Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 108-120, February 2010

ELA

SIGN PATTERNS THAT ALLOW EVENTUAL POSITIVITY*

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Abstract. Several necessary or sufficient conditions for a sign pattern to allow eventual positivity are established. It is also shown that certain families of sign patterns do not allow eventual positivity. These results are applied to show that for $n \geq 2$, the minimum number of positive entries in an $n \times n$ sign pattern that allows eventual positivity is $n+1$, and to classify all 2×2 and 3×3 sign patterns as to whether or not the pattern allows eventual positivity. A 3×3 matrix is presented to demonstrate that the positive part of an eventually positive matrix need not be primitive, answering negatively a question of Johnson and Tarazaga.

Key words. Eventually positive matrix, Potentially eventually positive sign pattern, Perron-Frobenius, Directed graph.

AMS subject classifications. (2010) 15B48, 15B35, 05C50, 15A18.

1. Introduction.

A real square matrix A is *eventually positive* (respectively, *eventually nonnegative*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k > 0$ (respectively, $A^k \geq 0$), where these inequalities are entrywise. For an eventually positive matrix, we call the least such k_0 the *power index* of A. It follows

*Received by the editors May 15, 2009. Accepted for publication February 17, 2010. Handling Editor: Judith J. McDonald. This research began at the American Institute of Mathematics workshop, “Nonnegative Matrix Theory: Generalizations and Applications,” and the authors thank AIM and NSF for their support. Author information: A. Berman, Faculty of Mathematics, Technion, Haifa 32000, Israel (berman@techunix-.technion.ac.il); M. Catral, Department of Mathematics, Iowa State University, Ames, IA 50011, USA (mrcatral@iastate.edu); L.M. DeAlba, Department of Mathematics and Computer Science, Drake University, Des Moines, IA 50311, USA (luz.dealba@drake.edu), partially funded by a Drake University Research Support Grant; A. Elhashash, Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA (abed@drexel.edu); F.J. Hall, Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA (matfjh@langate.gsu.edu); L. Hogben, Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu) and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org); I.-J. Kim, Department of Mathematics and Statistics, Minnesota State University, Mankato, MN 56001, USA (injae.kim@mnsu.edu); D. D. Olesky, Department of Computer Science, University of Victoria, Victoria, BC V8W 3P6, Canada (dolesky@cs.uvic.ca); P. Tarazaga, Department of Mathematics, Texas A&M University Corpus Christi, TX 78412, USA (pablo.tarazaga@tamucc.edu); M.J. Tsatsomeros, Department of Mathematics, Washington State University, Pullman, WA 99164, USA (tsat@wsu.edu,), P. van den Driessche, Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada (pvdd@math.uvic.ca).

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A publication of the International Linear Algebra Society
Volume 19, pp. 108-120, February 2010

from results in [5] that the power index k_0 of an eventually positive matrix may be arbitrarily large.

Given a real square matrix A, $\sigma(A)$ denotes the *spectrum* of A, i.e., the multiset of eigenvalues, and the *spectral radius* of A, $\max\{|\lambda| : \lambda \in \sigma(A)\}$, is denoted by $\rho(A)$. An eigenvalue λ of A is a *dominant* eigenvalue if $|\lambda| = \rho(A)$. The matrix A *has the Perron-Frobenius property* if A has a dominant eigenvalue that is positive and the corresponding eigenvector is nonnegative. The matrix A *has the strong Perron-Frobenius property* if A has a unique dominant eigenvalue (which is simple), the dominant eigenvalue is positive, and the corresponding eigenvector is positive. The definition of the Perron-Frobenius property given above is the one used in [8], which is slightly different from that used in [3]. The definition in [3] allows $\rho(A) \geq 0$ (i.e., A can be nilpotent). This distinction is important for the consideration of eventual nonnegativity (see Theorem 1.2 below). Since this paper is concerned primarily with eventually positive matrices, this distinction is not of great consequence here.

Theorem 1.1. [7, Theorem 1] *Let $A \in \mathbb{R}^{n\times n}$. Then the following are equivalent.*

1. *Both of the matrices A and A^T possess the strong Perron-Frobenius property.*
2. *The matrix A is eventually positive.*
3. *There exists a positive integer k such that $A^k > 0$ and $A^{k+1} > 0$.*

Note that having both of the matrices A and A^T possess the strong Perron-Frobenius property is equivalent to having A possess the strong Perron-Frobenius property and have a positive left eigenvector for $\rho(A)$.

Unlike the case of an eventually positive matrix, there is no known test for an eventually nonnegative matrix, although the following necessary condition is known (the requirement that A is not nilpotent in this theorem was pointed out in [3, p. 390]).

Theorem 1.2. [8, Theorem 2.3] *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both matrices A and A^T possess the Perron-Frobenius property.*

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ shows that the converse of Theorem 1.2 is false, as noted in [3, p. 392].

