Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

W-WEIGHTED GDMP INVERSE FOR RECTANGULAR MATRICES*

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Abstract. In this article, we introduce two new generalized inverses for rectangular matrices called W-weighted generalized-Drazin–Moore–Penrose (GDMP) and W-weighted generalized-Drazin-reflexive (GDR) inverses. The first generalized inverse can be seen as a generalization of the recently introduced GDMP inverse for a square matrix to a rectangular matrix. The second class of generalized inverse contains the class of the first generalized inverse. We then exploit their various properties and establish that the proposed generalized inverses coincide with different well-known generalized inverses under certain assumptions. We also obtain a representation of W-weighted GDMP inverse employing EP-core nilpotent decomposition. We define the dual of W-weighted GDMP inverse and obtain analogue results. Further, we discuss additive properties, reverse- and forward-order laws for GD, W-weighted GD, GDMP, and W-weighted GDMP generalized inverses.

Key words. Generalized inverse, GD inverse, W-weighted GD inverse, GDMP inverse, W-weighted GDMP inverse.

AMS subject classifications. 15A09.

1. Introduction and motivation. Let $\mathbb{C}^{m\times n}$ be the set of all complex matrices of size $m \times n$. Let A^*, $R(A)$, $N(A)$, and P_A denote the conjugate transpose of A, the range space of A, the null space of A, and the orthogonal projection onto the range space of A, respectively. Given $A \in \mathbb{C}^{m\times n}$, the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the following four matrix equations:

$$(1)AXA = A,\ (2)XAX = X,\ (3)(AX)^* = AX, \quad \text{and} \quad (4)(XA)^* = XA,$$

is called the *Moore–Penrose inverse* [26] of A, and is denoted as $A^\dagger$. The set of all matrices which satisfies any of the combinations of the above four matrix equations is denoted as $A\{i, j, k, l\}$, where $i, j, k, l \in \{1, 2, 3, 4\}$. For instance, if X satisfies equations (1) and (2), then $A\{1, 2\}$ denotes the set of all solutions of the first two matrix equations. We denote a member of $A\{1, 2\}$ as $A^{(1,2)}$, and it is called a *reflexive inverse* of the matrix A. The definition of the index of a matrix is recalled next. Let $A \in \mathbb{C}^{n\times n}$. The smallest nonnegative integer for which $rank(A^k) = rank(A^{k+1})$ is called the *index* of the matrix A, and we denote it by $ind(A)$. Let $A \in \mathbb{C}^{n\times n}$, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the matrix equations $XAX = X$, $XA = AX$, and $A^{k+1}X = A^k$ is called the *Drazin inverse* [5] of the matrix A. It is denoted as A^D. Here, k denotes the index of the matrix A. If $ind(A) = 1$, then the above equations reduce to $XAX = X$, $XA = AX$, and $A^2X = A$, and in this case, X is called the *group inverse* of A. The group inverse of a matrix A is denoted as $A^\#$. A matrix $A \in \mathbb{C}^{n\times n}$ is called EP (or range-Hermitian) if $R(A) = R(A^*)$. A matrix A is EP if and only if it commutes with its Moore–Penrose inverse, that is, $AA^\dagger = A^\dagger A$. EP matrices are also characterized as the class of matrices for which the Moore–Penrose inverse and the group inverse are the same.

In 1980, Cline and Greville [8] extended the Drazin inverse for square matrices to rectangular matrices, which is recalled next. Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$, the unique matrix $X = ((AW)^D)^2A \in \mathbb{C}^{m\times n}$ is the solution of equations: $(AW)^{k+1}XW = (AW)^k$, $XWAWX = X$, and $AWX = XWA$, where

*Received by the editors on March 9, 2022. Accepted for publication on September 16, 2022. Handling Editor: Oskar Baksalary. Corresponding Author: Vaibhav Shekhar.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

$k = ind(AW)$. In this case, X is called the *W-weighted Drazin inverse* of A. It is denoted as $A_{D,W}$. Throughout this article, we will consider a nonzero weight W. The literature for generalized inverses is quite rich due to their enormous applicability in several fields. In this direction, one of the most significant recent generalized inverses is the core inverse introduced by Baksalary and Trenkler [3], which is recalled next. Let $A \in \mathbb{C}^{n\times n}$, the matrix $X \in \mathbb{C}^{n\times n}$ is called the *core inverse* of A if $AX = P_A$ and $R(X) \subseteq R(A)$. The core inverse of a matrix is unique and is denoted by $A^{\textcircled{\#}}$. Motivated by the work of Baksalary and Trenkler [3], several authors introduced different generalized inverses and justified their application to linear equations (see [4], [19], [25], [27], [33], and the references cited therein). In 2014, Malik and Thome [19] introduced a new generalized inverse for square matrices, called DMP inverse, as follows: for any $A \in \mathbb{C}^{n\times n}$, the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the matrix equations $XAX = X$, $XA = A^D A$, and $A^k X = A^k A^\dagger$, where $k = ind(A)$, is called *DMP inverse* of A. It can be computed using the expression $X = A^D AA^\dagger$. In 2017, Meng [21] generalized the notion of the DMP inverse to a matrix of arbitrary order, the author called it W-weighted DMP inverse. The definition of W-weighted DMP inverse is recalled next. For any matrix $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ such that $ind(AW) = k$, the unique matrix $X = WA_{D,W}WAA^\dagger$ that satisfies the following equations: $XAX = X$, $XA = WA_{D,W}WA$ and $(WA)^{k+1}X = (WA)^{k+1}A^\dagger$, is called the *$W$-weighted DMP inverse* of A and is denoted as $A_W^{D,\dagger}$.

These generalized inverses have numerous applications. For example, the Moore–Penrose inverse is used to find the least-squares solution of a given linear system. The group inverse is applied to solve a problem involving Markov chains. The Drazin inverse helps to solve singular differential equations and has applications in numerical analysis, neural computing, partial orders, etc. The core inverse helps study partial order theory and find the Bott–Duffin inverse. MPCEP inverse is used to solve linear systems of equations arising in chemical equations, robotics, coding theory, etc. Interested readers are referred to [3], [5], [6], [14], [15], [17], [23], [25], and [32].

