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UNICYCLIC 3-COLORED DIGRAPHS WITH BICYCLIC INVERSES*

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Abstract. The class of unicyclic 3-colored digraphs with the cycle weight $\pm\mathrm{i}$ and with a unique perfect matching, denoted by $\mathcal{U}_g$, is considered in this article. Kalita & Sarma [On the inverse of unicyclic 3-coloured digraphs, *Linear and Multilinear Algebra*, DOI: 10.1080/03081087.2021.1948956] introduced the notion of inverse of 3-colored digraphs. They characterized the unicyclic 3-colored digraphs in $\mathcal{U}_g$ possessing unicyclic inverses. This article provides a complete characterization of the unicyclic 3-colored digraphs in $\mathcal{U}_g$ possessing bicyclic inverses.

Key words. Adjacency matrix, Corona graph, Inverse Graph, Bicyclic graph, 3-colored digraph, Weighted directed graph.

AMS subject classifications. 05C50, 15A18, 15A09, 05C20, 05C22.

1. Introduction. The notion of weighted directed graph was first introduced by Bapat et al. in [2]. A *weighted directed graph* is a directed graph with a simple underlying undirected graph and edges having complex weights of unit modulus, see [2]. Let G be a weighted directed graph. Throughout the article, $(i, j) \in E(G)$ means there exists a directed edge from the vertex i to j in G. We denote the weight of an edge (i, j) by w_{ij}. The adjacency matrix $A(G) = [a_{ij}]$ of G is the matrix with

$$a_{ij} = \begin{cases} \mathsf{w}_{ij} & \text{if } (i,j) \in E(G), \\ \overline{\mathsf{w}}_{ji} & \text{if } (j,i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

A 3-colored digraph is a weighted directed graph with edge weights chosen from the set $\{1, -1, \mathrm{i}\}$, where $\mathrm{i} = \sqrt{-1}$, see [2]. The edges with weight 1, -1, and i in a 3-colored digraph can be viewed as red edges, blue edges, and green edges, respectively, see [2]. In particular, if all the edges in a 3-colored digraph are red, then entries of $A(G)$ are 0 or 1, that is, $A(G)$ coincides with the adjacency matrix of the underlying unweighted undirected graph. Note that, as far as the study of the adjacency matrix of such a 3-colored digraph is concerned, the directions of the edges do not play any role.

The adjacency matrix of a 3-colored digraph is either singular or nonsingular. A 3-colored digraph G is said to be *singular* (respectively, *nonsingular*) if $A(G)$ is singular (respectively, nonsingular). The study of inverse of an unweighted undirected graph was first started by Harray and Minc, see [4]. They supplied a definition of inverse graph which says that 'an unweighted undirected graph G is invertible if $A(G)$ is nonsingular and $A(G)^{-1}$ is a matrix with entries 0 or 1'. Using this definition, they found that there is only one invertible connected graph, namely $G = P_2$, see [4]. This led Godsil [1] to introduce a different notion of inverse of a graph which is quite similar to that of Harray and Minc. Godsil supplied the following definition of inverse: 'a graph G has an *inverse* if $A(G)$ is nonsingular and there exists a signature matrix (a diagonal matrix with diagonal entries ±1) S such that the matrix $SA(G)^{-1}S$ is nonnegative'. Using

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this definition of inverse, several researchers have studied the inverses of bipartite graphs. The inverses of bipartite graphs with a unique perfect matching have been a subject of study for many researchers, see for example [10, 11, 8, 9, 1, 12, 13] and the references therein. A graph G on n vertices is said to be k-cyclic if G has $n-1+k$ edges. In particular, for $k=1,2$, we call G to be unicyclic and bicyclic graph, respectively. In [6], the authors have studied the inverses of non-bipartite, unicyclic unweighted undirected graphs with a unique perfect matching. They characterized non-bipartite, unicyclic graphs U with unique perfect matching for which U^+ is unicyclic and bicyclic. The notion of inverse of a 3-colored digraph was first studied in [7]. The authors adapted the following definition of inverse in [7].

Let G be a 3-colored digraph. Suppose that $A(G)$ is nonsingular and $A(G)^{-1}=(\alpha_{ij})$. The weighted directed graph denoted by G^+ is the graph on the same vertex set as that of G constructed as follows: *two distinct vertices i and j are adjacent in G^+ if and only if $\alpha_{ij}\neq 0$.* The graph G^+ is known as the *inverse* graph of G, see [7]. One can view $A(G)^{-1}$ as the adjacency matrix $A(G^+)$ of G^+.

The following definitions are essentially contained in [2] which is crucial for further development.

Definition 1.1. [2] *A i_1-i_k-walk W in a 3-colored digraph G is a finite sequence $i_1,\ldots,i_k$ of vertices such that, for $1\le p\le k-1$, either $(i_p,i_{p+1})\in E(G)$ or $(i_{p+1},i_p)\in E(G)$. Then $\mathsf{w}_W=a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$, where a_{ij} are entries of $A(G)$, is called the weight of the walk W. It is convenient to write the walk W as $W=[i_1,i_2,\ldots,i_k]$.*

Definition 1.2. *[5] Let H and G be 3-colored digraphs. Then H is said to be I-diagonally similar to G if there is a diagonal matrix D with diagonal entries from the set $I=\{\pm1,\pm\mathrm{i}\}$ such that $A(H)=D^*A(G)D$.*

The following lemma is crucial for further development.

