known that a nonnegative matrix is primitive if and only if it is eventually positive. Let $B \geq 0$ be irreducible with index of imprimitivity $r \geq 2$. Then r is the cyclic index of B, $\Gamma(B)$ is cyclically r-partite with ordered partition $\Pi = (V_1, \dots, V_r)$, and the sets V_i are uniquely determined (up to cyclic permutation of the V_i) (see, for example, [2, p. 70]). Furthermore, $\Gamma(B^r)$ is the disjoint union of r primitive digraphs on the sets of vertices $V_i, i = 1, \dots, r$ (see, for example, [8, Fact 29.7.3]).

Section 2 presents the definition of eventually r-cyclic matrices and some of their properties, including that if $\operatorname{rank} A^2 = \operatorname{rank} A$ and A is eventually r-cyclic, then A is r-cyclic. It is also shown there that a strongly eventually nonnegative matrix is eventually r-cyclic or eventually positive. These results are used in Section 3 to establish the validity of Algorithm 3.1, which tests whether a matrix is strongly eventually nonnegative; examples illustrating the use of the algorithm are included.

2. Eventually r-cyclic matrices.

Definition 2.1. For an ordered partition $\Pi = (V_1, \dots, V_r)$ of $\{1, \dots, n\}$ into r nonempty sets, the *cyclic characteristic matrix* $C_\Pi = [c_{ij}]$ of Π is the $n \times n$ matrix such that $c_{ij} = 1$ if there exists $\ell \in \{1, \dots, r\}$ such that $i \in V_\ell$ and $j \in V_{\ell+1}$, and $c_{ij} = 0$ otherwise.

Note that for any ordered partition $\Pi = (V_1, \dots, V_r)$ of $\{1, \dots, n\}$ into r nonempty sets, C_Π is r-cyclic, and $\Gamma(C_\Pi)$ contains every arc (v, w) for $v \in V_\ell$ and $w \in V_{\ell+1}$.

Definition 2.2. For matrices $A = [a_{ij}], C = [c_{ij}] \in \mathbb{R}^{n\times n}$, matrix A is *conformal* with C if for all $i, j = 1, \dots, n$, $c_{ij} = 0$ implies $a_{ij} = 0$. Equivalently, A is conformal with C if $\Gamma(A)$ is a subdigraph of $\Gamma(C)$ (with the same set of vertices).

Let Π be an ordered partition into r nonempty sets. Then A is r-cyclic with partition Π if and only if A is conformal with C_Π.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

OBSERVATION 2.3. If $A, B, C, D \in \mathbb{R}^{n\times n}$, $C, D \geq 0$, A is conformal with C and B is conformal with D, then AB is conformal with CD. If A is an r-cyclic matrix with partition Π, then A^k is conformal with ${C_\Pi}^k$.

OBSERVATION 2.4. Let $B \geq 0$ be irreducible with index of imprimitivity $r \geq 2$ and let Π describe the r-cyclic structure of B. Then for d large enough, C_Π is conformal with B^{dr+1}, i.e., $\Gamma(B^{dr+1}) = \Gamma(C_\Pi)$.

DEFINITION 2.5. A matrix A is *eventually r-cyclic* if there exists an ordered partition Π of $\{1, \dots, n\}$ into $r \geq 2$ nonempty sets, and a positive integer m such that for all $k \geq m$, A^k is conformal with ${C_\Pi}^k$. In this case, we say that Π *describes the eventually r-cyclic structure of* A. The *eventually cyclic index* of A is the largest r for which A is eventually r-cyclic.

Many eventual properties, such as eventual positivity or eventual nonnegativity, can be established by establishing the property for two consecutive powers of a matrix. The following proposition shows this is sufficient for eventually r-cyclic matrices.