In 2016, Wang and Liu [29] proposed a new generalized inverse called generalized Drazin (or GD) inverse as follows. For $A \in \mathbb{C}^{n\times n}$, a matrix $X \in \mathbb{C}^{n\times n}$ is called *GD inverse* of A if

$$AXA = A,\ XA^{k+1} = A^k \text{ and } A^{k+1}X = A^k,$$

where $k = ind(A)$. It is denoted by $X = A^{GD}$. In general, this inverse is not unique. We denote the set of all GD inverse of a matrix A by $A\{GD\}$. In 2018, Coll *et al.* [9] introduced weighted generalized Drazin inverse (WG-Drazin), which is an extension of the generalized Drazin inverse for a square matrix to a rectangular matrix. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, and $k = max\{k_1, k_2\}$. A matrix $X \in \mathbb{C}^{m\times n}$ is *W-weighted G-Drazin inverse* of A if it satisfies the following matrix equations:

$$WAWXWAW = WAW, \tag{1.1}$$

$$(AW)^{k+1}(XW) = (AW)^k, \tag{1.2}$$

$$(WX)(WA)^{k+1} = (WA)^k. \tag{1.3}$$

It is denoted by $A^{GD,W}$ [9]. The set of all weighted generalized Drazin inverse of matrix A is denoted by $A\{GD, W\}$. We further refer interested readers to [9], [11], [24], and [28] for more works on generalized inverses and their extensions.

In 2020, Hernández *et al.* [13] introduced another generalized inverse called generalized-Drazin–Moore–Penrose (*GDMP) inverse*. The definition of GDMP inverse is stated next. Let $A \in \mathbb{C}^{n\times n}$ and $k = ind(A)$. For each $A^{GD} \in A\{GD\}$, a GDMP inverse of A, denoted by $A^{GD\dagger}$, is an $n \times n$ matrix $A^{GD\dagger} = A^{GD}AA^\dagger$.

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This inverse is also not unique. The symbol $A\{GD\dagger\}$ stands for the set of all GDMP inverses of A. However, the notion of GDMP is limited to square matrices only. This article aims at expanding the applicability of this generalized inverse. To do this, we redefine it so that the new generalized inverse exists for a larger class of matrices.

Let A and B be invertible matrices. Then, $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)^{-1} = A^{-1}B^{-1}$ are known as reverse-order law and forward-order law, respectively. The first expression is known to be always true, while the later expression is not always true. These laws also do not hold for generalized inverses in general. In 1966, Greville [12] first obtained some sufficient conditions under which the reverse-order law holds for the Moore–Penrose inverse, that is, $(AB)^\dagger = B^\dagger A^\dagger$. The same problem was considered by several authors for other generalized inverses also. For example, Xiong and Zheng [31] provided some sufficient conditions for the reverse-order laws of {1,2,3}- and {1,2,4}-inverses. In 2016, Wang *et al.* [30] obtained a few results of the reverse-order law for the Drazin inverse. Deng [10] studied the reverse-order law for the group inverse on Hilbert space. The reverse-order law is used to analyze Markov chains [22]. Also, it directly applies to the celebrated Karmarkar algorithm [16]. Similarly, the forward-order law has many applications in numerical linear algebra (see [1]). In 2018, Castro-González and Hartwig [7] provided some sufficient conditions for the forward-order law for the Moore–Penrose inverse, that is, $(AB)^\dagger = A^\dagger B^\dagger$. In the same year, Liu and Xiong [18] presented forward-order laws for {1,2,3}- and {1,2,4}-inverses.

In 2022, Baksalary *et al.* [2] provided certain sufficient conditions under which the Moore–Penrose inverse is additive, that is, $(A+B)^\dagger = A^\dagger + B^\dagger$. Motivated by the works of these authors, we obtain various sufficient conditions for the reverse-order law, the forward-order law and the additive property for GD, W-weighted GD, GDMP, and W-weighted GDMP inverse.

This article aims to propose two new generalized inverses and investigate their properties by imposing certain conditions. To fulfill our objective, the rest of this article is organized as follows. In Section 2, we recall some preliminary results. We then propose two new generalized inverses in Section 3, which are extensions of GDMP inverse, and we call it W-weighted GDMP inverse and W-weighted GDR inverse, respectively. After that, we investigate some of their properties and obtain their representations. Section 4 discusses some results on dual W-weighted GDMP, which are analogous to those established in the previous sections. Section 5 establishes the reverse-order law, the forward-order law and the additive property for GD inverse, W-weighted GD inverse, GDMP inverse, and W-weighted GDMP inverse, respectively.

2. Preliminaries.

This section recalls a few terminologies that this article uses frequently and also collects some established results from the literature that play a significant role while proving our main results in the next sections. $O \in \mathbb{C}^{m\times n}$ is the null matrix of size $m \times n$. I_r denotes the identity matrix of size $r \times r$. Let $A_1 \in \mathbb{C}^{r_1\times r_2}$ and $A_2 \in \mathbb{C}^{r_3\times r_4}$. Then, $A_1 \oplus A_2$ represents the diagonal block matrix $\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} \in \mathbb{C}^{(r_1+r_3)\times(r_2+r_4)}$. Now, we state the first result, which is proved by Coll *et al.* [9], and is for W-weighted G-Drazin inverse of a matrix.

Lemma 2.1 (Remark 2.1, [9]). *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$.*

(i) *If $A = O$, then any matrix of suitable size is a W-weighted G-Drazin inverse of A.*
(ii) $A\{GD, W\} \subseteq WAW\{1\}$.
(iii) *If AW and WA are nilpotent matrices, then $WAW\{1\} \subseteq A\{GD, W\}$.*

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Next, we state a result that gives a representation of W-weighted G-Drazin inverse of a matrix.