Lemma 1.3. [5, *Lemma* 5] *Let G be a connected, unicyclic, 3-colored digraph with the cycle C. Then the following hold:*

(a) *If $\mathsf{w}_C=1$, then G is I-diagonally similar to a unicyclic graph with all edges red.*
(b) *If $\mathsf{w}_C=-1$, then G is I-diagonally similar to a unicyclic graph with all edges red except anyone edge on C which is blue.*
(c) *If $\mathsf{w}_C=\pm\mathrm{i}$, then G is I-diagonally similar to a unicyclic graph with all edges red except anyone edge on C which is green.*

It follows from Lemma 1.3 that the class of unicyclic 3-colored digraphs can be partitioned into three distinct classes. In view of Lemma 1.3, the study of inverses on the class of unicyclic 3-colored digraphs with the cycle weight 1 is the same as that of the class of unicyclic unweighted undirected graphs, which can be found in [10, 9, 6, 11, 8] and the references therein. Furthermore, the study of the inverses of the class of unicyclic 3-colored digraphs with the cycle weight -1 is quite similar to the unweighted undirected case. Thus in [7], the authors studied the inverses of unicyclic 3-colored digraphs with the cycle weight $\pm\mathrm{i}$. They considered the class of such unicyclic 3-colored digraphs having a unique perfect matching, denoted by $\mathcal{U}_g$. In [7], authors characterized the 3-colored digraphs in $\mathcal{U}_g$ whose inverses are again 3-colored digraphs. They supplied a complete characterization of unicyclic 3-colored digraphs in $\mathcal{U}_g$ possessing unicyclic inverses. So it is natural to wonder which graphs in $\mathcal{U}_g$ possess bicyclic inverses. In this article, we characterize the unicyclic graphs $U\in\mathcal{U}_g$ for which U^+ is bicyclic.

The article is organized as follows. In Section 2, we state some preliminary results and definitions. In Section 3, we supply constructions of four different types of unicyclic graphs in $\mathcal{U}_g$ possessing bicyclic

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inverses. Finally, we prove that these four types of graphs are the only possible unicyclic 3-colored graphs in $\mathcal{U}_g$ that possess bicyclic inverses.

2. Preliminaries. In this section, we discuss some preliminary results that are essential for the main purpose of the article. In defining graph properties such as subgraph, walk, path, matching in a 3-colored digraph, we focus only on the underlying unweighted undirected graph.

The following lemma contained in [7] guarantees that a unicyclic 3-colored digraph in $\mathcal{U}_g$ is always nonsingular.

LEMMA 2.1. [7, *Corollary* 2.3] *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$. Then* $\det(A(U)) = (-1)^{|\mathcal{M}|}$, *where $|\mathcal{M}|$ is the total number of edges in the unique perfect matching $\mathcal{M}$.*

Let G be a 3-colored digraph with a unique perfect matching $\mathcal{M}$. An edge e of G is called a *matching* edge if $e \in \mathcal{M}$, otherwise e is called a *non-matching* edge. Let i be a vertex of G. A vertex i' of G is called a *matching mate* of i if $[i, i'] \in \mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_k]$ in G is called an *alternating* path if the edges on P are alternately matching and non-matching edges. An alternating path $P = [u_1, u_2, \ldots, u_k]$ is called *mm-alternating* if $[u_1, u_2]$, $[u_{k-1}, u_k] \in \mathcal{M}$.

The next result is crucial for further development which describes the entries of the inverse of the adjacency matrix of a unicyclic 3-colored digraph.

LEMMA 2.2. [7, *Theorem* 3.3] *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$. If $A(U^+) = (\alpha_{ij})$, then*

$$\alpha_{ij} = \begin{cases} \mathsf{w}_P(-1)^{\frac{\|P\|-1}{2}} & \text{if } U \text{ has a unique } mm\text{-alternating } i-j \text{ path,} \\ \\ \mathsf{w}_P(-1)^{\frac{\|P\|-1}{2}} + \mathsf{w}_Q(-1)^{\frac{\|Q\|-1}{2}} & \text{if } U \text{ has exactly two } mm\text{-alternating } i-j \text{ paths,} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Here P and Q are mm-alternating paths in U between i and j, and $||P||$ and $||Q||$ are the lengths of P and Q, respectively.

The following lemma is essentially contained in [7, Corollary 3.4] which is an immediate consequence of Lemma 2.2.

LEMMA 2.3. [7, *Corollary* 3.4] *Let U be a unicyclic 3-colored digraph with a unique perfect matching. Let i and j be two vertices of U. Then $i \sim j$ in U^+ if and only if there is at least one mm-alternating path between i and j.*

Recall that weight of a path in a 3-colored digraph is either $1, -1$, or $\pm \mathrm{i}$. It follows from Lemma 2.2 that there is a possibility that ij-th entry α_{ij} of $A(U^+)$ is equal to ± 2 or $\pm(1 \pm \mathrm{i})$ for some i, j, if there are two mm-alternating paths between i and j. That is, U^+ might not be 3-colored digraph always. In [7], authors characterized the unicyclic 3-colored digraphs $U \in \mathcal{U}_g$ for which U^+ is a 3-colored digraph.

To proceed further, we need the following definition which can be found in [10].

DEFINITION 2.4. *Let U be a unicyclic 3-colored digraph with a unique perfect matching. Let Γ be the unique cycle in U. A matching edge e of U is said to be a* **peg** *if it is incident with exactly one vertex of Γ.*

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The following result provides a characterization of unicyclic 3-colored digraphs $U \in \mathcal{U}_g$ for which U^+ is again 3-colored digraph.