PROPOSITION 2.6. *If A is a matrix and for some nonnegative integer d, A^{dr+1} is r-cyclic with partition Π and A^{dr} is conformal with C_Π^r, then A is eventually r-cyclic and Π describes the eventually r-cyclic structure of A.*

Proof. For every positive integer k sufficiently large, there exist $a, b \geq 0$ such that $k = a(dr) + b(dr + 1)$ (see e.g., [2, Lemma 3.5.5]). Fix $k = a(dr) + b(dr + 1)$. Then $A^k = A^{a(dr)+b(dr+1)} = (A^{dr})^a(A^{dr+1})^b$ is conformal with $({C_\Pi}^r)^a {C_\Pi}^b$, which is conformal with ${C_\Pi}^{adr}{C_\Pi}^{b(dr+1)} = {C_\Pi}^k$. ∎

For any square matrix A, $\operatorname{rank} A^2 = \operatorname{rank} A$ if and only if the degree of 0 as a root of the minimal polynomial of A is at most 1. The combinatorial structure of eventually nonnegative matrices with this property was studied in [3], where it is shown that if A is an irreducible eventually nonnegative matrix such that $\operatorname{rank} A^2 = \operatorname{rank} A$, then some power of A is irreducible and nonnegative, i.e., A is strongly eventually nonnegative. A matrix with the property that $\operatorname{rank} A^2 = \operatorname{rank} A$ behaves very nicely in regard to being eventually r-cyclic, because this property eliminates issues caused by a nonzero nilpotent part. The following notation will be used in the next proof. The *nullspace* of a (possibly rectangular) $p \times q$ matrix M is $\text{NS}(M) = \{\mathbf{v} \in \mathbb{R}^q : M\mathbf{v} = 0\}$, and the *left nullspace* of M is $\text{LNS}(M) = \{\mathbf{w} \in \mathbb{R}^p : \mathbf{w}^T M = 0\}$.

THEOREM 2.7. *If $A \in \mathbb{R}^{n\times n}$, $\operatorname{rank} A^2 = \operatorname{rank} A$, and there is a positive integer m divisible by r such that A^{m+1} is r-cyclic with partition Π and A^m is conformal with ${C_\Pi}^r$, then A is r-cyclic with partition Π.*

Proof. Assume that A, m, r and $\Pi = (V_1, \dots, V_r)$ satisfy the hypotheses. Since $\operatorname{rank} A^2 = \operatorname{rank} A$, for every positive integer k, $\operatorname{rank} A^k = \operatorname{rank} A$. Thus, $\text{NS}(A^k) =$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

NS(A) and LNS(A^k) = LNS(A).

Initially, we assume that Π is consecutive. Partition $A = [A_{ij}]$ where $A_{ij} = A[V_i|V_j]$. By hypothesis, $A^m = B_1 \oplus \cdots \oplus B_r$ is a block diagonal matrix, and thus

$$\begin{aligned}\mathrm{NS}(A^m) &= \{[\mathbf{v}_1^T, \ldots, \mathbf{v}_r^T]^T : \mathbf{v}_\ell \in \mathrm{NS}(B_\ell), \ell = 1, \ldots, r\},\\ \mathrm{LNS}(A^m) &= \{[\mathbf{w}_1^T, \ldots, \mathbf{w}_r^T]^T : \mathbf{w}_\ell \in \mathrm{LNS}(B_\ell), \ell = 1, \ldots, r\}.\end{aligned}$$

For $\mathbf{v}_\ell \in \mathrm{NS}(B_\ell)$, define $\hat{\mathbf{v}}_\ell = [0^T, \ldots, 0^T, \mathbf{v}_\ell^T, 0^T, \ldots, 0^T]^T$, so $A^m \hat{\mathbf{v}}_\ell = 0$. Since $\mathrm{NS}(A) = \mathrm{NS}(A^m)$,

$$0 = A\hat{\mathbf{v}}_\ell = \begin{bmatrix} A_{1\ell}\mathbf{v}_\ell \\ \vdots \\ A_{r\ell}\mathbf{v}_\ell \end{bmatrix},$$

and so $A_{i\ell}\mathbf{v}_\ell = 0, i = 1, \ldots, r$. Similarly, $\mathbf{w}_\ell^T A_{\ell j} = 0^T, j = 1, \ldots, r$ for $\mathbf{w}_\ell \in \mathrm{LNS}(B_\ell)$. That is, for all $i, j = 1, \ldots, r$,

$$\mathrm{NS}(B_\ell) \subseteq \mathrm{NS}(A_{i\ell}) \qquad \text{and} \qquad \mathrm{LNS}(B_\ell) \subseteq \mathrm{LNS}(A_{\ell j}). \tag{2.1}$$