Theorem 2.2 (Theorem 2.1, [9]). *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$ and $k_2 = ind(WA)$. Then, there exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$A = P(A_1 \oplus A_2)Q^{-1} \text{ and } W = Q(W_1 \oplus W_2)P^{-1},$$

where A_1 and W_1 are $t \times t$ nonsingular matrices, and A_2W_2 and W_2A_2 are nilpotent matrices of indices k_1 and k_2, respectively. Moreover, $X \in A\{GD, W\}$ if and only if

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix} Q^{-1},$$

with $X_{12}W_2 = O, W_2X_{21} = O$ and $X_2 \in W_2A_2W_2\{1\}$. In particular, if $m = n$ and $AW = WA$, then $Q = P$. In this case, if $W = I_n$, then $W_1 = I_t$ and $W_2 = I_{n-t}$.

Theorem 2.3 (Proposition 2.13, [2]). *Let $P, Q \in \mathbb{C}^{n\times n}$ be orthogonal projections. Then, $(P + Q)^\dagger = P + Q$ if and only if $PQ = O$.*

3. Main results. In this section, we discuss the main results of this article. In particular, we propose two new generalized inverses and then investigate their properties. We first define W-weighted GDMP inverse of a matrix of arbitrary order.

Definition 3.1. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. Let $A^{GD,W} \in A\{GD, W\}$, a W-weighted GDMP inverse of A, denoted by $A^{GD\dagger,W}$, be an $n \times m$ matrix*

$$A^{GD\dagger,W} = WA^{GD,W}WP_A,$$

where P_A denotes the orthogonal projection onto the space $R(A)$.

Remark 3.2. *From the above definition, it is clear that a W-weighted GDMP inverse of A coincides with a GDMP inverse when $W = I$. Let $A = O \in \mathbb{C}^{m\times n}$. Then, $O \in \mathbb{C}^{n\times m}$ is a W-weighted GDMP inverse of A.*

The following example demonstrates Definition 3.1.

Example 3.3. *Let $A = \begin{bmatrix} 1 & 1 \end{bmatrix} \in \mathbb{C}^{1\times 2}$ and $W = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \mathbb{C}^{2\times 1}$. Clearly, $A^\dagger = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, $AW = 0$, $WA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ and $k = max\{1 = ind(AW),\ 2 = ind(WA)\} = 2$. Then, by the definition of W-weighted GD inverse, we get $A^{GD,W} = \begin{bmatrix} a & b \end{bmatrix} \in \mathbb{C}^{1\times 2}$, where $a, b \in \mathbb{C}$ are arbitrary. Further, we have $P_A = 1$. Now, $A^{GD\dagger,W} = WA^{GD,W}WP_A = \begin{bmatrix} a - b \\ b - a \end{bmatrix}$, where $a, b \in \mathbb{C}$ are arbitrary.*

From the above example, it is clear that, in general, a W-weighted GDMP inverse of a matrix A is not unique. If we replace $A^\dagger$ by $A^{(1,2)}$ in the Definition 3.1, then we will have another generalized inverse that contains the class of previous generalized inverse, and we call it W-weighted GDR, where R stands for reflexive (since $A^{(1,2)}$ is a reflexive inverse of A). This definition is produced below.

Definition 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. Let $A^{GD,W} \in A\{GD, W\}$, a W-weighted generalized-Drazin-reflexive (GDR) inverse of A, denoted by $A^{GDR,W}$, be an $n \times m$ matrix*

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$$A^{GDR,W} = WA^{GD,W}WT_A,$$

where $T_A = AA^{(1,2)}$.

REMARK 3.5. *Note that in the above definition,* T_A *is used just for the notational simplicity to make it similar to* W*-weighted GDMP inverse definition. Also, from the above definitions, it is clear that* $A\{GD\dagger, W\} \subseteq A\{GDR, W\}$.

Now, we prove our first main result of this section with the help of Definition 3.1.

LEMMA 3.6. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ *and* $k = max\{ind(AW), ind(WA)\}$. *If* $A^{GD\dagger,W} \in A\{GD\dagger, W\}$ *and* $A^{GD,W} \in A\{GD, W\}$, *then a* W*-weighted GDMP inverse of the matrix* A *satisfies the following properties:*

(i) $WAA^{GD\dagger,W}AW = WAW$.
(ii) $A^{GD\dagger,W}A = WA^{GD,W}WA$.
(iii) $A^{GD\dagger,W}A(WA)^k = (WA)^k$.
(iv) $WAA^{GD\dagger,W} = W$, *if* $W = A^*$.
(v) $A^{GD\dagger,W}(AW)^{k+1} = (WA)^kW = W(AW)^k$.
(vi) $AA^{GD\dagger,W}(AW)^{k+1} = (AW)^{k+1}$ *and* $A^{GD\dagger,W}(AW)^{k+1}A = (WA)^{k+1}$.
(vii) $(WA)^{k+1}A^{GD\dagger,W} = W(AW)^kP_A = (WA)^kWP_A$.
(viii) $A^{GD\dagger,W}P_A = A^{GD\dagger,W}$.
(ix) $A^{GD\dagger,W}A(WA)^{k+1}A^{GD\dagger,W} = (WA)^{k+1}A^\dagger$.
(x) $A^{GD\dagger,W}AA^{GD\dagger,W}AW = A^{GD\dagger,W}AW$.
(xi) $WAA^{GD\dagger,W}AA^{GD\dagger,W} = WAA^{GD\dagger,W}$.
(xii) $(A^{GD\dagger,W}A)(WA)^{k+1} = (WA)^{k+1}(A^{GD\dagger,W}A)$.