LEMMA 2.5. [7, *Theorem* 3.10] *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$. Then U^+ is a 3-colored digraph if and only if either U is non-bipartite or U has at least four pegs.*

LEMMA 2.6. *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$ such that U^+ is a 3-colored digraph. Then there is at most one mm-alternating path between any two vertices of U.*

Proof. Let $U \in \mathcal{U}_g$ and $A(U^+) = (\alpha_{ij})$. Suppose that P and Q are two distinct mm-alternating paths between the vertices i and j of U. Then the symmetric difference of P and Q yields the unique cycle, say Γ of U. Since weight of Γ is $\pm\mathrm{i}$, we see that Γ contains an odd number of green edges (edges of weight i). Thus, the number of green edges in P and in Q has the opposite parity. It follows that one of P and Q must have a weight ± 1 and the other have a weight $\pm\mathrm{i}$. Without loss of generality, assume that $\mathsf{w}_P = \pm 1, \mathsf{w}_Q = \pm\mathrm{i}$. Hence by Lemma 2.2, $A(U^+)$ is a matrix with entries from the set $\{0, \pm 1, \pm\mathrm{i}, \pm(1 \pm \mathrm{i})\}$, a contradiction. Thus, there is at most one mm-alternating path between any two vertices U. $\square$

The following result provides crucial information about the structure of the inverse graph. It says that the inverse graph always contains a cycle.

LEMMA 2.7. [7, *Lemma* 3.11] *Let $U \in \mathcal{U}_g$ be a unicyclic 3-colored digraph. Suppose that U has r pegs. Then the following holds:*

(a) *For $r \geq 3$, U^+ contains a cycle of length r and weight of the cycle is $\pm\mathrm{i}$.*
(b) *If $r = 1$, then U^+ contains a triangle with a weight $\pm\mathrm{i}$.*
(c) *If $r = 2$, then U^+ contains a cycle of length 4 with a weight $\pm(1 \pm \mathrm{i})$.*

3. Graphs in $\mathcal{U}_g$ with bicyclic inverses. In [7], the authors characterized the graphs $U \in \mathcal{U}_g$ that possess unicyclic inverses. It is therefore natural to ask : *Does there exist a graph $U \in \mathcal{U}_g$ whose inverse U^+ is bicyclic?* The following example guarantees that the answer is *affirmative*.

EXAMPLE 3.1. *Consider the graph $U \in \mathcal{U}_g$ as shown in Figure 1. The inverse graph U^+ is also shown in Figure 1. Observe that U^+ is bicyclic.*

The following result is essentially contained in [[7], Lemma 4.2].

LEMMA 3.2. *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$ such that U^+ is a 3-colored digraph. Then U^+ is unicyclic if and only if length of each mm-alternating path in U is at most 3.*

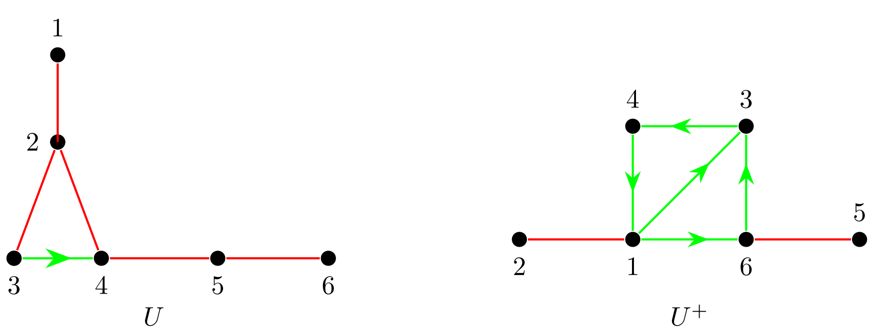

FIGURE 1. *U is unicyclic, U^+ is bicyclic.*

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The next lemma provides a necessary and sufficient condition for the unicyclic graphs in $\mathcal{U}_g$ possessing bicyclic inverses. The lemma is analogous to [11, Theorem 3.2] and [5, Theorem 4.5]. The proof is similar to that of [5, Theorem 4.5]. However, we supply the proof for completeness.

LEMMA 3.3. *Let U be a unicyclic 3-colored digraph in $\mathcal{U}_g$ such that U^+ is a 3-colored digraph. Then U^+ is bicyclic if and only if U has exactly one mm-alternating path of length 5.*

Proof. Suppose that U^+ is bicyclic. First, we show that U has an mm-alternating path of length 5. To the contrary, assume that U has no mm-alternating path of length 5. Then the length of each mm-alternating path in U is either 1 or 3. Then by Lemma 3.2, we see that the underlying graph of U^+ is unicyclic, a contradiction. Hence, there exists an mm-alternating path of length 5 in U.