A *sign pattern matrix* (or *sign pattern* for short) is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\operatorname{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *sign pattern class* (or *qualitative class*) of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\operatorname{sgn}(A) = \mathcal{A}$. Let $\mathcal{A} = [\alpha_{ij}], \hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ be sign patterns. If $\alpha_{ij} \neq 0$ implies $\alpha_{ij} = \hat{\alpha}_{ij}$,

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then $\mathcal{A}$ is a *subpattern* of $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$.

If $\mathcal{P}$ is a property of a real matrix, then a sign pattern $\mathcal{A}$ *requires* $\mathcal{P}$ if every real matrix $A \in \mathcal{Q}(\mathcal{A})$ has property $\mathcal{P}$, and $\mathcal{A}$ *allows* $\mathcal{P}$ or is *potentially* $\mathcal{P}$ if there is some $A \in \mathcal{Q}(\mathcal{A})$ that has property $\mathcal{P}$. Numerous properties have been investigated from the point of view of characterizing sign patterns that require or allow a particular property (see, for example, [6, 2] and the references therein).

Sign patterns that require eventual positivity or eventual nonnegativity have been characterized in [4]. Here we are interested in *potentially eventually positive (PEP)* sign patterns, i.e. sign patterns $\mathcal{A}$ for which there exists $A \in \mathcal{Q}(\mathcal{A})$ such that A is eventually positive. We investigate the following question that was raised at the American Institute of Mathematics workshop on Nonnegative Matrix Theory: Generalizations and Applications [1].

Question 1.3. *What sign patterns are potentially eventually positive (PEP)?*

The study of this question led to the discovery of Example 2.2, presented in Section 2, that answers negatively a question of Johnson and Tarazaga [7]. Section 3 gives methods of modifying a PEP sign pattern to obtain another PEP sign pattern. In Section 4, we show that for $n \geq 2$, the minimum number of positive entries in an $n \times n$ PEP sign pattern is $n+1$. Section 5 presents several families of sign patterns that are not PEP. In Section 6 we use the results of the preceding sections to classify all 2×2 and 3×3 sign patterns as to whether or not they are PEP, and make some concluding remarks in Section 7.

We now introduce additional definitions and notation. Two $n \times n$ sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{B} = P^T \mathcal{A} P$, or $\mathcal{B} = P^T \mathcal{A}^T P$ (where P is a permutation matrix); if $\mathcal{B}$ is equivalent to $\mathcal{A}$, then $\mathcal{B}$ is PEP if and only if $\mathcal{A}$ is PEP.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, define the *positive part* of $\mathcal{A}$ to be $\mathcal{A}^+ = \left[\alpha_{ij}^+\right]$ and the *negative part* of $\mathcal{A}$ to be $\mathcal{A}^- = \left[\alpha_{ij}^-\right]$, where

$$\alpha_{ij}^+ = \begin{cases} + & \text{if} \quad \alpha_{ij} = +, \\ 0 & \text{if} \quad \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -, \end{cases} \qquad \text{and } \alpha_{ij}^- = \begin{cases} - & \text{if} \quad \alpha_{ij} = -, \\ 0 & \text{if} \quad \alpha_{ij} = 0 \text{ or } \alpha_{ij} = +. \end{cases}$$

It follows that $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$. Similarly, for $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, define the *positive part* of A to be $A^+ = \left[a_{ij}^+\right]$ and the *negative part* of A to be $A^- = \left[a_{ij}^-\right]$, where

$$a_{ij}^+ = \begin{cases} a_{ij} & \text{if} \quad a_{ij} > 0, \\ 0 & \text{if} \quad a_{ij} \leq 0, \end{cases} \qquad \text{and } a_{ij}^- = \begin{cases} a_{ij} & \text{if} \quad a_{ij} < 0, \\ 0 & \text{if} \quad a_{ij} \geq 0. \end{cases}$$

If $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ sign pattern, the *digraph of* $\mathcal{A}$ is

$$\Gamma(\mathcal{A}) = \left(\{1, \ldots, n\}, \{(i,j) : \alpha_{ij} \neq 0\}\right).$$

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An $n \times n$ sign pattern $\mathcal{A}$ (or matrix) is *reducible* if there exists a permutation matrix P such that

$$P\mathcal{A}P^T = \begin{bmatrix} \mathcal{A}_{11} & O \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square non-empty sign patterns, and O denotes a block consisting entirely of zero entries. If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is called *irreducible*. We define [0] to be reducible. It is well known that for $n \geq 2$, $\mathcal{A}$ is irreducible if and only if $\Gamma(\mathcal{A})$ is strongly connected.