Proof. (i) $WAA^{GD\dagger,W}AW = WAWA^{GD,W}WAA^\dagger AW = WAWA^{GD,W}WAW = WAW$.
(ii) $A^{GD\dagger,W}A = WA^{GD,W}WP_AA = WA^{GD,W}WA$.
(iii) $A^{GD\dagger,W}A(WA)^k = WA^{GD,W}WP_AA(WA)^k = WA^{GD,W}WA(WA)^k = WA^{GD,W}(WA)^{k+1} = (WA)^k$.
(iv) Since $W = A^*$, we have $WW^\dagger = A^\dagger A$ and $W^\dagger W = AA^\dagger$. Thus, $WAA^{GD\dagger,W} = WAWA^{GD,W}W AA^\dagger AA^\dagger = WAWA^{GD,W}WAWW^\dagger A^\dagger = WAWW^\dagger A^\dagger = WAA^\dagger AA^\dagger = WAA^\dagger = WW^\dagger W = W$.
(v) $A^{GD\dagger,W}(AW)^{k+1} = WA^{GD,W}WP_AAW(AW)^k = WA^{GD,W}W(AW)^{k+1} = WA^{GD,W}(WA)^{k+1}W = (WA)^kW = W(AW)^k$.
(vi) Pre-multiplying $A^{GD\dagger,W}(AW)^{k+1} = W(AW)^k$ by A, we obtain $AA^{GD\dagger,W}(AW)^{k+1} = (AW)^{k+1}$. Similarly, post-multiplying $A^{GD\dagger,W}(AW)^{k+1} = (WA)^kW$ by A, we get $A^{GD\dagger,W}(AW)^{k+1}A = (WA)^{k+1}$.
(vii) $(WA)^{k+1}A^{GD\dagger,W} = (WA)^{k+1}WA^{GD,W}WP_A = W(AW)^{k+1}A^{GD,W}WP_A = W(AW)^kP_A = (WA)^kWP_A$.
(viii) $A^{GD\dagger,W}P_A = WA^{GD,W}WAA^\dagger AA^\dagger = WA^{GD,W}WP_A = A^{GD\dagger,W}$.
(ix) From (vii), $A^{GD\dagger,W}A(WA)^{k+1}A^{GD\dagger,W} = A^{GD\dagger,W}(AW)^{k+1}P_A$. Now, applying (v), we get $A^{GD\dagger,W}A(WA)^{k+1}A^{GD\dagger,W} = (WA)^kWP_A = (WA)^{k+1}A^\dagger$.
(x) $A^{GD\dagger,W}AA^{GD\dagger,W}AW = WA^{GD,W}WAA^\dagger AWA^{GD,W}WAA^\dagger AW = WA^{GD,W}WAW\ A^{GD,W}WAW = WA^{GD,W}WAW = A^{GD\dagger,W}AW$.
(xi) Similar to part (x).
(xii) $(A^{GD\dagger,W}A)(WA)^{k+1} = WA^{GD,W}WAA^\dagger A(WA)^{k+1} = WA^{GD,W}WA(WA)^{k+1} = WA^{GD,W}(WA)^{k+1}(WA) = (WA)^k(WA) = (WA)^{k+1}$. And $(WA)^{k+1}(A^{GD\dagger,W}A) =$

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

$(WA)^{k+1}WA^{GD,W}WAA^{\dagger}A = (WA)^{k+1}WA^{GD,W}WA = W(AW)^{k+1}A^{GD,W}WA = W(AW)^{k}A = (WA)^{k+1}$. So, $(A^{GD\dagger,W}A)(WA)^{k+1} = (WA)^{k+1}(A^{GD\dagger,W}A)$. □

THEOREM 3.7. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. If $W = A^*$, then $WAA^{GD\dagger,W}W^{\dagger}$ is an orthogonal projection onto $R(W)$.*

Proof. By Lemma 3.6 (iv), we have $WAA^{GD\dagger,W} = W$. Post-multiplying by $W^{\dagger}$, we get $WAA^{GD\dagger,W}W^{\dagger} = WW^{\dagger}$. But, $WW^{\dagger}$ is an orthogonal projection onto $R(W)$. Hence, $WAA^{GD\dagger,W}W^{\dagger}$ is an orthogonal projection onto $R(W)$. □

Under certain fixed weight W, a W-weighted GDMP inverse coincides with different well-known generalized inverses. This fact is investigated in the following result.

THEOREM 3.8. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. For each $A^{GD,W} \in A\{GD, W\}$, a W-weighted GDMP inverse of the matrix A satisfies the following properties:*

(i) *If $W \in A\{1\}$, then $A^{GD\dagger,W} \in A\{1\}$.*
(ii) *If $W \in A\{2\}$, and $WA^{GD,W}$ is idempotent, then $A^{GD\dagger,W} \in A\{2\}$.*
(iii) *If $W = A^*$, then $A^{GD\dagger,W} = A^*A^{GD,W}A^*$.*
(iv) *If $W = A^{\dagger}$, then $A^{GD\dagger,W}AA^{GD\dagger,W} = A^{GD\dagger,W} = A^{\dagger}$.*

Proof. (i) If $W \in A\{1\}$, then we have

$$\begin{aligned} AA^{GD\dagger,W}A &= AA^{(1)}A^{GD,W}A^{(1)}P_AA \\ &= AA^{(1)}A^{GD,W}A^{(1)}A. \end{aligned} \tag{3.4}$$

By Lemma 3.6 (i), we get $A^{(1)}AA^{(1)}A^{GD,W}A^{(1)}AA^{(1)} = A^{(1)}AA^{(1)}$, pre and post-multiplying by A, we have $AA^{(1)}A^{GD,W}A^{(1)}A = A$. Now, using this expression in (3.4), we get $AA^{GD\dagger,W}A = A$, and thus $A^{GD\dagger,W} \in A\{1\}$.

(ii) As $W \in A\{2\}$, we have $WAW = W$. Also, $WA^{GD,W}$ is idempotent, so $(WA^{GD,W})^2 = WA^{GD,W}$. Now,

$$\begin{aligned} A^{GD\dagger,W}AA^{GD\dagger,W} &= WA^{GD,W}WP_AAWA^{GD,W}WP_A \\ &= WA^{GD,W}WAWA^{GD,W}WP_A \\ &= (WA^{GD,W})^2WP_A \\ &= WA^{GD,W}WP_A \\ &= A^{GD\dagger,W}. \end{aligned}$$

(iii) If $W = A^*$, then

$$\begin{aligned} A^{GD\dagger,W} &= A^*A^{GD,W}A^*P_A \\ &= A^*A^{GD,W}A^*. \end{aligned}$$

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(iv) If $W = A^\dagger$, then

$$\begin{aligned}A^{GD\dagger,W}AA^{GD\dagger,W} &= A^\dagger A^{GD,W}A^\dagger AA^\dagger AA^\dagger A^{GD,W}A^\dagger AA^\dagger \\ &= A^\dagger AA^\dagger A^{GD,W}A^\dagger AA^\dagger A^{GD,W}A^\dagger AA^\dagger \\ &= A^\dagger AA^\dagger A^{GD,W}A^\dagger AA^\dagger \\ &= A^{GD\dagger,W} \\ &= WA^{GD,W}WP_A \\ &= A^\dagger AA^\dagger \\ &= A^\dagger.\end{aligned}$$

□

The following result provides a sufficient condition under which a W-weighted GDMP inverse of an EP matrix coincides with its Moore–Penrose inverse.

