Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

JORDAN CHAINS OF H-CYCLIC MATRICES, II*

ANDREW L. NICKERSON AND PIETRO PAPARELLA†

Abstract. McDonald and Paparella [*Linear Algebra Appl.* 498 (2016), 145–159] gave a necessary condition on the structure of the Jordan chains of h-cyclic matrices. In this work, that necessary condition is shown to be sufficient. As a consequence, we provide a spectral characterization of nonsingular, h-cyclic matrices.

Key word. Bipartite digraph, Digraph, Circulant matrix, Cyclically h-partite digraph, Jordan canonical form

AMS subject classifications. 15A18, 15A20, 15B99.

1. Introduction.

A celebrated result in spectral graph theory states that a graph is bipartite if and only if the spectrum of its adjacency matrix is *symmetric*, i.e., $-\lambda$ is an eigenvalue whenever λ is. However, as noted by Nikiforov [5, p. 3], a bipartite *digraph* cannot, in general, be characterized by its spectrum alone, and restrictions on the Jordan structure of the matrix must be considered.

Recall that a *directed graph* (or *digraph*, for brevity) $\Gamma = (V, E)$ consists of a finite, nonempty set V of *vertices*, together with a set $E \subseteq V \times V$ of *arcs*. If A is an n-by-n matrix with entries over a field $\mathbb{F}$, then the *digraph of* A, denoted by $\Gamma = \Gamma_A$, has vertex set $V = \{1, \ldots, n\}$ and arc set $E = \{(i,j) \in V \times V \mid a_{ij} \neq 0\}$.

A digraph $\Gamma = (V, E)$ is called *h-partite* if there is a partition $P = \{V_1, \ldots, V_h\}$ of V such that, for each arc $(i,j) \in E$, there are positive integers $k, \ell \in \{1, \ldots, h\}$ such that $i \in V_\ell$ and $j \in V_k$. A digraph $\Gamma = (V, E)$ is called *cyclically h-partite* if there is a partition $P = \{V_1, \ldots, V_h\}$ of V such that, for each arc $(i,j) \in E$, there is a positive integer $\ell \in \{1, \ldots, h\}$ such that $i \in V_\ell$ and $j \in V_{\ell+1}$ (where, for convenience, $V_{h+1} := V_1$). Notice that there is no distinction between a cyclically bipartite digraph and a bipartite digraph, but there is if $h > 2$. Furthermore, notwithstanding its more restrictive nature, characterizing the spectral properties of cyclically h-partite digraphs subsumes the case of bipartite digraphs.

A matrix A is called *h-cyclic* or *cyclically h-partite* if Γ_A is cyclically h-partite. McDonald and Paparella [3, Lemma 4.1] showed that if A is a nonsingular, h-cyclic matrix with complex entries, then a given Jordan chain corresponding to an eigenvalue λ of A determines a Jordan chain corresponding to $\lambda\omega^k$, $1 \leq k \leq h-1$, where $\omega := \exp(2\pi \mathrm{i}/h)$ (for full details, see Lemma 4.1 below). As an immediate consequence, if the nonsingular Jordan block $J_p(\lambda)$ appears in any Jordan canonical form of A, then the nonsingular Jordan block $J_p(\lambda\omega^k)$ also appears in any Jordan canonical form of A, $\forall k \in \{1, \ldots, h-1\}$ [3, Corollary 4.2].

The central aim of this work is to establish a converse of this result—i.e., if a given Jordan chain corresponding to an eigenvalue λ of a matrix $A \in \mathsf{M}_n(\mathbb{C})$ determines a Jordan chain for $\lambda\omega^k$, for every eigenvalue λ of A, then A is h-cyclic (see Theorem 4.5 below). As a consequence, we provide a spectral characterization of nonsingular, h-cyclic matrices.

*Received by the editors on March 11, 2022. Accepted for publication on July 30, 2022. Handling Editor: Adam Berliner. Corresponding Author: Pietro Paparella.

†Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA. (andrewlewis-nickerson@gmail.com, pietrop@uw.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

2. Notation and background. For $n \in \mathbb{N}$, denote by

- $\langle n \rangle$ the set $\{1, \dots, n\}$;
- $R(n)$ the set $\{0, 1, \dots, n-1\}$; and
- $\omega = \omega_n$ the complex number $\exp(2\pi \mathrm{i}/n)$.

The *symmetric group of degree* n is denoted by S_n.