Now consider

$$A^{m+1} = A^m A = \begin{bmatrix} B_1A_{11} & B_1A_{12} & \ldots & B_1A_{1r} \\ B_2A_{21} & B_2A_{22} & \ldots & B_2A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_rA_{r1} & B_rA_{r2} & \ldots & B_rA_{rr} \end{bmatrix}.$$

Since A^{m+1} is conformal with C_Π,

$$B_\ell A_{\ell j} = 0 \ \text{ unless } \ j \equiv \ell + 1 \mod r.$$

Since $B_\ell \mathbf{v} = 0$ implies $A_{i\ell}\mathbf{v} = 0, i = 1, \ldots, r$,

$$A_{i\ell}A_{\ell j} = 0 \ \text{ unless } \ j \equiv \ell + 1 \mod r. \tag{2.2}$$

By considering $A^{m+1} = AA^m$ and the left null space,

$$A_{i\ell}A_{\ell j} = 0 \ \text{ unless } \ i \equiv \ell - 1 \mod r. \tag{2.3}$$

So the only product of the form $A_{i\ell}A_{\ell j}$ that is not required to be 0 is $A_{\ell-1,\ell}A_{\ell,\ell+1}$ (with indices mod r). Thus,

$$B_\ell = (A_{\ell,\ell+1} \cdots A_{r1}A_{12} \cdots A_{\ell-1,\ell})^{m/r},$$

so $\mathrm{NS}(A_{\ell-1,\ell}) \subseteq \mathrm{NS}(B_\ell)$ and $\mathrm{LNS}(A_{\ell,\ell+1}) \subseteq \mathrm{LNS}(B_\ell)$. Then by (2.1),

$$\mathrm{NS}(A_{\ell-1,\ell}) = \mathrm{NS}(B_\ell) \qquad \text{and} \qquad \mathrm{LNS}(A_{\ell,\ell+1}) = \mathrm{LNS}(B_\ell). \tag{2.4}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

ELA

So by (2.1), $\mathrm{NS}(A_{\ell-1,\ell}) \subseteq \mathrm{NS}(A_{i,\ell})$ for $i = 1, \ldots, r$. This implies that for each i there exists a (possibly rectangular) matrix M_i such that

$$A_{i,\ell} = M_i A_{\ell-1,\ell}. \tag{2.5}$$

So for $i \not\equiv \ell - 1 \mod r$,

$$\begin{aligned}
0 &= \mathrm{rank}(A_{i\ell} A_{\ell,\ell+1}) && \text{by (2.3)}\\
&= \mathrm{rank}(M_i A_{\ell-1,\ell} A_{\ell,\ell+1}) && \text{by (2.5)}\\
&\geq \mathrm{rank}(M_i A_{\ell-1,\ell}) + \mathrm{rank}(A_{\ell-1,\ell} A_{\ell,\ell+1}) - \mathrm{rank}(A_{\ell-1,\ell}) && \text{by [9, (2.7)]}\\
&= \mathrm{rank}(M_i A_{\ell-1,\ell}) && \text{because } \mathrm{LNS}(A_{\ell-1,\ell} A_{\ell,\ell+1}) = \mathrm{LNS}(A_{\ell-1,\ell}) \text{ from (2.4)}\\
&= \mathrm{rank}(A_{i\ell}) && \text{by (2.5).}
\end{aligned}$$

Thus, $A_{i\ell} = 0$ for $i \not\equiv \ell - 1 \mod r$, and A is r-cyclic with partition Π.