THEOREM 3.9. *Let $A \in \mathbb{C}^{m\times m}$ be an EP matrix and $W = P_A$. Then, $A^{GD\dagger,W} = A^\dagger$.*

Proof. If $W = P_A$, then the equation $WAWA^{GD,W}WAW = WAW$ implies that $A^{GD,W} \in A\{1\}$. Now, $A^{GD\dagger,W} = P_AA^{GD,W}P_A^2 = P_AA^{GD,W}P_A = AA^\dagger A^{GD,W}AA^\dagger$. Since A is an EP matrix, therefore, A and $A^\dagger$ commute and thus, $A^{GD\dagger,W} = A^\dagger AA^{GD,W}AA^\dagger$. Further, as $A^{GD,W} \in A\{1\}$, we have $A^{GD\dagger,W} = A^\dagger AA^\dagger = A^\dagger$. □

If we know the nilpotency of the matrices $A^{GD\dagger,W}(AW)^{k+1}A$ and $AA^{GD\dagger,W}(AW)^{k+1}$, then we can guarantee that the sets $WAW\{1\}$ and $A\{GD,W\}$ are the same. This is shown in the next result.

THEOREM 3.10. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. If $A^{GD\dagger,W}(AW)^{k+1}A$ and $AA^{GD\dagger,W}(AW)^{k+1}$ are both nilpotent, then $WAW\{1\} = A\{GD,W\}$.*

Proof. By (vi) of Lemma 3.6, we have $AA^{GD\dagger,W}(AW)^{k+1} = (AW)^{k+1}$ and $A^{GD\dagger,W}(AW)^{k+1}A = (WA)^{k+1}$ which imply that $(AW)^{k+1}$ and $(WA)^{k+1}$ both are nilpotent. So, AW and WA both are nilpotent. By Lemma 2.1, we get $WAW\{1\} \subseteq A\{GD,W\}$. Also, $A\{GD,W\} \subseteq WAW\{1\}$ is true by the definition of $A\{GD,W\}$. Hence, $WAW\{1\} = A\{GD,W\}$. □

The following result shows that if X is a $\{1\}$-inverse of WAW, then it is also a $\{1\}$-inverse of $WAA^{GD\dagger,W}AW$.

THEOREM 3.11. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. Then, $WAW\{1\} \subseteq WAA^{GD\dagger,W}AW\{1\}$.*

Proof. Let $X \in WAW\{1\}$. By the Lemma 3.6 (i), we have $WAA^{GD\dagger,W}AW = WAW$. So,

$$WAA^{GD\dagger,W}AWXWAA^{GD\dagger,W}AW = WAWXWAW.$$

Since, $X \in WAW\{1\}$, using the above equation, we have

$$WAA^{GD\dagger,W}AWXWAA^{GD\dagger,W}AW = WAW = WAA^{GD\dagger,W}AW.$$

Hence, $WAW\{1\} \subseteq WAA^{GD\dagger,W}AW\{1\}$. □

Now, we obtain some representations of the introduced generalized inverses. In this direction, we first prove the following result which talks about the form of W-weighted GDR for a particular form of $\{1,2\}$-inverse of $A_1 \oplus A_2$. For proving this, we will use a result from [20] which says that $A_1^{(1,2)} \oplus A_2^{(1,2)} \in (A_1 \oplus A_2)\{1,2\}$. Clearly, if A_1 is nonsingular, then $A_1^{(1,2)}$ coincides with the usual inverse A_1^{-1}.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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THEOREM 3.12. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k = max\{k_1 = ind(AW),\ k_2 = ind(WA)\}$. Then, there exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$A = P(A_1 \oplus A_2)Q^{-1} \text{ and } W = Q(W_1 \oplus W_2)P^{-1},$$

where A_1 and W_1 are $t \times t$ nonsingular matrices, and A_2W_2 and W_2A_2 are nilpotent matrices of indices k_1 and k_2, respectively. Moreover, for a particular $A_1^{-1} \oplus A_2^{(1,2)} \in (A_1 \oplus A_2)\{1, 2\}$, the representation of GDR is

$$A^{GDR,W} = Q\begin{bmatrix} A_1^{-1} & O \\ O & W_2X_2W_2T_{A_2} \end{bmatrix} P^{-1},$$

where $X_2 \in W_2A_2W_2\{1\}$ and $T_{A_2} = A_2A_2^{(1,2)}$.

Proof. From Theorem 2.2, we have

$$A = P(A_1 \oplus A_2)Q^{-1} \text{ and } W = Q(W_1 \oplus W_2)P^{-1},$$

where A_1 and W_1 are $t \times t$ nonsingular matrices, and A_2W_2 and W_2A_2 are nilpotent matrices of indices k_1 and k_2, respectively. Now,

$$\begin{aligned}
A^{GDR,W} &= WA^{GD,W}WAA^{(1,2)} \\
&= Q\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix}\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}\begin{bmatrix} A_1^{-1} & O \\ O & A_2^{(1,2)} \end{bmatrix} P^{-1} \\
&= Q\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix}\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} I & O \\ O & A_2A_2^{(1,2)} \end{bmatrix} P^{-1} \\
&= Q\begin{bmatrix} W_1(W_1A_1W_1)^{-1}W_1 & W_1X_{12}W_2T_{A_2} \\ W_2X_{21}W_1 & W_2X_2W_2T_{A_2} \end{bmatrix} P^{-1} \\
&= Q\begin{bmatrix} A_1^{-1} & W_1X_{12}W_2T_{A_2} \\ W_2X_{21}W_1 & W_2X_2W_2T_{A_2} \end{bmatrix} P^{-1}.
\end{aligned}$$

By Theorem 2.2, we have $X_{12}W_2 = O$ and $W_2X_{21} = O$, so we get

$$A^{GDR,W} = Q\begin{bmatrix} A_1^{-1} & O \\ O & W_2X_2W_2T_{A_2} \end{bmatrix} P^{-1},$$

where $X_2 \in W_2A_2W_2\{1\}$ and $T_{A_2} = A_2A_2^{(1,2)}$. ∎

The following example is provided in support of Theorem 3.12.