The algebra of m-by-n matrices over a field $\mathbb{F}$ is denoted by $\mathsf{M}_{m\times n}(\mathbb{F})$; when $m = n$, the set $\mathsf{M}_{m\times n}(\mathbb{F})$ is abbreviated to $\mathsf{M}_n(\mathbb{F})$. If $A \in \mathsf{M}_{m\times n}(\mathbb{F})$, then the (i, j)-entry of A is denoted by $[A]_{ij}$, a_{ij}, or $a_{i,j}$.

If $A \in \mathsf{M}_n(\mathbb{C})$ and $R, C \subseteq \{1, \dots, n\}$, then $A(R, C)$ denotes the submatrix of A whose rows and columns are indexed by R and C, respectively.

If $A, B \in \mathsf{M}_{m\times n}(\mathbb{F})$, then the *Hadamard product of* A *and* B, denoted by $A \circ B$, is the $m \times n$ matrix such that $[A \circ B]_{ij} = a_{ij}b_{ij}$.

If $\lambda \in \mathbb{F}$, then the n*-by-*n *Jordan block with eigenvalue* λ, denoted by $J_n(\lambda)$, is defined by

$$J_n(\lambda) = \lambda \sum_{i=1}^{n} E_{ii} + \sum_{i=1}^{n-1} E_{i,i+1} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathsf{M}_n(\mathbb{F}),$$

where E_{ij} is the n-by-n matrix whose (i, j)-entry equals one and zero otherwise.

2.1. Cyclically h-partite matrices. If Γ_A is cyclically h-partite with partition P, then A is said to be *h-cyclic with partition* P or that P *describes the h-cyclic structure of* A. The partition $P = \{V_1, \dots, V_h\}$ is *consecutive* if $V_1 = \{1, \dots, i_1\}$, $V_2 = \{i_1 + 1, \dots, i_2\}, \dots, V_h = \{i_{h-1} + 1, \dots, n\}$. If A is h-cyclic with consecutive partition P, then A is of the form

$$\begin{array}{c} \\ 1 \\ \vdots \\ h-1 \\ h \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & h \end{array} \\ \begin{bmatrix} & A_{12} & & \\ & & \ddots & \\ & & & A_{h-1,h} \\ A_{h1} & & & \end{bmatrix} \end{array}, \tag{2.1}$$

where $A_{i,i+1} = A(V_i, V_{i+1}) \in \mathsf{M}_{|V_i|\times|V_{i+1}|}(\mathbb{C})$, $\forall i \in \langle h \rangle$ [1, p. 71]. If P is not consecutive, then there is a permutation matrix Q such that $Q^\top AQ$ is h-cyclic with consecutive partition [1, p. 71]. Given that the eigenvalues of a matrix do not change with respect to permutation similarity, hereinafter it is assumed, without loss generality, that every h-cyclic matrix is of the form (2.1).

If $x \in \mathbb{C}^n$ and

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix},$$

where $x_i \in \mathbb{C}^{|V_i|}$, then x is said to be *conformably partitioned with* A *(or* P*)*.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

Suppose that $x_1, \ldots, x_p \in \mathbb{C}^n$. Recall that if x_1 is an eigenvector corresponding to $\lambda \in \mathbb{C}$ and $Ax_i = \lambda x_i + x_{i-1}$, $1 < i \leq p$, then $\{x_1, \ldots, x_p\}$ is called a *Jordan chain of A corresponding to λ*. Furthermore, it can be shown that if $\{x_1, \ldots, x_p\}$ is a Jordan chain, then $\{x_1, \ldots, x_p\}$ is linearly independent and $x_k = (A - \lambda I)^{p-k} x_p$, $1 \leq k \leq p$.

The following partial products of submatrices of an h-cyclic matrix will be of use in the sequel.

Let $A \in \mathsf{M}_n(\mathbb{C})$ and suppose that A is of the form (2.1). For $i \in \langle h \rangle$ and $p \in \mathbb{N}$, let

$$B_{ip} := \prod_{j=h+1-p}^{h} A_{\gamma^{j-1}(i),\gamma^{j}(i)},$$

where $\gamma \in S_h$ is the h-cycle of order h defined by $\gamma(i) = i \bmod h + 1$. For ease of notation, the matrix B_{ih} is abbreviated to B_i. Notice that B_i is a square matrix of order $|V_i|$ and

$$A^h = \bigoplus_{j=1}^{h} B_j = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_h \end{bmatrix}, \tag{2.2}$$

(see, e.g., Brualdi and Ryser [1, p. 73]).