Without the assumption that Π is consecutive, there exists a permutation matrix P such that $(PAP^T)^{m+1} = PA^{m+1}P^T$ is r-cyclic with consecutive partition Π' and $(PAP^T)^m = PA^mP^T$ is conformal with ${C_{\Pi'}}^r$. Since $\mathrm{rank}(PAP^T)^2 = \mathrm{rank}(PAP^T)$, $(PAP^T)_{ij} = 0$ unless $j \equiv i + 1 \mod r$ (using the block structure of $C_{\Pi'}$). Thus, A is r-cyclic with partition Π. ☐

COROLLARY 2.8. *Let $A \in \mathbb{R}^{n\times n}$ have $\mathrm{rank}\, A^2 = \mathrm{rank}\, A$. Then A is eventually r-cyclic if and only if A is r-cyclic.*

We now return to strongly eventually nonnegative matrices. We need a preliminary lemma.

LEMMA 2.9. *If A and B are $n \times n$ nonnegative matrices having all diagonal entries positive, then $\Gamma(A) \cup \Gamma(B) \subseteq \Gamma(AB)$.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. If $(u, v) \in \Gamma(A)$, then

$$(AB)_{uv} = \sum_{i=1}^{n} a_{ui} b_{iv} \geq a_{uv} b_{vv} > 0,$$

so $(u, v) \in \Gamma(AB)$. Thus, $\Gamma(A) \subseteq \Gamma(AB)$. The case $\Gamma(B) \subseteq \Gamma(AB)$ is similar. ☐

REMARK 2.10. Let A be a strongly eventually nonnegative matrix with power index k_0 and r dominant eigenvalues. Since A is eventually nonnegative, $\rho(A)$ is an eigenvalue of A [5]. There is a positive integer k such that $A^k \geq 0$ and A^k is irreducible. For any such k, A^k has positive left and right eigenvectors for its spectral radius; the same is true for every power of A, including A itself. If $r = 1$, then A is eventually positive [6]. Now assume $\ell \geq k_0$ such that $\rho(A^\ell)$ is simple. Then A^ℓ is irreducible [1, Corollary 2.3.15], and so the r dominant eigenvalues of A^ℓ are $\{\rho(A^\ell), \rho(A^\ell)\omega, \ldots, \rho(A^\ell)\omega^{r-1}\}$ where ω is a primitive rth root of unity [1, Theorem

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

2.2.20]. Furthermore, A^ℓ is r-cyclic [1, Theorem 2.2.20]. Note that $\ell \geq k_0$ with $\rho(A^\ell)$ simple necessarily exists (e.g., $\ell = kk_0r+1$). If $u \geq k_0$ and $\rho(A^u)$ is simple, then any positive integers x, y divisible by r, the multiplicity of $\rho(A^{\ell x+uy}) = \rho(A^\ell)^x\rho(A^u)^y)$ is r, so $\Gamma(A^{\ell x+uy})$ has r strong components [1, Theorem 2.3.14].

THEOREM 2.11. *Let A be strongly eventually nonnegative matrix A having $r \geq 2$ dominant eigenvalues and power index k_0. Then there exists a positive integer $m \geq k_0$ divisible by r such that A^{m+1} is r-cyclic with partition Π and A^m is conformal with ${C_\Pi}^r$.*

Proof. Choose a positive integer $\ell \geq k_0$ such that $\rho(A^\ell)$ is simple, so A^ℓ is irreducible and r-cyclic, and let $\Pi = (V_1, \dots, V_r)$ denote an ordered partition that describes the r-cyclic structure of A^ℓ. Let $m = \ell r$; the spectral radius of $A^{m+1} \geq 0$ is simple, and thus A^{m+1} is irreducible and r-cyclic. Let $\Psi = (W_1, \dots, W_r)$ be an ordered partition that describes the r-cyclic structure of A^{m+1}. It suffices to show that A^m is conformal with ${C_\Psi}^m$. Note that for an r-cyclic matrix, in any power that is a multiple of r, the order of the sets in the partition is irrelevant, since all arcs are within partition sets. Thus, it suffices to show that the unordered sets $\{V_1, \dots, V_r\}$ and $\{W_1, \dots, W_r\}$ are equal.