EXAMPLE 3.13. *Let* $A = \begin{bmatrix} -1 & 3 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{bmatrix} \in \mathbb{C}^{3\times 4}$ *and*

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

$W = \begin{bmatrix} -1 & 3/2 & 3/2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \in \mathbb{C}^{4\times 3}$. *Decomposing* A *and* W, *we have*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and

$$W = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Clearly, A *and* W *have the forms* $A = P(A_1 \oplus A_2)Q^{-1}$ *and* $W = Q(W_1 \oplus W_2)P^{-1}$, *respectively, where* $A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $W_1 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, $A_2 = [0 \quad 1]$, $W_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *Here,* $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$ *are invertible matrices. Further,* $W_2A_2W_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Thus, taking* $X_2 = \begin{bmatrix} a & b \end{bmatrix} \in W_2A_2W_2\{1\}$, *where* $a, b \in \mathbb{C}$ *and using Theorem 3.12, we get*

$$A^{GDR,W} = Q \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} P^{-1},$$

where $a \in \mathbb{C}$ *is arbitrary.*

Since $R(AA^{GD\dagger,W}) = R(AWA^{GD,W}WAA^{\dagger}) \subseteq R(AWA^{GD,W})$ and $N(A^{GD,W}WA) \subseteq N(AWA^{GD,W}WA) = N(A^{GD\dagger,W}A)$. So, we have the following remark.

Remark 3.14. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$. *Then,*

(i) $R(AA^{GD\dagger,W}) \subseteq R(AWA^{GD,W})$.
(ii) $N(A^{GD,W}WA) \subseteq N(A^{GD\dagger,W}A)$.

If we assume WA and AW to be EP matrices, then with the help of EP-core nilpotent decomposition, we can compute a W-weighted GDMP inverse of a matrix.

Theorem 3.15. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *be such that both* AW *and* WA *are EP matrices,* $k_1 = ind(AW)$, *and* $k_2 = ind(WA)$. *Then, there exist unitary matrices* $P \in \mathbb{C}^{m\times m}$ *and* $Q \in \mathbb{C}^{n\times n}$ *such that*

$$A = P(A_1 \oplus A_2)Q^* \text{ and } W = Q(W_1 \oplus W_2)P^*,$$

where A_1 *and* W_1 *are* $t \times t$ *nonsingular matrices, and* A_2W_2 *and* W_2A_2 *are null matrices.*

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

Further, a W-weighted GDMP inverse of A is given by

$$A^{GD\dagger,W} = Q\begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} P^*.$$

In particular, if $m = n$ and $AW = WA$, then $Q = P$. In this case, if $W = I_n$, then $W_1 = I_t$ and $W_2 = I_{n-t}$.

Proof. Suppose that one of the following two disjoint situations $m \neq n$ or $m = n$ with $AW \neq WA$ holds. As AW and WA are EP matrices, there exist unitary matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that

$$AW = P(T \oplus O)P^* \text{ and } WA = Q(S \oplus O)Q^*,$$

where T and S are nonsingular matrices [6]. It is obvious that $(AW)^k A = A(WA)^k$, where $k = max\{k_1, k_2\}$. Let

$$A = P\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} Q^* \text{ and } W = Q\begin{bmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{bmatrix} P^*.$$

Then,

$$(AW)^k A = P\begin{bmatrix} T^k & O \\ O & O \end{bmatrix}\begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} Q^* = P\begin{bmatrix} T^k A_1 & T^k A_{12} \\ O & O \end{bmatrix} Q^*$$

and

$$A(WA)^k = P\begin{bmatrix} A_1 S^k & O \\ A_{21} S^k & O \end{bmatrix} P^*.$$

So, $A_{12} = O$ and $A_{21} = O$, and therefore $A = P\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} Q^*$. Now,

$$AW = P\begin{bmatrix} A_1 W_1 & A_1 W_{12} \\ A_2 W_{21} & A_2 W_2 \end{bmatrix} P^* = P\begin{bmatrix} T & O \\ O & O \end{bmatrix} P^*.$$

Thus, $W_{12} = O$, $A_2 W_{21} = O$ and $A_2 W_2 = O$. Similarly, when we calculate WA, we get $W_{21} = O$ and $W_2 A_2 = O$. Using the equation $WAWA^{GD,W}WAW = WAW$, we get

$$A^{GD,W} = P\begin{bmatrix} (W_1 A_1 W_1)^{-1} & X_{21} \\ X_{21} & X_2 \end{bmatrix} Q^*,$$

such that $X_{12} W_2 = O$ and $W_2 X_{21} = O$. Hence,

$$\begin{aligned}
A^{GD\dagger,W} &= WA^{GD,W}WP_A \\
&= Q\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix}\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}\begin{bmatrix} A_1^{-1} & O \\ O & A_2^{\dagger} \end{bmatrix} P^* \\
&= Q\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} (W_1A_1W_1)^{-1} & X_{12} \\ X_{21} & X_2 \end{bmatrix}\begin{bmatrix} W_1 & O \\ O & W_2 \end{bmatrix}\begin{bmatrix} I & O \\ O & A_2A_2^{\dagger} \end{bmatrix} P^* \\
&= Q\begin{bmatrix} W_1(W_1A_1W_1)^{-1}W_1 & W_1X_{12}W_2 \\ W_2X_{21}W_1 & W_2X_2W_2 \end{bmatrix}\begin{bmatrix} I & O \\ O & A_2A_2^{\dagger} \end{bmatrix} P^* \\
&= Q\begin{bmatrix} A_1^{-1} & O \\ O & W_2X_2W_2A_2A_2^{(1,2)} \end{bmatrix} P^* \\
&= Q\begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} P^* \qquad (\because W_2A_2{=}0).
\end{aligned}$$

∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

The following example demonstrates Theorem 3.15.