We note the following useful theorem due to Mirsky.

THEOREM 2.1 (Mirsky [4, Theorem 1]). *Let $A \in \mathsf{M}_n(\mathbb{C})$ and suppose that A is of the form (2.1). If $\lambda_1, \ldots, \lambda_m$ are the nonzero eigenvalues of B_1, then the spectrum of A consists of $n - hm$ zeros and the hm hth roots of $\lambda_1, \ldots, \lambda_m$.*

2.2. Circulant matrices. We digress to briefly discuss circulant matrices (for a general reference, see Davis [2]).

DEFINITION 2.2 ([2, p. 66]). If $r := (r_1, \ldots, r_n)$, where $r_i \in \mathbb{C}$, $i \in \langle n \rangle$, and $C \in \mathsf{M}_n(\mathbb{C})$, then C is called a *circulant* or a *circulant matrix* with *reference vector* r, denoted $\operatorname{circ}(r)$, if

$$c_{ij} = r_{((j-i) \bmod n)+1},$$

for every $(i,j) \in \langle n \rangle^2$.

DEFINITION 2.3. Denote by e_i the i^{th} canonical basis vector of $\mathbb{C}^n$. For $n \in \mathbb{N}$, $n \geq 2$, the *basic circulant of order n*, denoted by K_n, is the circulant matrix defined by $K_n = \operatorname{circ}(e_2)$.

For example,

$$K_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and, in general,

$$K_n = \begin{bmatrix} & I_{n-1} \\ 1 & \end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

3. Nonsingular h-cyclic matrices. We briefly digress to address some properties of nonsingular, h-cyclic matrices.

THEOREM 3.1. *Let $A \in \mathsf{M}_n(\mathbb{C})$ be h-cyclic and nonsingular. If $P = \{V_1, \ldots, V_h\}$ describes the h-cyclic structure of A, then $|V_i| = |V_j|$, $\forall i, j \in \langle h \rangle$.*

Proof. For contradiction, suppose that $|V_i| \neq |V_j|$. Without losing generality, it may be assumed that $|V_j| > |V_i|$. Notice that $B_j \in \mathsf{M}_{|V_j|}(\mathbb{C})$ and

$$\begin{aligned} \operatorname{rank}(B_j) &\leq \operatorname{rank}\left(\prod_{k=1}^{h} A_{\gamma^{k-1}(j),\gamma^k(j)}\right) \\ &\leq \min\left\{\operatorname{rank}(A_{12}), \ldots, \operatorname{rank}(A_{h-1,h}), \operatorname{rank}(A_{h,1})\right\} \\ &\leq |V_i| \\ &< |V_j|, \end{aligned}$$

i.e., B_j is rank deficient. Thus, $0 \in \sigma(B_j)$, but

$$0 \in \sigma(B_j) \implies 0 \in \sigma(A^h) \iff 0 \in \sigma(A),$$

which contradicts the assumption that A is nonsingular. ◻

COROLLARY 3.2. *If $A \in \mathsf{M}_n(\mathbb{C})$ is h-cyclic and nonsingular, then h divides n.*

Remark 3.3. The converse of Theorem 3.1 does not hold; indeed, consider the singular bipartite matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

4. Main results. Given that it will be used in several of the subsequent results, hereinafter, $\alpha_{ij} := (i - j) \bmod h$, $\forall i, j \in \mathbb{Z}$ and $\forall h \in \mathbb{N}$.

The following result was established by McDonald and Paparella [3, Lemma 4.1] for nonsingular matrices; however, examining the proof reveals that the supposition is unnecessary.

LEMMA 4.1 ([3, Lemma 4.1]). *Let $A \in \mathsf{M}_n(\mathbb{C})$ and suppose that A is of the form (2.1).*

1. *If $\left\{x_{\langle 0,j \rangle}\right\}_{j=1}^{p}$ is a right Jordan chain corresponding to $\lambda \in \sigma(A)$, where $x_{\langle 0,j \rangle}$ is partitioned conformably with respect to A as*

$$x_{\langle 0,j \rangle} = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{hj} \end{bmatrix},\ j \in \langle p \rangle,$$

then, for $k \in R(h)$, the set

$$\left\{ x_{\langle k,j \rangle} := \begin{bmatrix} (\omega^k)^{\alpha_{1j}} x_{1j} \\ \vdots \\ (\omega^k)^{\alpha_{hj}} x_{hj} \end{bmatrix} \right\}_{j=1}^{p},$$

is a right Jordan chain corresponding to $\lambda\omega^k$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