A digraph is *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. A nonnegative sign pattern or nonnegative matrix is *primitive* if its digraph is primitive. It is well known that a digraph $\Gamma = (V_\Gamma, E_\Gamma)$ is primitive if and only if there exists $k \in \mathbb{N}$ such that for all $i, j \in V_\Gamma$, there is a walk of length k from i to j, and a nonnegative sign pattern $\mathcal{A}$ is primitive if and only if there exists a positive integer k_0 such that for all $A \in \mathcal{Q}(\mathcal{A})$, $A^k > 0$ for all $k \geq k_0$. The least such k_0 is called the *exponent* of $\mathcal{A}$. If $\mathcal{A} \geq 0$ is primitive, then the exponent of $\mathcal{A}$ is equal to the power index of A for all $A \in \mathcal{Q}(\mathcal{A})$.

2. Eventually positive matrices and sign patterns that allow eventual positivity. In this section we establish a condition sufficient to ensure that a sign pattern is PEP, and provide an example of an eventually positive matrix that shows that this condition is not necessary for a sign pattern to be PEP. In addition, this example answers negatively an open question about eventually positive matrices.

The idea used in the proof of the next theorem was observed in [7, pp. 330, 335], although it was not articulated in terms of sign patterns.

Theorem 2.1. *Let $\mathcal{A}$ be a sign pattern such that $\mathcal{A}^+$ is primitive. Then $\mathcal{A}$ is PEP.*

Proof. Let $A(\varepsilon) = B + \varepsilon C$ where B (respectively, C) is the matrix obtained from $\mathcal{A}^+$ (respectively, $\mathcal{A}^-$) by replacing $+$ by 1 (respectively, $-$ by -1). Since $\mathcal{A}^+$ is primitive, by Theorem 1.1 there exists a positive integer k such that $B^k > 0$ and $B^{k+1} > 0$. Since for a fixed k, the entries of $A(\varepsilon)^k$ depend continuously on ε, we can choose ε small enough so that $A\left(\varepsilon\right)^k > 0$ and $A\left(\varepsilon\right)^{k+1} > 0$. By Theorem 1.1, $A(\varepsilon)$ is eventually positive. ☐

In [7, Question 8.1], Johnson and Tarazaga asked whether an eventually positive matrix A must have A^+ primitive, and Example 2.2 below shows this is not true.

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EXAMPLE 2.2. The matrix

$$B = \begin{bmatrix} \frac{13}{10} & -\frac{3}{10} & 0 \\ \frac{13}{10} & 0 & -\frac{3}{10} \\ -\frac{31}{100} & \frac{3}{10} & \frac{101}{100} \end{bmatrix}$$

is eventually positive. The eigenvalues of B are $1, \frac{1}{200}\left(131 \pm i\sqrt{2159}\right)$, the vectors $[1,1,1]^T$ and $[\frac{8}{9}, \frac{1}{30}, 1]^T$ are right and left eigenvectors for $\rho(B) = 1$, and $B^k > 0$ for $k \geq k_0 = 10$. Notice that $B^+ = \begin{bmatrix} \frac{13}{10} & 0 & 0 \\ \frac{13}{10} & 0 & 0 \\ 0 & \frac{3}{10} & \frac{101}{100} \end{bmatrix}$ is reducible.

Taking $\mathcal{A} = \operatorname{sgn}(B)$, it follows that $\mathcal{A}$ is a PEP sign pattern having $\mathcal{A}^+$ reducible and thus is not primitive. Consequently, Example 2.2 shows that the converse of Theorem 2.1 is false.

3. Modifications of potentially eventually positive sign patterns. In this section we establish ways to modify a PEP sign pattern to obtain additional PEP sign patterns. As is the case in many sign pattern problems, subpatterns and superpatterns play an important role in the study of PEP sign patterns.

THEOREM 3.1. *If $\mathcal{A}$ is PEP, then every superpattern is PEP. If $\mathcal{A}$ is not PEP, then every subpattern is not PEP.*

Proof. Let $\mathcal{A} = [\alpha_{ij}]$ be a sign pattern that is PEP and let $\hat{\mathcal{A}}$ be obtained from $\mathcal{A}$ by changing one entry $\alpha_{st} = 0$ to $-$ or $+$. Let A be an eventually positive matrix in $\mathcal{Q}(\mathcal{A})$. Then there exists a positive integer k such that $A^k > 0$ and $A^{k+1} > 0$. Let $A(\varepsilon) = A + \varepsilon E_{st}$. Since for a fixed k, the entries of $A(\varepsilon)^k$ depend continuously on ε, we can choose ε small enough so that $A(\varepsilon)^k > 0$ and $A(\varepsilon)^{k+1} > 0$. By Theorem 1.1, $A(\varepsilon)$ is eventually positive, and so $\hat{\mathcal{A}}$ is PEP. For the second statement, if $\mathcal{A}$ is not PEP but has a PEP subpattern, then the first statement is contradicted. ◻

REMARK 3.2. If A is eventually positive, then so is $B = A + tI$ for every $t \geq 0$, because eventual positivity of B is equivalent to B and B^T having the strong Perron-Frobenius property.