By Observation 2.4, we can choose s large enough so that the diagonal blocks $A^{\ell rs}[V_i]$ and $A^{(m+1)rs}[W_i]$ are positive for $i = 1, \dots, r$. Since $A^{\ell rs}A^{(m+1)rs} = A^{\ell(rs)+(m+1)(rs)}$, $\Gamma(A^{\ell rs}A^{(m+1)rs})$ has r strong components. Since all diagonal entries of $\Gamma(A^{\ell rs})$ and $\Gamma(A^{(m+1)rs})$ are positive, by Lemma 2.9,

$$\Gamma(A^{\ell rs}) \cup \Gamma(A^{(m+1)rs}) \subseteq \Gamma(A^{\ell rs}A^{(m+1)rs}).$$

But $\Gamma(A^{\ell rs}) \cup \Gamma(A^{(m+1)rs})$ contains the complete digraphs on $V_i, i = 1, \dots, r$ and $W_i, i = 1, \dots, r$, so the only way for $\Gamma(A^{\ell rs}A^{(m+1)rs})$ to have r strong components is to have $\{V_1, \dots, V_r\} = \{W_1, \dots, W_r\}$. □

COROLLARY 2.12. *If $A \in \mathbb{R}^{n\times n}$ is strongly eventually nonnegative with $r \geq 2$ dominant eigenvalues, then A is eventually r-cyclic.*

COROLLARY 2.13. *If $A \in \mathbb{R}^{n\times n}$ is strongly eventually nonnegative with $r \geq 2$ dominant eigenvalues and* $\operatorname{rank} A^2 = \operatorname{rank} A$, *then A is r-cyclic.*

3. Testing for strong eventual nonnegativity.

In this section, we provide an algorithm to test whether a matrix is strongly eventually nonnegative and prove that it works, illustrate the algorithm with examples, and discuss computational issues related to the algorithm.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

ELA

3.1. Algorithm and proof.

ALGORITHM 3.1. Test a matrix for strong eventual nonnegativity.
Let A be an $n \times n$ real matrix.

1. *Compute the spectrum $\sigma(A)$, set r equal to the number of dominant eigenvalues, and set $\omega = e^{2\pi i/r}$.*
2. *If the multiset of dominant eigenvalues is not $\{\rho(A), \rho(A)\omega, \dots, \rho(A)\omega^{r-1}\}$, then A is not strongly eventually nonnegative, else continue.*
3. *Compute eigenvectors $\mathbf{v}$ and $\mathbf{w}$ for $\rho(A)$ for A and A^T.*
4. *If $\mathbf{v}$ or $\mathbf{w}$ is not a multiple of a positive eigenvector, then A is not strongly eventually nonnegative, else continue.*
5. *If $r = 1$, then A is eventually positive (and thus is strongly eventually nonnegative), else continue.*
6. *Compute a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that*
$$A = S(\operatorname{diag}(\rho(A), \rho(A)\omega, \dots, \rho(A)\omega^{r-1}) \oplus M)S^{-1}.$$
7. *Set $B_1 = S(\operatorname{diag}(1, \omega, \dots, \omega^{r-1}) \oplus 0)S^{-1}$.*
8. *If B_1 is not nonnegative or B_1 is not r-cyclic, then A is not strongly eventually nonnegative, else continue.*
9. *Set $q = \lceil \frac{n}{r} \rceil r$. Then A is strongly eventually nonnegative if and only if A^q and A^{q+1} are conformal with ${B_1}^r$ and B_1, respectively.*

The following result will be used in the proof of Algorithm 3.1.