Next, we show that U does not contain more than one mm-alternating path of length 5. To the contrary, assume that U has at least two mm-alternating paths of length 5. Let $P = [u_1, u_2, u_3, u_4, u_5, u_6]$ and $Q = [v_1, v_2, v_3, v_4, v_5, v_6]$ be two mm-alternating paths of length 5. By Lemma 2.3, $u_1 \sim u_6$ in U^+. Observe that $[u_3, u_4, u_5, u_6]$ is an mm-alternating path. So $u_6 \sim u_3$ in U^+, by Lemma 2.3. Similarly, we see that $u_3 \sim u_4$, $u_4 \sim u_1$ in U^+. Thus, U^+ contains the cycle $C_1 = [u_1, u_6, u_3, u_4, u_1]$. Similarly, U^+ contains the cycle $C_2 = [v_1, v_6, v_3, v_4, v_1]$. By Lemma 2.6, the end vertices of P and Q cannot be the same, that is, $\{u_1, u_6\} \neq \{v_1, v_6\}$. We now show that C_1 and C_2 are distinct. To the contrary, assume that C_1 and C_2 are not distinct. Then $\{u_1, u_6, u_3, u_4\} = \{v_1, v_6, v_3, v_4\}$. Observe that $u_1 \neq v_3$, otherwise, $u_2 = v_4$ which is not possible. Similarly, $u_1 \neq v_4$. Thus $u_1 = v_1$ or v_6. Similarly, we can show that $v_1 = u_1$ or u_6. If $u_1 = v_6$, then $v_1 = u_6$, a contradiction to $\{u_1, u_6\} \neq \{v_1, v_6\}$. Thus, $u_1 = v_1$. Since $\{u_1, u_6\} \neq \{v_1, v_6\}$ it follows that $u_6 \neq v_6$. Then $u_6 = v_3$ or v_4, which implies that $u_5 = v_4$ or v_3, which is not possible. Thus, U^+ contains two distinct 4-cycles, namely C_1 and C_2. Moreover, U^+ is non-bipartite whenever U is non-bipartite, by Lemma 2.7. Thus, U^+ contains an odd cycle. Now suppose that U is bipartite. Since U^+ is 3-colored digraph so U contains at least 4 pegs, by Theorem 2.5. By Lemma 2.7, we see that U^+ contains a cycle of length at least 4 and this cycle is formed by the end vertices of the pegs that do not lie on Γ. So for any graph $U \in \mathcal{U}_g$, U^+ contains a cycle which is distinct from C_1, C_2. Hence, U^+ is not bicyclic, a contradiction.

Conversely, assume that U has exactly one mm-alternating path of length 5. So the lengths of all other mm-alternating paths in U are either 1 or 3. By Lemma 2.6, there is at most one mm-alternating path between any two vertices of U. Observe that the mm-alternating paths of length 1 are just the matching edges of U. Let $\mathcal{Q}$ and $\mathcal{M}'$ denote the set of all mm-alternating paths of length 3 and the set of all non-matching edges in U, respectively. Define a map f from $\mathcal{Q}$ to $\mathcal{M}'$ that maps a mm-alternating path in $\mathcal{Q}$ to the non-matching edge contained in that path. Note that each mm-alternating path of length 3 contains a unique non-matching edge and a non-matching edge is contained in a unique mm-alternating path of length 3. Thus, f is a one-one and onto map. So $|\mathcal{Q}| = \frac{n}{2}$, where $n = |V(U)|$. Therefore, the total number of mm-alternating paths in U is $\frac{n}{2} + \frac{n}{2} + 1 = n + 1$. Thus by Lemma 2.3, the size of U^+ is $n + 1$. Hence, U^+ is bicyclic. ∎

Let $U \in \mathcal{U}_g$ such that U^+ is a bicyclic 3-colored digraph. By Lemma 3.3, U has exactly one mm-alternating path of length 5. Henceforth, we denote this path by $P = [u, u', x, x', v, v']$. Furthermore, we denote the unique cycle of U by $\Gamma = [v_1, v_2, \ldots, v_n, v_1]$.

The following lemmas provide crucial information about unicyclic 3-colored digraphs in $\mathcal{U}_g$ possessing bicyclic inverses.

LEMMA 3.4. *Let $U \in \mathcal{U}_g$ such that U^+ is a 3-colored digraph and U^+ is bicyclic. If U has exactly one peg, then the length of Γ is 3.*

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Proof. Since U^+ is a 3-colored digraph and U has exactly one peg, by Theorem 2.5, it follows that U is non-bipartite. Assume that length of Γ is at least 5. Let $[z, z']$ be the peg of U where z lies in Γ. Let a, b be two vertices of Γ such that $[z, a]$ and $[z, b]$ are edges in Γ. Note that both $[z, a]$ and $[z, b]$ are non-matching edges. Thus, there exist vertices $a', b' \in V(\Gamma)$ such that $[a, a']$ and $[b, b']$ are matching edges in U. If $a' \sim b'$ in U, then we get two distinct mm-alternating paths of length 5, namely $P_1 = [z', z, b, b', a', a]$ and $P_2 = [z', z, a, a', b', b]$, which is not possible since U^+ is bicyclic. Now if $a' \nsim b'$, then by the similar argument as given in above, we see that U contains two distinct mm-alternating paths of length 5, a contradiction. Hence, the length of Γ is 3. ∎

Lemma 3.5. *Let $U \in \mathcal{U}_g$ such that U^+ is a 3-colored digraph and U^+ is bicyclic. Suppose that $P = [u, u', x, x', v, v']$ is the mm-alternating path of length 5 in U. Then*

(a) $\deg(x) = 3, \deg(x') = 2, \deg(u) = \deg(v') = 1$ *in U if $[x, x']$ lies in Γ and $[v, v']$ is the only peg of U.*
(b) $\deg(x) = \deg(x') = 2, \deg(u) = \deg(v') = 1$ *in U if $[x, x']$ lies in Γ and U contains at least three pegs.*
(c) $\deg(x) = \deg(x') = 2, \deg(u) = \deg(v') = 1$ *in U if $[x, x']$ lies outside Γ.*

Proof. By Lemma 3.3, P is the unique mm-alternating path of length 5.