THEOREM 3.3. *Let $\hat{\mathcal{A}}$ be the sign pattern obtained from a sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$. If $\mathcal{A}$ is a PEP sign pattern, then $\hat{\mathcal{A}}$ is PEP.*

Proof. Since $\mathcal{A}$ is PEP, there exists $A \in \mathcal{Q}(\mathcal{A})$ such that A is eventually positive. There exists $t > 0$ such that $\hat{A} = tI + A \in \mathcal{Q}(\hat{\mathcal{A}})$, where $\hat{\mathcal{A}}$ is obtained from $\mathcal{A}$ by changing all diagonal entries to $+$. By Remark 3.2, $\hat{A}$ is eventually positive and $\hat{\mathcal{A}}$ is

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A publication of the International Linear Algebra Society
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PEP. □

4. Minimum number of positive entries in a potentially eventually positive sign pattern. In this section we establish that for $n \geq 2$, the minimum number of positive entries in an $n \times n$ PEP sign pattern is $n+1$.

The next observation gives obvious necessary conditions for a sign pattern to allow eventual positivity.

Observation 4.1. [7, p. 327] *If $\mathcal{A}$ is PEP, then*

1. *Every row of $\mathcal{A}$ has at least one $+$.*
2. *Every column of $\mathcal{A}$ has at least one $+$.*

Corollary 4.2. *If $\mathcal{A}$ is PEP, then $\Gamma(\mathcal{A})$ has a cycle (of length one or more) consisting entirely of $+$ entries.*

Proof. If $\Gamma(\mathcal{A})$ does not have a cycle consisting of all $+$, then there is a permutation matrix P such that $P^T \mathcal{A}^+ P$ or $P^T (\mathcal{A}^+)^T P$ is a strictly upper triangular sign pattern, and thus $\mathcal{A}$ does not have a $+$ in some row (and column). □

Lemma 4.3. *If $\mathcal{A}$ is a PEP sign pattern, then there is an eventually positive matrix $C \in \mathcal{Q}(\mathcal{A})$ such that*

(a) $\rho(C) = 1$, and
(b) $C\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is the $n \times 1$ all ones vector.

If $n \geq 2$, the sum of all the off-diagonal entries of C is positive.

Proof. There exists $A \in \mathcal{Q}(\mathcal{A})$ that is eventually positive, which implies $\rho(A) \in \sigma(A)$ is a simple eigenvalue, $|\lambda| < \rho(A)$ for all $\lambda \in \sigma(A) \setminus \{\rho(A)\}$, and $\rho(A)$ has a positive eigenvector $v = [v_1 \ v_2 \ \dots \ v_n]^T$. Let $B = \frac{1}{\rho(A)} A$. Then $B \in Q(\mathcal{A})$, B is eventually positive, and $\rho(B) = 1$ with $Bv = v$. Now let $C = D^{-1} B D$ for $D = \operatorname{diag}(v_1, \dots, v_n)$. Then $C \in Q(\mathcal{A})$ is eventually positive and C satisfies conditions (a) and (b). Let $C = [c_{ij}]$. Since $C\mathbb{1} = \mathbb{1}$, for $i = 1, \dots, n$, $c_{ii} = 1 - \sum_{j \neq i} c_{ij}$. Since $1 > |\lambda|$ for every eigenvalue $\lambda \neq 1$ and 1 is a simple eigenvalue, $n > \operatorname{tr} C$. Thus the sum of all the off-diagonal elements is positive. □

Theorem 4.4. *For $n \geq 2$, an $n \times n$ sign pattern that has exactly one positive entry in each row and each column is not PEP.*

Proof. Suppose to the contrary that $\mathcal{C}$ is an $n \times n$ sign pattern that has exactly one positive entry in each row and each column and is PEP. Then there exists $C \in Q(\mathcal{C})$ satisfying conditions (a) and (b) of Lemma 4.3. Since C is eventually positive, 1 is a simple eigenvalue and $|\lambda| < 1$ for all $\lambda \in \sigma(C) \setminus \{1\}$, so $|\det C| < 1$. Note that each row i has exactly one positive entry, say in column k_i, and if the other entries in the

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A publication of the International Linear Algebra Society
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row are denoted $-c_{ij}, j \neq k_i$, then $c_{ij} \geq 0$ and $c_{i,k_i} = 1 + \sum_{j \neq k_i} c_{ij}$.