EXAMPLE 3.16. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 3}$ *and* $W = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$. *Then,* $AW = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $WA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *are EP matrices. The EP decompositions of the matrices A and W are given as*

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$W = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix},$$

which are in forms $A = P(A_1 \oplus A_2)Q^*$ and $W = Q(W_1 \oplus W_2)P^*$, where $A_1 = [2]$, $W_1 = [1]$, $A_2 = [0 \ \ 0]$, $W_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ *and* $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Therefore, by Theorem 3.15, we have*

$$A^{GD\dagger,W} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \\ 0 & 0 \end{bmatrix}.$$

Note that the representation of a W-weighted GDMP inverse need not be unique as the matrices P and Q need not be unique in EP-core nilpotent decomposition.

4. Dual W-weighted GDMP or W-weighted MPGD. Similar to the definition of W-weighted GDMP inverse, we can define its dual. This section is devoted to the brief discussion on dual W-weighted GDMP (or W-weighted MPGD). Most of the results obtained in the previous section can be obtained analogously for this particular generalized inverse. Therefore, we discuss only some of the important results without its proof. First, we define dual W-weighted GDMP inverse.

DEFINITION 4.1. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. Let $A^{GD,W} \in A\{GD, W\}$, a dual W-weighted GDMP inverse of A, denoted by $A^{\dagger GD,W}$, be an $n \times m$ matrix*

$$A^{\dagger GD,W} = P_{A^*} W A^{GD,W} W,$$

where P_{A^*} denotes the orthogonal projection onto the space $R(A^*)$.

We present our first result of this section below.

LEMMA 4.2. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. For each $A^{GD,W} \in A\{GD, W\}$, a dual W-weighted GDMP inverse of the matrix A satisfies the following properties:*

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(i) $WAA^{\dagger GD,W}AW = WAW$.
(ii) $WAA^{\dagger GD,W}AA^{\dagger GD,W} = WAA^{\dagger GD,W}$.
(iii) $A^{\dagger GD,W}AA^{\dagger GD,W}AW = A^{\dagger GD,W}AW$.
(iv) $P_{A^*}A^{\dagger GD,W} = A^{\dagger GD,W}$.
(v) $(WA)^{k+1}A^{\dagger GD,W} = W(AW)^k$.
(vi) $(WA)^{k+1}A^{\dagger GD,W}A = (WA)^{k+1}$.
(vii) $A^{\dagger GD,W}(AW)^{k+1} = P_{A^*}(WA)^kW$.
(viii) $AA^{\dagger GD,W}(AW)^{k+1} = (AW)^{k+1}$.
(ix) $(AW)^kAA^{\dagger GD,W} = (AW)^k$.

The following result can be proved proceeding similarly as in Theorem 3.8.

Theorem 4.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. For each $A^{GD,W} \in A\{GD, W\}$, a dual W-weighted GDMP inverse of the matrix A satisfies the following properties:*

(i) *If WA and AW both are idempotent matrices, then $AA^{\dagger GD,W}A = AWA$.*
(ii) *If $W \in A\{1\}$, then $A^{\dagger GD,W} \in A\{1\}$.*
(iii) *If $W = A^*$, then $A^{\dagger GD,W} = A^*A^{GD,W}A^*$.*
(iv) *If $W = A^\dagger$, then $A^{\dagger GD,W}AA^{\dagger GD,W} = A^{\dagger GD,W} = WAW = A^\dagger$.*

Similarly, we can prove the next result as in Theorem 3.9.

Theorem 4.4. *Let $A \in \mathbb{C}^{m\times m}$ be an EP matrix and $W = P_{A^*}$. Then, $A^{\dagger GD,W} = A^\dagger$.*

We end this section by stating the next result which is an analogue of Theorem 3.15.

Theorem 4.5. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ be such that both AW and WA are EP matrices, $k_1 = ind(AW)$, and $k_2 = ind(WA)$. Then, there exist unitary matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$A = P(A_1 \oplus A_2)Q^* \text{ and } W = Q(W_1 \oplus W_2)P^*,$$

where A_1 and W_1 are $t \times t$ nonsingular matrices, and A_2W_2 and W_2A_2 are null matrices.
Further, a dual W-weighted GDMP inverse of A is given by

$$A^{\dagger GD,W} = Q\begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix}P^*.$$

In particular, if $m = n$ and $AW = WA$, then $Q = P$. In this case, if $W = I_n$, then $W_1 = I_t$ and $W_2 = I_{n-t}$.

5. Reverse-order law, forward-order law, and additive property. In this section, we present some sufficient conditions under which the reverse-order law, forward-order law, and additive property hold for GD, W-weighted GD, GDMP, and W-weighted GDMP inverse. Also, we discuss some results in which the absorption law holds for GD inverse.