2. *If* $\left\{y_{\langle j,0\rangle}\right\}_{j=1}^{p}$ *is a left Jordan chain corresponding to* $\lambda \in \sigma(A)$*, where* $y_{\langle j,0\rangle}$ *is partitioned conformably with respect to* A *as*

$$y_{\langle j,0\rangle}^\top = \begin{bmatrix} y_{j1}^\top & \cdots & y_{jh}^\top \end{bmatrix},\ j \in \langle p\rangle,$$

then, for $k \in R(h)$*, the set*

$$\left\{ y_{\langle j,k\rangle}^\top := \begin{bmatrix} (\omega^k)^{\alpha_{j1}} y_{j1}^\top & \cdots & (\omega^k)^{\alpha_{jh}} y_{jh}^\top \end{bmatrix} \right\}_{j=1}^{p},$$

is a left Jordan chain corresponding to $\lambda\omega^k$*.*

Remark 4.2. Although McDonald and Paparella stated the aforementioned result for nonzero eigenvalues, the proof given is valid when $\lambda = 0$. However, if any Jordan canonical form of A has a singular Jordan block of size p-by-p, then the result does not guarantee another singular Jordan block in the Jordan canonical form of A; e.g., the 3-cyclic matrix

$$A := \left[\begin{array}{c|ccc|cc} 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

has one singular Jordan block of order two and one singular Jordan block of order one.

The following results were presented by McDonald and Paparella [3, Lemma 4.3 and Remark 4.4].

Lemma 4.3 (c.f. [3, Lemma 4.3]). *If* $k \in R(h)$ *and* $\ell \in \langle p\rangle$*, then*

$$\begin{aligned} W_{k\ell}^1 &:= \omega^k \begin{bmatrix} (\omega^k)^{\alpha_{1\ell}} \\ \vdots \\ (\omega^k)^{\alpha_{h\ell}} \end{bmatrix} \begin{bmatrix} (\omega^k)^{\alpha_{\ell 1}} & \dots & (\omega^k)^{\alpha_{\ell h}} \end{bmatrix} \\ &= \operatorname{circ}(\omega^k, 1, (\omega^k)^{h-1}, \dots, (\omega^k)^2), \end{aligned}$$

and

$$\begin{aligned} W_{k\ell}^2 &:= \begin{bmatrix} (\omega^k)^{\alpha_{1\ell}} \\ \vdots \\ (\omega^k)^{\alpha_{h\ell}} \end{bmatrix} \begin{bmatrix} (\omega^k)^{\alpha_{(\ell+1)1}} & \dots & (\omega^k)^{\alpha_{(\ell+1)h}} \end{bmatrix} \\ &= \operatorname{circ}(\omega^k, 1, (\omega^k)^{h-1}, \dots, (\omega^k)^2). \end{aligned}$$

Lemma 4.4 (c.f. [3, Remark 4.4]). *If* $k \in R(h)$ *and*

$$C_k := \operatorname{circ}(\omega^k, 1, (\omega^k)^{h-1}, \dots, (\omega^k)^2) \in \mathsf{M}_h(\mathbb{C}), \tag{4.3}$$

then

$$\sum_{k=0}^{h-1} C_k = hK_h = \operatorname{circ}(0, h, 0, \dots, 0).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

THEOREM 4.5. *Let $A \in \mathsf{M}_n(\mathbb{C})$ and let $P = \{V_1, \ldots, V_h\}$ be a partition of $\langle n \rangle$.*

(i) *Suppose that, for every eigenvalue $\lambda \in \sigma(A)$ with corresponding right Jordan chain $\{x_{\langle 0,j\rangle}\}_{j=1}^{p}$, and whenever the vector $x_{\langle 0,j\rangle}$ is partitioned conformably with respect to P as*

$$x_{\langle 0,j\rangle} = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{hj} \end{bmatrix},\ j \in \langle p \rangle,$$

the set

$$\left\{ x_{\langle k,j\rangle} := \begin{bmatrix} (\omega^k)^{\alpha_{1j}} x_{1j} \\ \vdots \\ (\omega^k)^{\alpha_{hj}} x_{hj} \end{bmatrix} \right\}_{j=1}^{p},$$

is a right Jordan chain corresponding to $\lambda\omega^k$ for every $k \in R(h)$.