Let $\widehat{C}$ be the matrix obtained from C by permuting the columns so that the positive entry of each column is on the diagonal. We apply Gershgorin's Theorem, noting that the ith disk has center $c_{i,k_i} = 1 + \sum_{j \neq k_i} c_{ij}$ and radius $\sum_{j \neq k_i} c_{ij}$. It follows that the eigenvalues of $\widehat{C}$ have absolute value that is at least 1, and thus that $|\det \widehat{C}| \geq 1$. Since $|\det C| = |\det \widehat{C}|$, a contradiction is obtained. □

A 1×1 sign pattern is PEP if and only if the sole entry is $+$, so assume $n \geq 2$. It follows from Theorem 2.1 that for every such n, there is an $n \times n$ sign pattern that is PEP and has exactly $n + 1$ positive entries, namely a sign pattern having all the entries of an n-cycle and one diagonal entry equal to $+$, and all other entries equal to 0 or $-$. From Observation 4.1, any $n \times n$ sign pattern that is PEP must have at least n positive entries, and by Theorem 4.4, an $n \times n$ sign pattern with exactly n positive entries is not PEP. Thus for $n \geq 2$, we obtain the following result.

Corollary 4.5. *For $n \geq 2$, the minimum number of $+$ entries in an $n \times n$ PEP sign pattern is $n + 1$.*

5. Sign patterns that are not potentially eventually positive. Note that if $\mathcal{A}$ is reducible, then $\mathcal{A}$ is not PEP. In this section we establish that several families of irreducible sign patterns are not PEP. We use the notation $[-]$ (respectively, $[+]$) for a (rectangular) sign pattern consisting entirely of negative (respectively, positive) entries. A square sign pattern $\mathcal{A}$ is a Z *sign pattern* if $\alpha_{ij} \neq +$ for all $i \neq j$. The next result follows from Corollary 4.5.

Proposition 5.1. *If $\mathcal{A}$ is an $n \times n$ Z sign pattern with $n \geq 2$, then $\mathcal{A}$ is not PEP.*

Theorem 5.2. *If $\mathcal{A}$ is the block sign pattern $\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$ with $\mathcal{A}_{12} = \mathcal{A}_{12}^-$, $\mathcal{A}_{21} = \mathcal{A}_{21}^+$, and $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ square, then $\mathcal{A}$ is not PEP.*

Proof. Suppose $\mathcal{A}$ is PEP. Then $\mathcal{A}$ cannot be reducible, so $\mathcal{A}_{12}^- \neq O$ and $\mathcal{A}_{21}^+ \neq O$. Let $A \in \mathcal{Q}(\mathcal{A})$ be eventually positive with spectral radius ρ, and let $v > 0$ and $w > 0$ be right and left eigenvectors for ρ, respectively. Then $(A - \rho I)v = 0$, and $w^T(A - \rho I) = 0$. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, partitioned conformally with the partition of $\mathcal{A}$. Then,

$$(A_{11} - \rho I)v_1 + A_{12}^- v_2 = 0, \tag{5.1}$$

$$w_1^T(A_{11} - \rho I) + w_2^T A_{21}^+ = 0. \tag{5.2}$$

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A publication of the International Linear Algebra Society
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Equation (5.1) implies $(A_{11} - \rho I)v_1 \gneq 0$, while (5.2) implies $w_1^T(A_{11} - \rho I) \lneq 0$. But then, $w_1^T(A_{11} - \rho I)v_1 > 0$ and $w_1^T(A_{11} - \rho I)v_1 < 0$, which is a contradiction. □

Proposition 5.3. *Let*

$$\mathcal{A}_0 = \begin{bmatrix} [+] & [-] & [+] & \dots \\ [-] & [+] & [-] & \dots \\ [+] & [-] & [+] & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be a square checkerboard block sign pattern with square diagonal blocks. Then $-\mathcal{A}_0$ is not PEP, and if $\mathcal{A}_0$ has a negative entry, then $\mathcal{A}_0$ is not PEP.

Proof. $\mathcal{A}_0^2 = \mathcal{A}_0$. □

Corollary 5.4. *Let $\mathcal{A} = [\mathcal{A}_{ij}]$ be a square block sign pattern with square diagonal blocks. Let $\mathcal{D}$ be a diagonal pattern that when partitioned conformally with $\mathcal{A}$ as $\mathcal{D} = [\mathcal{D}_{ij}]$ has all of the diagonal entries of $\mathcal{D}_{ii}$ equal to + or all − for each i. If $\mathcal{DAD} \leq 0$, or if $\mathcal{DAD} \geq 0$ and $\mathcal{D}$ has at least one − entry, then $\mathcal{A}$ is not PEP.*

Proof. This follows from Theorem 3.1 and Proposition 5.3, since $\mathcal{A}$ is a subpattern of $-\mathcal{A}_0$ (if $\mathcal{DAD} \leq 0$) or $\mathcal{A}_0$ (if $\mathcal{DAD} \geq 0$). □

Lemma 5.5. *Let $A = [a_{ij}]$ be an $n \times n$ matrix with $n \geq 2$ such that for a fixed $k \in \{1, ..., n\}$,*

1. *$a_{kk} > 0$, and*
2. *in row k (respectively, column k) every off-diagonal entry is nonpositive and some off-diagonal entry is negative.*

Suppose that $\rho(A)$ is an eigenvalue of A that has a positive eigenvector (respectively, left eigenvector) $x = [x_i]$. Then $\rho(A) < a_{kk}$.