(a) Since $[v, v']$ is the only peg of U, by Lemma 3.4, it follows that length of Γ is 3. Also Γ contains $[x, x']$. Since U is unicyclic, so $\Gamma = [x, x', v, x]$. First we show that $\deg(x) = 3$ in U. To the contrary, assume that $\deg(x) \geq 4$. Let y be a vertex adjacent to x in U such that $y \neq u'$, $y \neq x'$ and $y \neq v$. Note that there exists a vertex y' in U such that $[y, y']$ is a matching edge. If $y = v'$, then U has more than one perfect matching, a contradiction. Thus, $y \neq v'$. If $y = u$, then $[u', u, x, x', v, v']$ is another mm-alternating path of length 5 in U, which is not possible. Thus, $y \neq u$. Hence $[y, y']$ lies outside P. Thus, $[y', y, x, x', v, v']$ is another mm-alternating path of length 5, a contradiction. Hence, $\deg(x) = 3$. Similarly, we can prove that $\deg(x') = 2$ and $\deg(u) = 1 = \deg(v')$.

(b) Since $[x, x']$ lies in Γ and U contains at least three pegs, it follows that length of Γ is at least 5. First, we show that $\deg(x) = 2$ in U. To the contrary, assume that $\deg(x) \geq 3$. Let y be a vertex adjacent to x in U such that $y \neq u'$ and $y \neq x'$. Note that there exists a vertex y' in U such that $[y, y']$ is a matching edge. If $y = v'$, then U has more than one perfect matching, a contradiction. Thus, $y \neq v'$. If $y = u$, then $[u', u, x, x', v, v']$ is another mm-alternating path of length 5 in U, which is not possible. Thus, $y \neq u$. If $y = v$, then $C = [x, x', v, x]$ is a cycle of length 3 in U. Since U is unicyclic, we see that $\Gamma = C$, a contradiction because length of Γ is at least 5. Thus, $y \neq v$. Hence, $[y, y']$ lies outside P. Thus, $[y', y, x, x', v, v']$ is another mm-alternating path of length 5, a contradiction. Hence, $\deg(x) = 2$. Similarly, we can prove that $\deg(x') = 2$ and $\deg(u) = 1 = \deg(v')$.

(c) First, we show that $\deg(x) = 2$ in U. To the contrary, assume that $\deg(x) \geq 3$. Let y be a vertex adjacent to x in U such that $y \neq u', y \neq x'$. Note that there exists a vertex y' in U such that $[y, y']$ is the matching edge. If $y = v'$, then U has more than one perfect matching, a contradiction. Thus, $y \neq v'$. If $y = u$, then $[u', u, x, x', v, v']$ is another mm-alternating path of length 5 in U, which is not possible. Thus, $y \neq u$. If $y = v$, then $C = [x, x', v, x]$ is a cycle of length 3 in U. Note that C contains the edge $[x, x']$. So $\Gamma \neq C$. Thus, U contains two distinct cycles, namely Γ and C, which is not possible because U is unicyclic. So $y \neq v$. Hence, $[y, y']$ lies outside P. Thus, $[y', y, x, x', v, v']$ is another mm-alternating path of length 5, a contradiction. Hence, $\deg(x) = 2$. Similarly, we can prove that $\deg(x') = 2$ and $\deg(v') = 1 = \deg(u)$.

3.1. Constructions of unicyclic 3-colored digraphs with bicyclic inverses. In this subsection, we provide constructions of four different classes of unicyclic 3-colored digraphs in $\mathcal{U}_g$ and prove that these are the classes of 3-colored digraph in $\mathcal{U}_g$ possessing bicyclic inverses.

Recall that a *quasi-pendant* vertex of a graph is a vertex adjacent to a vertex of degree 1. A *simple corona tree* is a tree obtained from another tree T' by adding a new vertex of degree 1 to every vertex of T'.

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Observe that a simple corona does not contain any mm-alternating path of length more than 3.

We now provide the construction of a class of 3-colored digraphs in $\mathcal{U}_g$ below and call them Type (A) graphs.

Type (A):

Let $\mathcal{U}_1$ be the class of unicyclic 3-colored digraphs constructed in the following steps:

1. Take a triangle $\Gamma = [v_1, v_2, v_3, v_1]$ of weight $\pm$i.
2. Take a corona tree T. Choose a quasi-pendant vertex (a vertex which is adjacent to a pendant vertex) u in T. Attach T at a vertex, say v_1 of Γ by identifying the vertex u of T.
3. Take another corona tree T'. Let w' be a quasi-pendant vertex in T'. Now connect w' either to v_2 or v_3 by adding an edge. By Lemma 1.3, the resulting 3-colored digraph is I-diagonally similar to the 3-colored digraph as shown in Figure 2.

EXAMPLE 3.6. *The graph shown in Figure 2 is an example of a graph in $\mathcal{U}_1$.*

REMARK 3.7. *Let $U \in \mathcal{U}_1$. From the construction above, it follows that U has a unique perfect matching and U is non-bipartite having exactly one peg. Thus, U^+ is a 3-colored digraph, by Theorem 2.5. Observe that the unique cycle Γ contains exactly one matching edge, namely $[v_2, v_3]$. Furthermore, U contains exactly one mm-alternating path of length 5. Hence, U^+ is bicyclic, by Lemma 3.3. Furthermore, if $U \in \mathcal{U}_1$, then the path P passes through Γ and Γ contains exactly one matching edge, namely $[v_2, v_3]$ which lies in P. The following lemma says that the converse is also true.*

LEMMA 3.8. *Let $U \in \mathcal{U}_g$ such that U^+ is 3-colored bicyclic graph and satisfies the following:*

(i) *U has exactly one peg,*
(ii) *P has a matching edge lying in Γ.*

Then $U \in \mathcal{U}_1$.