(ii) *Suppose that, for every eigenvalue $\lambda \in \sigma(A)$ with corresponding left Jordan chain $\{y_{\langle j,0\rangle}\}_{j=1}^{p}$, and whenever the vector $y_{\langle j,0\rangle}$ is partitioned conformably with respect to P as*

$$y_{\langle j,0\rangle}^\top = \begin{bmatrix} y_{j1}^\top & \cdots & y_{jh}^\top \end{bmatrix},\ j \in \langle p \rangle,$$

the set

$$\left\{ y_{\langle j,k\rangle}^\top := \begin{bmatrix} (\omega^k)^{\alpha_{j1}} y_{j1}^\top & \cdots & (\omega^k)^{\alpha_{jh}} y_{jh}^\top \end{bmatrix} \right\}_{j=1}^{p},$$

is a left Jordan chain corresponding to $\lambda\omega^k$ for every $k \in R(h)$

If the above hold, then A is h-cyclic with partition P.

Proof. The proof that follows is similar to the proof of Theorem 4.6 given by McDonald and Paparella [3, pp. 155–156], which we include for completeness.

By hypothesis, any Jordan canonical form of A is of the form

$$S^{-1}AS = \bigoplus_{i=1}^{m} \left(\bigoplus_{k=0}^{h-1} J_{n_i}\left(\lambda_i \omega^k\right) \right), 1 \le m < n.$$

For $i \in \langle m \rangle$, let

$$D_i := \operatorname{diag}\left(0, \ldots, 0, \overbrace{\bigoplus_{k=0}^{h-1} J_{n_i}(\lambda_i\omega^k)}^{i}, 0, \ldots, 0 \right),$$

and $A_i := SD_iS^{-1} \in \mathsf{M}_n(\mathbb{C})$, where $\operatorname{diag}(M_1, \ldots, M_k)$ denotes the block-diagonal matrix with block-diagonal entries $M_1, \ldots, M_k$.

For $k \in R(h)$, let C_k be defined as in (4.3). By Lemmas 4.3 and 4.4 and properties of the Hadamard product, notice that

$$A_i = \sum_{k=0}^{h-1} \left(\sum_{j=1}^{p_i} \lambda_i \omega^k x_{\langle k,j\rangle} y_{\langle j,k\rangle}^\top + \sum_{j=1}^{p_i - 1} x_{\langle k,j\rangle} y_{\langle j+1,k\rangle}^\top \right),$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

$$
\begin{aligned}
&= \sum_{k=0}^{h-1}\left(\sum_{j=1}^{p_i}\lambda_i\omega^k\begin{bmatrix}(\omega^k)^{\alpha_{1j}}x_{1j}\\ \vdots\\ (\omega^k)^{\alpha_{hj}}x_{hj}\end{bmatrix}\begin{bmatrix}(\omega^k)^{\alpha_{j1}}y_{j1}^\top & \cdots & (\omega^k)^{\alpha_{jh}}y_{jh}^\top\end{bmatrix} + \right.\\
&\qquad \left.\sum_{j=1}^{p_i-1}\begin{bmatrix}(\omega^k)^{\alpha_{1j}}x_{1j}\\ \vdots\\ (\omega^k)^{\alpha_{hj}}x_{hj}\end{bmatrix}\begin{bmatrix}(\omega^k)^{\alpha_{j+1,1}}y_{j+1,1}^\top & \cdots & (\omega^k)^{\alpha_{j+1,h}}y_{j+1,h}^\top\end{bmatrix}\right)\\
&= \lambda_i\sum_{k=0}^{h-1}\sum_{j=1}^{p_i}W_{kj}^1\circ x_{\langle 0,j\rangle}y_{\langle j,0\rangle}^\top + \sum_{k=0}^{h-1}\sum_{j=1}^{p_i-1}W_{kj}^2\circ x_{\langle 0,j\rangle}y_{\langle j+1,0\rangle}^\top\\
&= \lambda_i\sum_{j=1}^{p_i}\left(\sum_{k=0}^{h-1}C_k\circ x_{\langle 0,j\rangle}y_{\langle j,0\rangle}^\top\right) + \sum_{j=1}^{p_i-1}\left(\sum_{k=0}^{h-1}C_k\circ x_{\langle 0,j\rangle}y_{\langle j+1,0\rangle}^\top\right)\\
&= \lambda_i\sum_{j=1}^{p_i}\left[\left(\sum_{k=0}^{h-1}C_k\right)\circ x_{\langle 0,j\rangle}y_{\langle j,0\rangle}^\top\right] + \sum_{j=1}^{p_i-1}\left[\left(\sum_{k=0}^{h-1}C_k\right)\circ x_{\langle 0,j\rangle}y_{\langle j+1,0\rangle}^\top\right]\\
&= \lambda_i h\sum_{j=1}^{p_i}K_h\circ x_{\langle 0,j\rangle}y_{\langle j,0\rangle}^\top + h\sum_{j=1}^{p_i-1}K_h\circ x_{\langle 0,j\rangle}y_{\langle j+1,0\rangle}^\top\\
&= \lambda_i h\sum_{j=1}^{p_i}\begin{bmatrix} & x_{1j}y_{j2}^\top & & \\ & & \ddots & \\ & & & x_{(h-1)j}y_{jh}^\top\\ x_{hj}y_{j1}^\top & & & \end{bmatrix} + \\
&\qquad h\sum_{j=1}^{p_i-1}\begin{bmatrix} & x_{1j}y_{j+1,2}^\top & & \\ & & \ddots & \\ & & & x_{h-1,j}y_{j+1,h}^\top\\ x_{hj}y_{j+1,1}^\top & & & \end{bmatrix}\\
&= \begin{bmatrix} & A_{12}^{(i)} & & \\ & & \ddots & \\ & & & A_{h-1,h}^{(i)}\\ A_{h1}^{(i)} & & & \end{bmatrix},
\end{aligned}
$$