Proof. The result is clear if $\rho(A) = 0$, so suppose $\rho(A) > 0$. The equation $Ax = \rho(A)x$ implies that

$$\sum_{j=1}^{n} a_{kj}x_j = \rho(A)x_k.$$

Thus,

$$(a_{kk} - \rho(A))x_k = \sum_{j \neq k}(-a_{kj})x_j > 0.$$

Since $x_k > 0$, it follows that $\rho(A) < a_{kk}$. The result for the column hypothesis follows by considering A^T. □

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THEOREM 5.6. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern with $n \geq 2$ such that for every $k = 1, \ldots, n$,*

1. *$\alpha_{kk} = +$, and*
2. *(a) no off-diagonal entry in row k is $+$, or*
   (b) no off-diagonal entry in column k is $+$.

Then $\mathcal{A}$ is not PEP.

Proof. If $A \in \mathcal{Q}(\mathcal{A})$ is eventually positive, then A is not reducible, so the hypotheses of Lemma 5.5 are satisfied for $k = 1, \ldots, n$. Thus $a_{kk} > \rho(A)$ for $k = 1, \ldots, n$. But then $\operatorname{tr} A > n\rho(A)$, which is a contradiction. □

COROLLARY 5.7. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern with $n \geq 2$ that requires a positive eigenvalue and suppose that $\alpha_{kk} = +$ and α_{kj} is 0 or $-$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$, and $\alpha_{ii} = -$ for all $i \neq k$. Then $\mathcal{A}$ is not PEP.*

Proof. Suppose that there is an eventually positive matrix $A = [a_{ij}]$ in $Q(\mathcal{A})$, so A is not reducible. Then by Theorem 1.1, $\rho(A)$ is a positive eigenvalue of A with a positive eigenvector. By Lemma 5.5, it follows that $a_{kk} > \rho(A)$. Hence, $A - \rho(A)I \in Q(\mathcal{A})$, but $A - \rho(A)I$ cannot have a positive eigenvalue, giving a contradiction. □

6. Classification of sign patterns of orders 2 and 3. In this section we use results in the previous sections to show that a 2×2 sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive, and determine which 3×3 sign patterns are PEP. We use ? to denote one of 0, $+$, $-$, and $\ominus$ to denote one of 0, $-$. The following theorem is a consequence of Theorem 2.1, Corollary 4.5, and Theorem 5.2.

THEOREM 6.1. *Every PEP 2×2 sign pattern is equivalent to a sign pattern of the form $\begin{bmatrix} + & + \\ + & ? \end{bmatrix}$. Thus a 2×2 sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive.*

LEMMA 6.2. *If $\mathcal{A}$ is a PEP 3×3 sign pattern such that $\mathcal{A}^+$ is irreducible, then $\mathcal{A}^+$ is primitive.*

Proof. If $\mathcal{A}^+$ is irreducible and not primitive, then $\Gamma(\mathcal{A}^+)$ has one 3-cycle and no 2-cycles or 1-cycles, or $\Gamma(\mathcal{A}^+)$ has two 2-cycles and no 3-cycles or 1-cycles. Then $\mathcal{A}$ is not PEP by Theorem 4.4, or Corollary 5.4. □

The sign patterns in the next lemma, which all have reducible positive parts, are used in the classification of 3×3 sign patterns in Theorem 6.4 below.

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LEMMA 6.3. *Any sign pattern of the form $\mathcal{A}_1, \mathcal{A}_2$, or $\mathcal{A}_3$ below is not PEP.*

$$\mathcal{A}_1 = \begin{bmatrix} + & \ominus & \ominus \\ \ominus & ? & + \\ \ominus & + & ? \end{bmatrix},\ \mathcal{A}_2 = \begin{bmatrix} + & \ominus & \ominus \\ + & ? & + \\ \ominus & + & ? \end{bmatrix},\ \mathcal{A}_3 = \begin{bmatrix} + & 0 & \ominus \\ + & ? & \ominus \\ \ominus & + & + \end{bmatrix}.$$

Proof. The sign pattern $\begin{bmatrix} + & - & - \\ - & + & + \\ - & + & + \end{bmatrix}$ is not PEP by Proposition 5.3, so any sign pattern of the form $\mathcal{A}_1$ is not PEP by Theorem 3.1 and the contrapositive of Theorem 3.3.