Proof. By Lemma 3.4, it follows that length of Γ is 3. Again by Lemma 3.5, $\deg(x) = 3$, $\deg(x') = 2$ and $\deg(u) = 1 = \deg(v')$ in U. Since P has a matching edge lying in Γ, we see that the edge $[x, x']$ lies on Γ. Without loss of generality, assume that $x = v_2, v = v_1$ and $x' = v_3$. Note that $[v_1, v']$ is the only peg of U at the vertex v_1 of Γ. Thus, a tree say, T, is attached at v_1 and T is a simple corona tree. Since

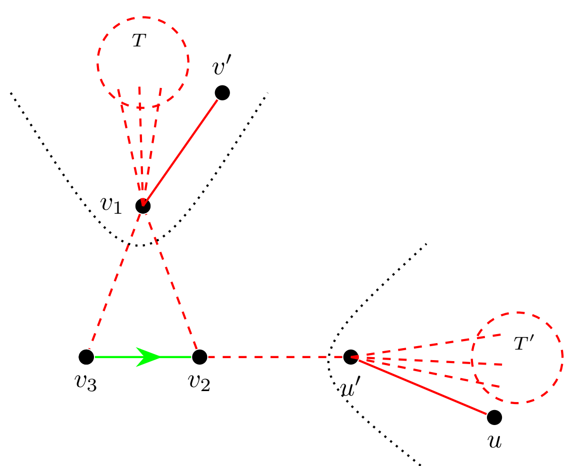

FIGURE 2. *Here the solid edges denote the matching edges.*

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$\deg(x') = 2$, $\deg(x) = 3$, that is, $\deg(v_3) = 2$ and $\deg(v_2) = 3$, it follows that there is no tree attached at the vertex v_3 of Γ while there is a tree attached at the vertex v_2 of Γ. Let T_0 be the tree attached at the vertex v_2 of Γ. Note that T_0 contains the non-matching edge $[u', v_2]$ of P. Now consider the graph $T_0 - [u', v_2]$. We see that $T_0 - [u', v_2]$ consists of two components, one of which is a simple corona tree say F_1 containing the vertex u' and the other component is a unicyclic graph, say F_2 containing Γ and T. Note that both F_1, F_2 have a unique perfect matching. It follows that U can be constructed by following the steps as given in the construction of Type (A) graph. Hence, $U \in \mathcal{U}_1$. $\square$

We now provide a construction of graph in $\mathcal{U}_g$ which is different from that of Type (A) graphs constructed above. We call these graphs as Type (B).

Type (B):

Let $\mathcal{U}_2$ be the class of unicyclic 3-colored digraphs constructed in the following steps:

1. Take a cycle $\Gamma = [v_1, v_2, \ldots, v_n, v_1]$ of weight $\pm$i, where $n \geq 5$.
2. Take $n-2$ simple corona trees $T_1, T_2, \ldots, T_{n-2}$. Choose a quasi-pendant vertex u_i in T_i for $i = 1, \ldots, n-2$. Attach T_i at the vertex v_i of Γ by identifying the vertex u_i for $i = 1, \ldots, n-2$. By Lemma 1.3, the resulting 3-colored digraph is I-diagonally similar to the 3-colored digraph as shown in Figure 3.

Example 3.9. *The graph shown in Figure 3 is an example of a graph in $\mathcal{U}_2$.*

Remark 3.10. *Let $U \in \mathcal{U}_2$. From the construction above, it follows that U has a unique perfect matching with at least three pegs. Thus, U^+ is a 3-colored digraph, by Theorem 2.5. Observe that U has exactly one mm-alternating path of length 5. Hence, U^+ is bicyclic, by Lemma 3.3. Furthermore, if $U \in \mathcal{U}_2$, then the path P passes through Γ and Γ contains exactly one matching edge, namely $[v_{n-1}, v_n]$ which lies in P. The following lemma says that the converse is also true.*

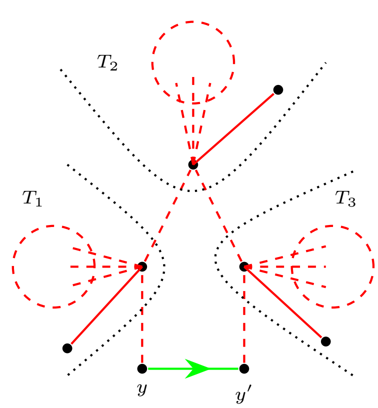

Figure 3. *The solid edges denotes the matching edges.*

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LEMMA 3.11. *Let $U \in \mathcal{U}_g$ such that U^+ is 3-colored bicyclic graph and satisfies the following:*

(i) *U has at least three pegs,*
(ii) *P has a matching edge lying in Γ.*

Then $U \in \mathcal{U}_2$.

Proof. Since U has at least three pegs and P has a matching edge lying in Γ, it follows that length of Γ is at least 5. By Lemma 3.5, $\deg(x) = 2 = \deg(x')$ and $\deg(u) = 1 = \deg(v')$ in U. Note that $[x, x']$ lies on Γ. Without loss of generality, assume that $x = v_{n-1}$ and $x' = v_n$. Since $\deg(x) = 2 = \deg(x')$, that is, $\deg(v_{n-1}) = 2 = \deg(v_n)$, it follows that there is no tree attached at the vertices v_{n-1} and v_n of Γ. Observe that $[v_{n-1}, v_n]$ is a matching edge. Furthermore, apart from $[v_{n-1}, v_n]$, if Γ contains another matching edge, then U must contain more than one mm-alternating path of length 5, which is not possible. Thus, $[v_{n-1}, v_n]$ is the only matching edge contained in Γ. So U has exactly $n-2$ pegs incident with the vertices $v_1, v_2, \ldots, v_{n-2}$. Thus, a tree is attached at each v_i for $i = 1, \ldots, n-2$. Moreover, each such tree is simple corona tree, otherwise U contains mm-alternating path of length 5 distinct from P, which is not possible. Thus, U can be obtained by attaching a tree at the vertices $v_1, v_2, \ldots, v_{n-2}$ of Γ by identifying a quasi-pendant vertex of the tree, where all such trees are simple corona trees. Hence, $U \in \mathcal{U}_2$. $\square$

We now provide a construction of graph in $\mathcal{U}_g$ which is different from that of Type (A) and Type (B) graphs constructed above. We call these graphs as Type (C).