THEOREM 3.2. [4] *If A is strongly eventually nonnegative and has r dominant eigenvalues, then the dominant eigenvalues of A are $\{\rho(A), \rho(A)\omega, \dots, \rho(A)\omega^{r-1}\}$ where $\omega = e^{2\pi i/r}$.*

THEOREM 3.3. *Algorithm 3.1 is correct.*

Proof. The first three assertions that A is or is not strongly eventually nonnegative are justified by the following theorems:

2. Theorem 3.2.
4. Remark 2.10.
5. Remark 2.10.

There are two remaining assertions, in Steps 8 and 9. Define $B = \frac{1}{\rho(A)}A$, and note that B has one of the following properties if and only if A has the same property: nonnegative, r-cyclic, strongly eventually nonnegative, conformal with a matrix C.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

Thus, we establish the results for B rather than A. There exists a nonsingular $T \in \mathbb{R}^{(n-r)\times(n-r)}$ such that $M = T(G \oplus N)T^{-1}$ where N is nilpotent and G is nonsingular. Define $B_0 = S(0 \oplus T(G \oplus 0)T^{-1})S^{-1}$. From the definitions of B_1 and B_0,

$$\begin{aligned} {B_1}^{dr+1} &= B_1 \text{ for } d \geq 0, \qquad \rho(B_1) = 1, \qquad \rho(B_0) < 1, \\ B^k &= {B_1}^k + {B_0}^k \text{ for } k \geq n, \qquad \text{and } \operatorname{rank}(B_1 + B_0)^2 = \operatorname{rank}(B_1 + B_0). \end{aligned}$$

Thus, $\lim_{k\to\infty} {B_0}^k = 0$, and

$$\lim_{d\to\infty} B^{dr+1} = B_1. \tag{3.1}$$

Thus, if B_1 has an negative entry or is not r-cyclic, B^{dr+1} retains this property for arbitrarily large d and so B and thus A are not eventually nonnegative. This establishes the validity of Step 8.

For Step 9, we may assume that $B_1 \geq 0$ is r-cyclic with partition Π. By (3.1), for k large enough, $({B_1}^k)_{ij} > 0$ implies $(B^k)_{ij} > 0$. By the construction of B_1 from S, B_1 and B_1^T have positive eigenvectors for simple eigenvalue 1, so by [1, Corollary 2.3.15], B_1 is irreducible. Then by Observation 2.4 and the fact that ${B_1}^{dr+1} = B_1$, C_Π is conformal with B_1.

First assume B^q and B^{q+1} are conformal with ${B_1}^r$ and B_1, respectively. By Proposition 2.6, B is eventually r-cyclic and Π describes the eventually r-cyclic structure of B. So for k large enough, (3.1) implies $B^k \geq 0$ and if $\gcd(r, k) = 1$, then $\rho(B^k)$ is simple so B^k is irreducible. Thus, B is strongly eventually nonnegative.

For the converse, assume that B is strongly eventually nonnegative, so $B_1 + B_0$ is strongly eventually nonnegative. By Theorem 2.11, there exists a positive integer $m \geq k_0$ divisible by r such that $(B_1 + B_0)^{m+1}$ is r-cyclic with partition Π and $(B_1 + B_0)^m$ is conformal with ${C_\Pi}^r$. Since $\operatorname{rank}(B_1 + B_0)^2 = \operatorname{rank}(B_1 + B_0)$, by Theorem 2.7, $B_1 + B_0$ is conformal with C_Π. As a consequence of (3.1), B_1 must be r-cyclic with the same partition Π. Since $B_1 \geq 0$ and C_Π is conformal with B_1, a matrix is conformal with ${C_\Pi}^k$ if and only if it is conformal with ${B_1}^k$. Thus, $(B_1 + B_0)^q$ and $(B_1 + B_0)^{q+1}$ are conformal with ${B_1}^r$ and B_1, respectively. Since $q \geq n$, $B^q = (B_1 + B_0)^q$ and $B^{q+1} = (B_1 + B_0)^{q+1}$. □