To show that any sign pattern of the form $\mathcal{A}_2$ is not PEP, we first show that $\mathcal{A} = \begin{bmatrix} + & - & - \\ + & + & + \\ - & + & + \end{bmatrix}$ is not PEP. Assume to the contrary that $\mathcal{A}$ is PEP; then by Lemma 4.3 there exists an eventually positive matrix $A \in \mathcal{Q}(\mathcal{A})$ of the form

$$A = \begin{bmatrix} 1 + a_{12} + a_{13} & -a_{12} & -a_{13} \\ a_{21} & 1 - a_{21} - a_{23} & a_{23} \\ -a_{31} & a_{32} & 1 - a_{32} + a_{31} \end{bmatrix},$$

where $a_{12}, a_{13}, a_{21}, a_{23}, a_{31}$ and a_{32} are positive, $\rho(A) = 1$ and

$$a_{21} + a_{23} + a_{32} - a_{12} - a_{13} - a_{31} > 0. \tag{6.1}$$

Let $w = [w_i] > 0$ be such that $w^T A = w^T$ and $w_1 = 1$. Then

$$\begin{array}{rcrcr} a_{21}w_2 & - & a_{31}w_3 & = & -(a_{12} + a_{13}) \\ -(a_{21} + a_{23})w_2 & + & a_{32}w_3 & = & a_{12}. \end{array}$$

Because A is eventually positive, w_2 and w_3 are uniquely determined as

$$w_2 = \frac{-a_{32}(a_{12} + a_{13}) + a_{12}a_{31}}{a_{21}a_{32} - a_{31}(a_{21} + a_{23})}$$

and

$$w_3 = \frac{-[a_{12}a_{23} + a_{13}(a_{21} + a_{23})]}{a_{21}a_{32} - a_{31}(a_{21} + a_{23})},$$

where $a_{21}a_{32} - a_{31}(a_{21} + a_{23}) \neq 0$. Since $w_3 > 0$, it follows that

$$a_{21}a_{32} - a_{31}(a_{21} + a_{23}) < 0. \tag{6.2}$$

Thus since $w_2 > 0$,

$$a_{12}a_{31} - a_{32}(a_{12} + a_{13}) < 0. \tag{6.3}$$

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Because $\rho(A) = 1$ is a simple eigenvalue of A, 0 is a simple eigenvalue of $A - I$ and $A - I$ does not have any positive eigenvalues. A calculation of the characteristic polynomial of $A - I$ gives

$$\det(xI - (A - I)) = x(x^2 + \beta x + \gamma),$$

where

$$\gamma = -a_{13}a_{21} - a_{12}a_{23} - a_{13}a_{23} + a_{12}a_{31} - a_{21}a_{31} - a_{23}a_{31} - a_{12}a_{32} - a_{13}a_{32} + a_{21}a_{32}.$$

Now, using (6.2) and (6.3), $\gamma < -a_{13}a_{21} - a_{12}a_{23} - a_{13}a_{23} + a_{32}(a_{12} + a_{13}) - a_{21}a_{31} - a_{23}a_{31} - a_{12}a_{32} - a_{13}a_{32} + a_{31}(a_{21} + a_{23}) < 0$. Thus $\det(xI - (A - I))$ must have a positive root, so $A - I$ has a positive eigenvalue, which is a contradiction.

Since $\mathcal{A}$ is not PEP, no sign pattern of the form $\mathcal{A}_2$ is PEP by Theorem 3.1 and the contrapositive of Theorem 3.3.

It remains to show any sign pattern of the form $\mathcal{A}_3$ is not PEP. This can be established by a modification of the proof that $\mathcal{A}_2$ is not PEP. Specifically, consider $\mathcal{A} = \begin{bmatrix} + & 0 & - \\ + & ? & - \\ - & + & + \end{bmatrix}$ and use the same notation for the form of an eventually positive matrix $A \in \mathcal{Q}(\mathcal{A})$ with the properties specified in Lemma 4.3. Then by Lemma 5.5 applied to column 3,

$$a_{31} > a_{32}. \tag{6.4}$$

From inequality (6.4) and the $\mathcal{A}_3$ analog of (6.1),

$$a_{21} > a_{13} + a_{23} > a_{23}. \tag{6.5}$$

Then as in the proof for $\mathcal{A}_2$, solving for the second coordinate w_2 of the positive left eigenvector $w = [1, w_2, w_3]^T$ gives

$$w_2 = \frac{a_{13}a_{32}}{a_{21}a_{31} - a_{23}a_{31} - a_{21}a_{32}}.$$

Since $w_2 > 0$,

$$a_{21}a_{31} - a_{23}a_{31} - a_{21}a_{32} > 0. \tag{6.6}$$

Evaluate γ where $\det(xI - (A - I)) = x(x^2 + \beta x + \gamma)$ to obtain

$$\gamma = -a_{13}a_{21} - a_{31}a_{21} + a_{32}a_{21} + a_{13}a_{23} + a_{23}a_{31} - a_{13}a_{32}.$$

As in the proof for $\mathcal{A}_2$, substitution using inequalities (6.5) and (6.6) yields $\gamma < 0$ and a contradiction. □

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A publication of the International Linear Algebra Society
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THEOREM 6.4. *A 3×3 sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive or $\mathcal{A}$ is equivalent to a sign pattern of the form*

$$\mathcal{B} = \begin{bmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{bmatrix},$$

where $?$ *is one of* $0, +, -$ *and* $\ominus$ *is one of* $0, -$.