Type (C):

Let $\mathcal{U}_3$ be the class of unicyclic 3-colored digraphs constructed in the following steps:

1. Take a triangle $\Gamma = [v_1, v_2, v_3, v_1]$ of weight $\pm$i.
2. Take the path $P_2 = [x, x']$, two corona trees T and T'. Let w and w' be quasi-pendant vertices in T and T', respectively. Add the edges $[w, x]$ and $[x', w']$ to join T and T' via P_2. Let T_0 be the resulting tree. Finally, attach T_0 at the vertex v_1 of Γ, by identifying a quasi-pendant vertex of T_0. By Lemma 1.3, the resulting 3-colored digraph is I-diagonally similar to the 3-colored digraph as shown in Figure 4.

Observe that $\mathcal{U}_3 \subset \mathcal{U}_g$. By Theorem 2.5, it follows that U^+ is a 3-colored digraph for any $U \in \mathcal{U}_3$. Furthermore, any graph $U \in \mathcal{U}_3$ has exactly one mm-alternating path of length 5 and all the edges of this path are cut-edges.

EXAMPLE 3.12. *The graph shown in Figure 4 is an example of a graph in $\mathcal{U}_3$.*

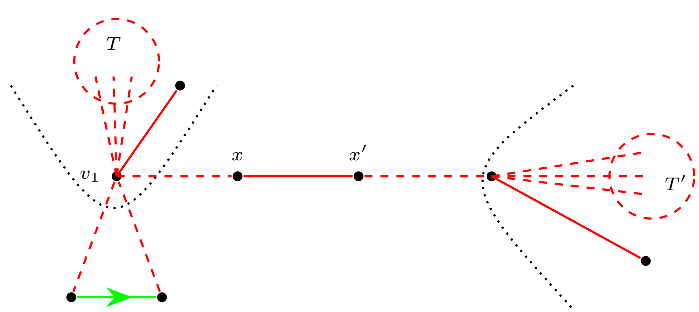

FIGURE 4. *Here the solid edges denote the matching edges.*

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Remark 3.13. *Let $U \in \mathcal{U}_3$. From the construction above, it follows that U has a unique perfect matching and U is non-bipartite having exactly one peg incident with the vertex v_1 of Γ. Thus, U^+ is a 3-colored digraph, by Theorem 2.5. Observe that the unique cycle Γ contains exactly one matching edge, namely $[v_2, v_3]$. Also U contains exactly one mm-alternating path of length 5. Hence, U^+ is bicyclic, by Lemma 3.3. Furthermore, if $U \in \mathcal{U}_3$, then all the edges of the path P are cut-edges, that is, P is contained in some tree attached at some vertex of Γ. More specifically, P is contained in the tree attached at the vertex v_1 of Γ. The following lemma says that the converse is also true.*

Lemma 3.14. *Let $U \in \mathcal{U}_g$ such that U^+ is 3-colored bicyclic graph and satisfies the following:*

(i) *U has exactly one peg,*
(ii) *Γ contains exactly one matching edge,*
(iii) *All the edges of P are cut-edges.*

Then $U \in \mathcal{U}_3$.

Proof. By Lemma 3.4, it follows that length of Γ is 3. Again by Lemma 3.5, $\deg(x) = 2$, $\deg(x') = 2$ and $\deg(u) = 1 = \deg(v')$ in U. Let $[v_1, y]$ be the peg of U incident with the vertex v_1 of Γ. Thus, a tree, say T, is attached at v_1. Moreover, T must be a simple corona tree. Now since U has a unique peg, it follows that $[v_2, v_3]$ is a matching edge. Also $\deg(v_2) = \deg(v_3) = 2$, otherwise U will contain an mm-alternating path of length 5 distinct from P, which is not possible. It follows that there is no tree attached at the vertices v_2, v_3. Let T_0 be the tree containing the path P, attached at the vertex v_1 of Γ. Since $\deg(x) = 2 = \deg(x')$, we see that $T_0 - \{x, x'\}$ consists of exactly two components, say T and T'. Observe that T and T' have unique perfect matching. Since U has no mm-alternating path of length more than 5, it follows that both T and T' do not contain mm-alternating path of length 5 or more. Thus, both T and T' are simple corona trees. It follows that T_0 can be constructed as described in step (2) of Type (C). Hence, $U \in \mathcal{U}_3$. ∎

We now provide a construction of graph in $\mathcal{U}_g$ which is different from that of Type (A), Type (B), and Type (C) graphs constructed above. We call these graphs as Type (D).