where

$$
A_{k,k+1}^{(i)} := \lambda_i h\sum_{j=1}^{p_i}x_{kj}y_{j,k+1}^\top + h\sum_{j=1}^{p_i-1}x_{kj}y_{j+1,k+1}^\top \in \mathsf{M}_{|V_k|\times|V_{k+1}|}(\mathbb{C}),
$$

$k \in \langle h\rangle$, and, for convenience, $h+1 = 1$.

Clearly, the matrices $A_1,\ldots,A_m$ are h-cyclic with partition P and since $A = \sum_{i=1}^m A_i$, it follows that A is h-cyclic with partition P. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 486-493, August 2022.

Example 4.6. Consider the matrices

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and

$$S = \begin{bmatrix} x & a & x & -a \\ y & b & y & -b \\ z & c & -z & c \\ w & d & -w & d \end{bmatrix},$$

where $\det(S) = -4(ay - bx)(cw - dz) \neq 0$.

If $A = SJS^{-1}$ and $P = \{\{1,2\},\{3,4\}\}$, then A satisfies the hypotheses of Theorem 4.5, so A is bipartite. Indeed, a calculation via a computer algebra system reveals that

$$A = \begin{bmatrix} 0 & 0 & \frac{wx}{cw-dz} & \frac{-xz}{cw-dz} \\ 0 & 0 & \frac{wy}{cw-dz} & \frac{-yz}{cw-dz} \\ \frac{yz}{ay-bx} & \frac{-xz}{ay-bx} & 0 & 0 \\ \frac{wy}{ay-bx} & \frac{-xw}{ay-bx} & 0 & 0 \end{bmatrix}.$$

The following result yields a complete characterization of the Jordan structure for invertible h-cyclic matrices.

THEOREM 4.7. *If $A \in \mathsf{M}_n(\mathbb{C})$ is nonsingular and $P = \{V_1, \ldots, V_h\}$ is a partition of $\langle n \rangle$, then A is h-cyclic with partition P if and only if the Jordan chains of A satisfy conditions 4.5 and 4.5 of Theorem 4.5.*

Proof. Follows by Lemma 4.1 and Theorem 4.5. ∎

Acknowledgements. The second author thanks Daniel Szyld for his talk at the 2019 Meeting of the International Linear Algebra Society that provided the impetus for this work and Michael Tsatsomeros for pointing out that Lemma 4.1 of McDonald and Paparella [3] generalized the result for bipartite graphs.

REFERENCES

[1] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*, vol. 39. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge, 1991.

[2] P.J. Davis. *Circulant Matrices*. A Wiley-Interscience Publication. John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[3] J.J. McDonald and P. Paparella. Jordan chains of h-cyclic matrices. *Linear Algebra Appl.*, 498:145–159, 2016.

[4] L. Mirsky. An inequality for characteristic roots and singular values of complex matrices. *Monatsh. Math.*, 70:357–359, 1966.

[5] V. Nikiforov. Hypergraphs and hypermatrices with symmetric spectrum. *Linear Algebra Appl.*, 519:1–18, 2017.