Proof. A sign pattern $\mathcal{A}$ such that $\mathcal{A}^+$ is primitive is PEP by Theorem 2.1. For B as in Example 2.2, it was shown that $\operatorname{sgn}(B)$ is PEP. Since every sign pattern of the form $\mathcal{B}$ is a superpattern of $\operatorname{sgn}(B)$, it follows that any such sign pattern is PEP by Theorem 3.1.

Conversely, assume $\mathcal{A} = [\alpha_{ij}]$ is PEP. If $\mathcal{A}^+$ is irreducible, then by Lemma 6.2, $\mathcal{A}^+$ is primitive. So assume $\mathcal{A}^+$ is reducible. By Observation 4.1, each row and each column must contain a $+$. Let

$$\mathcal{A}_a = \begin{bmatrix} + & \ominus & \ominus \\ ? & ? & + \\ ? & + & ? \end{bmatrix}, \quad \mathcal{A}_b = \begin{bmatrix} + & \ominus & \ominus \\ ? & ? & \ominus \\ ? & ? & + \end{bmatrix}.$$

If $\mathcal{A}^+$ contains a 2-cycle then $\mathcal{A}$ is equivalent to a pattern of the form $\mathcal{A}_a$; otherwise $\mathcal{A}$ is equivalent to a pattern of the form $\mathcal{A}_b$. Every pattern equivalent to a pattern of the form $\mathcal{A}_a$ is equivalent to a pattern in one of the three more explicit forms

$$\mathcal{A}_4 = \begin{bmatrix} + & \ominus & \ominus \\ + & ? & + \\ + & + & ? \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} + & \ominus & \ominus \\ + & ? & + \\ \ominus & + & ? \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} + & \ominus & \ominus \\ \ominus & ? & + \\ \ominus & + & ? \end{bmatrix}.$$

Any pattern of the form $\mathcal{A}_4$ is not PEP by Theorem 5.2. Any pattern of the form $\mathcal{A}_2$ or $\mathcal{A}_1$ is not PEP by Lemma 6.3. Thus $\mathcal{A}$ is not equivalent to a pattern the form $\mathcal{A}_a$.

Now assume $\mathcal{A}$ is equivalent to a pattern of the form $\mathcal{A}_b$. By Corollary 4.5, there must be at least four entries equal to $+$ in $\mathcal{A}$. Thus $\mathcal{A}$ is equivalent to a pattern of one of the three more explicit forms

$$\mathcal{A}_5 = \begin{bmatrix} + & \ominus & \ominus \\ + & ? & \ominus \\ + & ? & + \end{bmatrix}, \quad \mathcal{A}_6 = \begin{bmatrix} + & \ominus & \ominus \\ \ominus & + & \ominus \\ ? & ? & + \end{bmatrix}, \quad \mathcal{A}_7 = \begin{bmatrix} + & \ominus & \ominus \\ + & ? & \ominus \\ \ominus & ? & + \end{bmatrix}.$$

Any pattern of the form $\mathcal{A}_5$ is not PEP by Theorem 5.2. Any pattern of the form $\mathcal{A}_6$ is not PEP by Theorem 5.6.

So $\mathcal{A}$ must be equivalent to a pattern of the form $\mathcal{A}_7$. If the (3,2)-entry of $\mathcal{A}$ is $\ominus$, then the (2,2)-entry is $+$ and $\mathcal{A}$ is not PEP by Theorem 5.6. So the (3,2)-entry of

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 108-120, February 2010

$\mathcal{A}$ is + and $\mathcal{A}$ has the form $\begin{bmatrix} + & \ominus & \ominus \\ + & ? & \ominus \\ \ominus & + & + \end{bmatrix}$. If $\alpha_{31} = 0$, then $\mathcal{A}$ is not PEP by Theorem 5.2. If $\alpha_{12} = 0$, then $\mathcal{A}$ is the pattern $\mathcal{A}_3$ in Lemma 6.3, so is not PEP; the case $\alpha_{23} = 0$ is equivalent. Thus $\alpha_{12} = \alpha_{23} = \alpha_{31} = -$, and $\mathcal{A}$ is equivalent to form $\mathcal{B}$. □

7. Concluding remarks. We have shown that a sufficient condition for a sign pattern to allow eventual positivity, namely that its positive part is primitive, is also necessary for a 2×2 sign pattern. However, this condition is not necessary for a 3×3 sign pattern, as proved in Theorem 6.4. For $n \geq 4$, the identification of necessary and sufficient conditions for an $n \times n$ sign pattern to allow eventual positivity remains open. Also open is the classification of sign patterns that allow eventual nonnegativity. Such sign patterns clearly include those that allow eventual positivity, that allow nilpotency, or are nonnegative.

Acknowledgement. The authors thank the referee for helpful comments, especially with the exposition of Section 6.

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