3.2. Examples. We illustrate the algorithm with examples.

Example 3.4. Let

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 2 & -1 & 2 & 0 & 1 & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

Step 1: $\sigma(A) = \left\{4, -2+2i\sqrt{3}, -2-2i\sqrt{3}, 0, 0, 0\right\}$, so $r = 3$ and $\rho(A) = 4$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 4$ are both $[1,1,1,1,1,1,1,1]^T$. For Steps 6 and 7, a possible S and the resulting B_1 are

$$S = \begin{bmatrix} 1 & \frac{(-1-i\sqrt{3})}{2} & \frac{(-1+i\sqrt{3})}{2} & 0 & 0 & -1 \\ 1 & \frac{(-1+i\sqrt{3})}{2} & \frac{(-1-i\sqrt{3})}{2} & 0 & -\frac{1}{2} & 0 \\ 1 & \frac{(-1-i\sqrt{3})}{2} & \frac{(-1+i\sqrt{3})}{2} & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & \frac{(-1+i\sqrt{3})}{2} & \frac{(-1-i\sqrt{3})}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $B_1 \geq 0$. By examining $\Gamma(B_1)$ we see that B_1 is 3-cyclic with partition $(\{1,3\},\{2,5\},\{4,6\})$. Computations then verify that A^6 and A^7 are conformal with ${B_1}^6$ and B_1, respectively, so B is strongly eventually nonnegative.

Example 3.5. Let

$$A = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & -\frac{3}{4} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}.$$

Step 1: $\sigma(A) = \{2, -2, -1, -1, 1, 1, 0, 0\}$, so $r = 2$ and $\rho(A) = 2$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 2$ are both $[1,1,1,1,1,1,1,1]^T$. For Steps 6 and 7, a possible S and the resulting B_1 are

$$S = \begin{bmatrix} 1 & -1 & -1 & 8 & 0 & 0 & 0 & -4 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -8 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & -1 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \end{bmatrix},$$

and B_1 is clearly nonnegative and 2-cyclic. Step 9: Computations show A^9 is not conformal with B_1, so A is not strongly eventually nonnegative.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

EXAMPLE 3.6. Let

$$A = \begin{bmatrix} 0 & 0 & 45 & 1155 \\ 0 & 0 & 2097 & -897 \\ 871 & 329 & 0 & 0 \\ 187 & 1013 & 0 & 0 \end{bmatrix}.$$

Step 1: $\sigma(A) = \{1200, -1200, 684i\sqrt{3}, -684i\sqrt{3}\}$, so $r = 2$ and $\rho(A) = 1200$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 1200$ are $[1,1,1,1]^T$ and $[7,5,9,3]^T$, respectively. For Steps 6 and 7, a possible S and the resulting B_1 are

$$S = \begin{bmatrix} 1 & -1 & -\frac{5i}{3\sqrt{3}} & \frac{5i}{3\sqrt{3}} \\ 1 & -1 & \frac{7i}{3\sqrt{3}} & -\frac{7i}{3\sqrt{3}} \\ 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{7}{12} & \frac{5}{12} & 0 & 0 \\ \frac{7}{12} & \frac{5}{12} & 0 & 0 \end{bmatrix},$$

so $B_1 \geq 0$ and 2-cyclic. Since A is conformal with B_1, A^4 and A^5 are conformal with ${B_1}^4$ and B_1, respectively, and A is strongly eventually nonnegative.