Type (D):

Let $\mathcal{U}_4$ be the class of unicyclic 3-colored digraphs constructed in the following steps:

1. Take a cycle $\Gamma = [v_1, v_2, \ldots, v_n, v_1]$ of weight $\pm \mathrm{i}$, where $n \geq 4$.
2. Take $n-1$ simple corona trees $T_1, T_2, \ldots, T_{n-1}$. Choose a quasi-pendant vertex u_i in T_i for $i = 1, \ldots, n-1$. Attach T_i at the vertex v_i of Γ by identifying the vertex u_i for $i = 1, \ldots, n-1$.
3. Take the path $P_2 = [x, x']$, two corona trees T and T'. Let w and w' be quasi-pendant vertices in T and T', respectively. Add the edges $[w, x]$ and $[x', w']$ to join T and T' via P_2. Let T_0 be the resulting tree. Finally, attach T_0 at the vertex v_n of Γ, by identifying a quasi-pendant vertex of T_0. By Lemma 1.3, the resulting 3-colored digraph is I-diagonally similar to the 3-colored digraph as shown in Figure 5.

Example 3.15. *The graph shown in Figure 5 is an example of a graph in $\mathcal{U}_4$.*

Remark 3.16. *Let $U \in \mathcal{U}_4$. From the construction above, it follows that U has a unique perfect matching with at least four pegs. By Theorem 2.5, it follows that U^+ is a 3-colored digraph for any $U \in \mathcal{U}_4$. Furthermore, any graph $U \in \mathcal{U}_4$ has exactly one mm-alternating path of length 5. Hence, U^+ is bicyclic, by Lemma 3.3. Also we see that Γ does not contain any matching edges. Moreover, if $U \in \mathcal{U}_4$, then all the edges of the path P are cut-edges, that is, P is contained in some tree attached at some vertex of Γ. The following lemma says that the converse is also true.*

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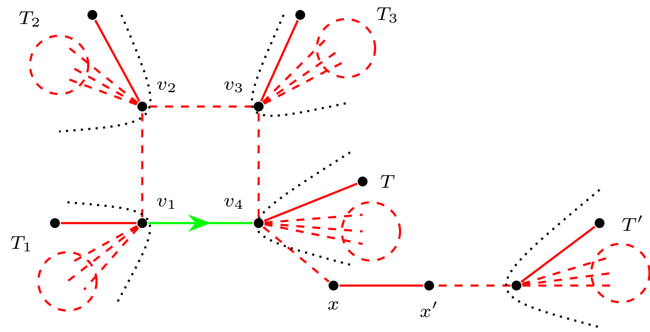

FIGURE 5. *Here the solid edges denote the matching edges.*

LEMMA 3.17. *Let $U \in \mathcal{U}_g$ such that U^+ is 3-colored bicyclic graph and satisfies the following:*

(i) *Γ does not contain a matching edge,*
(ii) *All the edges of P are cut-edges.*

Then $U \in \mathcal{U}_4$.

Proof. Since edges of P are cut-edges, let T_0 be the tree attached at the vertex, say v_n of Γ, which contains the path P. By Lemma 3.5, $\deg(x) = 2 = \deg(x')$ and $\deg(u) = \deg(v') = 1$. Thus, we see that $T_0 - \{x, x'\}$ consists of exactly two components, say T and T'. Observe that T and T' have unique perfect matching. Since U has no mm-alternating path of length more than 5, it follows that both T and T' do not contain mm-alternating path of length 5 or more. Thus, both T and T' are simple corona trees. It follows that T_0 can be constructed as described in step (3) of Type (D). Moreover, the trees attached at the vertices $v_1, v_2, \ldots, v_{n-1}$ of Γ are simple corona trees, otherwise U contains mm-alternating path of length 5 distinct from P, which is not possible. Note that Γ does not contain any matching edges. Therefore, U can be obtained by attaching a tree at each of the vertices of Γ by identifying a quasi-pendant vertex of the tree, where all such trees are simple corona trees, except the tree attached at the vertex v_n, which can be obtained as described in step (3) of type (D). Hence, $U \in \mathcal{U}_4$. ∎

We now prove our main result of this article which says that a unicyclic 3-colored digraph $U \in \mathcal{U}_g$ possessing bicyclic inverse must be either of Type (A), Type (B), Type(C), or Type (D) graphs constructed above.

THEOREM 3.18. *Let U be a unicyclic 3-colored digraph with a unique perfect matching such that U^+ is a 3-colored digraph. Then U^+ is bicyclic if and only if $U \in \mathcal{T}$, where $\mathcal{T} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$.*

Proof. Assume that $U \in \mathcal{T}$. Hence, U^+ is a 3-colored digraph. Also from the construction of $\mathcal{T}$, it follows that any graph $U \in \mathcal{T}$ contains exactly one mm-alternating path of length 5. Hence by Lemma 3.3, U^+ is bicyclic.

Conversely, assume that U^+ is bicyclic. By Lemma 3.3, U has exactly one mm-alternating path of length 5, namely the path P. Now either $V(P) \cap V(\Gamma) = \emptyset$ or $V(P) \cap V(\Gamma) \neq \emptyset$. First suppose that $V(P) \cap V(\Gamma) = \emptyset$. In this case, P is contained in some tree attached at some vertex of Γ. Then the following two cases occur.

(a) U has exactly one peg and Γ contains exactly one matching edge. In this case, $U \in \mathcal{U}_3$, by Lemma 3.14.
(b) Γ does not contain any matching edges. In this case, $U \in \mathcal{U}_4$, by Lemma 3.17.

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Now assume that $V(P) \cap V(\Gamma) \neq \emptyset$. Then the following two cases occur.

(i) U has exactly one peg. In this case, $U \in \mathcal{U}_1$, by Lemma 3.8.
(ii) U has at least three pegs. In this case, $U \in \mathcal{U}_2$, by Lemma 3.11.

Since $\mathcal{T} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$, so the proof is complete. ∎

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