In this particular case (because the spectrum consists entirely of real multiples of roots of unity), we can extend the spectral analysis in the algorithm to estimate the power index of A. Set $B = \frac{1}{1200}A$ and $\alpha = \rho(B - B_1)$, and define

$$\hat{B}_0 = \frac{1}{\alpha}(B - B_1) = \begin{bmatrix} 0 & 0 & -\frac{5}{4\sqrt{3}} & \frac{5}{4\sqrt{3}} \\ 0 & 0 & \frac{7}{4\sqrt{3}} & -\frac{7}{4\sqrt{3}} \\ \frac{1}{4\sqrt{3}} & -\frac{1}{4\sqrt{3}} & 0 & 0 \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 \end{bmatrix}.$$

Since $\sigma(\hat{B}_0) = \{i, -i, 0, 0\}$, ${\hat{B}_0}^{4k+1} = \hat{B}_0$. Solving $\alpha^k |(\hat{B}_0)_{24}| = (B_1)_{24}$ yields $k = 109.001$, and in fact $A^{109} \not\geq 0$, but A is nonnegative thereafter.

3.3. Computational issues. The computations in Examples 3.4, 3.5 and 3.6 were all done in exact arithmetic, so there was no issue of roundoff error. However, eigenvalues will generally need to be computed as decimal approximations, and round-off error is an issue. Fortunately, to implement Algorithm 3.1 it is not necessary to compute Jordan forms (or eigenvectors for repeated eigenvalues), which are difficult to do in decimal arithmetic. If the matrix A is eventually nonnegative, then the dominant eigenvalues are simple and well spread out. The accuracy of the computations will depend on the condition number of each dominant eigenvalue, which in turn depends on the angle between the eigenvectors of A and A^T (see, for example, [7, p. 323]). Step 6 of Algorithm 3.1 requires computing a matrix $S = [\mathbf{s}_1, \ldots, \mathbf{s}_n]$ such that

$$S^{-1}AS = \operatorname{diag}(\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}) \oplus M.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 129-140, November 2010

ELA

This can be done as follows:

- Compute eigenvectors $\mathbf{s}_1, \ldots, \mathbf{s}_r$ for the dominant eigenvalues $\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}$.
- Extend $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ to a basis $\{\mathbf{s}_1, \ldots, \mathbf{s}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$.
- Set $U = [\mathbf{s}_1, \ldots, \mathbf{s}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n]$. Then

$$U^{-1}AU = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix} \text{ where } H_{11} = \operatorname{diag}(\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}).$$

- Since $\sigma(H_{11}) \cap \sigma(H_{22}) = \emptyset$, by [7, Lemma 7.1.5], we can solve a system of linear equations to find a matrix $Z \in \mathbb{R}^{r\times(n-r)}$ such that $H_{11}Z - ZH_{22} = -H_{12}$.
- Then for $Y = \begin{bmatrix} I_r & Z \\ 0 & I_{n-r} \end{bmatrix}$, $Y^{-1}U^{-1}AUY = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}$, and $S = UY$ is a satisfactory matrix for Step 6.

Acknowledgements. The author thanks M. Catral, C. Erickson, D.D. Olesky, and P. van den Driessche for many enjoyable and fruitful discussions that led to the study of strongly eventually nonnegative matrices, and thanks the referee for helpful suggestions to improve the exposition.

REFERENCES

[1] A. Berman and B. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* SIAM, Philadelphia, 1994.

[2] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, New York, 1991.

[3] S. Carnochan Naqvi and J.J. McDonald. The combinatorial structure of eventuallynonnegative matrices. *Electronic Journal of Linear Algebra*, 9:255–269, 2002.

[4] M. Catral, C. Erickson, L. Hogben, D.D. Olesky, and P. van den Driessche. Eventually nonnegative matrices and related classes. Submitted.

[5] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29:43–60, 1978.

[6] D. Handelman. Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains. *Journal of Operator Theory*, 6:55-74, 1981.

[7] G.H. Golub and C.F. Van Loan. *Matrix Computations*, 3rd edition. Johns Hopkins University Press, Baltimore, 1996.

[8] J.L. Stuart. Digraphs and matrices. In L. Hogben, editor, *Handbook of Linear Algebra*, Chapman & Hall/CRC Press, Boca Raton, 2007.

[9] F. Zhang. *Matrix Theory: Basic Results and Techniques.* Springer, New York, 1999.