5.1. GD inverse. First, we illustrate the reverse-order law for GD inverse under the assumption of a few conditions.

Theorem 5.1. *Let $A, B \in \mathbb{C}^{m\times m}$ with $AB^2 = B^2A = BAB$ and $k = max\{ind(A), ind(B)\} \geq 2$. If $ABB^{GD} = BB^{GD}A$ and $BB^{GD}A^{GD} = A^{GD}BB^{GD}$, then $(AB)^{GD} = B^{GD}A^{GD}$.*

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

Proof. If $AB^2 = B^2A = BAB$, then $(AB)^k = A^kB^k = B^kA^k$, $A^{k+1}B^k = B^kA^{k+1}$ and $A^kB^{k+1} = B^{k+1}A^k$ for $k \geq 2$. Now, $ABB^{GD}A^{GD}AB = AA^{GD}BB^{GD}AB = AA^{GD}ABB^{GD}B = AB$. Further, we have

$$\begin{aligned}
B^{GD}A^{GD}(AB)^{k+1} &= B^{GD}A^{GD}A^{k+1}B^{k+1} \\
&= B^{GD}A^kB^{k+1} \\
&= B^{GD}B^{k+1}A^k \\
&= B^kA^k \\
&= A^kB^k \\
&= (AB)^k,
\end{aligned}$$

and

$$\begin{aligned}
(AB)^{k+1}B^{GD}A^{GD} &= A^{k+1}B^{k+1}B^{GD}A^{GD} \\
&= A^{k+1}B^kA^{GD} \\
&= B^kA^{k+1}A^{GD} \\
&= B^kA^k \\
&= (AB)^k.
\end{aligned}$$

Using the definition of GD inverse, it clearly shows that $(AB)^{GD} = B^{GD}A^{GD}$. ∎

If $AB = BA$ and $BB^{GD}A^{GD} = A^{GD}BB^{GD}$, then the reverse-order law for GD inverse is true for every positive integer k.

THEOREM 5.2. *Let $A, B \in \mathbb{C}^{m\times m}$ and $k = max\{ind(A), ind(B)\}$. If $AB = BA$ and $BB^{GD}A^{GD} = A^{GD}BB^{GD}$, then $(AB)^{GD} = B^{GD}A^{GD}$.*

Proof. We have $AA^{GD}A = A$, $A^{GD}A^{k+1} = A^k$, $A^{k+1}A^{GD} = A$ and $BB^{GD}B = B$, $B^{GD}B^{k+1} = B^k$, $B^{k+1}B^{GD} = B^k$. Now,

$$\begin{aligned}
ABB^{GD}A^{GD}AB &= AA^{GD}BB^{GD}AB \\
&= AA^{GD}BB^{GD}BA \\
&= AA^{GD}BA \\
&= AA^{GD}AB \\
&= AB, \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
B^{GD}A^{GD}(AB)^{k+1} &= B^{GD}A^{GD}A^{k+1}B^{k+1} \\
&= B^{GD}A^kB^{k+1} \\
&= B^{GD}B^{k+1}A^k \\
&= B^kA^k \\
&= A^kB^k \\
&= (AB)^k, \qquad (5.6)
\end{aligned}$$

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A publication of the International Linear Algebra Society
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and

$$\begin{aligned}(AB)^{k+1}B^{GD}A^{GD} &= A^{k+1}B^{k+1}B^{GD}A^{GD}\\ &= A^{k+1}B^{k}A^{GD}\\ &= B^{k}A^{k+1}A^{GD}\\ &= B^{k}A^{k}\\ &= (AB)^{k}. \end{aligned} \tag{5.7}$$

From (5.5), (5.6), and (5.7), we get $(AB)^{GD} = B^{GD}A^{GD}$. ◻

The next result discusses the forward-order law involving GD inverse.

THEOREM 5.3. *Let $A, B \in \mathbb{C}^{m\times m}$ and $k = max\{ind(A), ind(B)\}$. If $AB = BA$ and $B^{GD}BA = AB^{GD}B$, then $(AB)^{GD} = A^{GD}B^{GD}$.*

Proof. Clearly, $AA^{GD}A = A$, $A^{GD}A^{k+1} = A^k$, $A^{k+1}A^{GD} = A$, $BB^{GD}B = B$, $B^{GD}B^{k+1} = B^k$, and $B^{k+1}B^{GD} = B^k$. Now,

$$\begin{aligned}ABA^{GD}B^{GD}AB &= BAA^{GD}B^{GD}BA\\ &= BAA^{GD}AB^{GD}B\\ &= BAB^{GD}B\\ &= ABB^{GD}B\\ &= AB, \end{aligned} \tag{5.8}$$

$$\begin{aligned}A^{GD}B^{GD}(AB)^{k+1} &= A^{GD}B^{GD}B^{k+1}A^{k+1}\\ &= A^{GD}B^{k}A^{k+1}\\ &= A^{GD}A^{k+1}B^{k}\\ &= A^{k}B^{k}\\ &= (AB)^{k}, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}(AB)^{k+1}A^{GD}B^{GD} &= B^{k+1}A^{k+1}A^{GD}B^{GD}\\ &= B^{k+1}A^{k}B^{GD}\\ &= A^{k}B^{k+1}B^{GD}\\ &= A^{k}B^{k}\\ &= (AB)^{k}. \end{aligned} \tag{5.10}$$

From (5.8), (5.9), and (5.10), we get $(AB)^{GD} = A^{GD}B^{GD}$. ◻

A set of necessary conditions are obtained in the next result for the absorption law of a GD inverse (i.e., $A^{GD}(A + B)B^{GD} = A^{GD} + B^{GD}$).

THEOREM 5.4. *Let $A, B \in \mathbb{C}^{m\times m}$ and $k = max\{ind(A), ind(B)\}$. If $A^{GD}(A + B)B^{GD} = A^{GD} + B^{GD}$, then $AA^{GD}BB^{GD} = AA^{GD}$, $A^{GD}AB^{GD}B = B^{GD}B$, $A^kBB^{GD} = A^k$, and $A^{GD}AB^k = B^k$.*

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Proof. We have

$$A^{GD}(A+B)B^{GD} = A^{GD} + B^{GD}. \tag{5.11}$$

Pre-multiplying by A^{k+1} and A in equation (5.11), we get

$$\begin{aligned} A^{k+1}A^{GD}(A+B)B^{GD} &= A^{k+1}A^{GD} + A^{k+1}B^{GD} \\ A^k(A+B)B^{GD} &= A^k + A^{k+1}B^{GD} \\ A^{k+1}B^{GD} + A^kBB^{GD} &= A^k + A^{k+1}B^{GD} \\ A^kBB^{GD} &= A^k, \end{aligned}$$

and

$$\begin{aligned} AA^{GD}(A+B)B^{GD} &= AA^{GD} + AB^{GD} \\ AB^{GD} + AA^{GD}BB^{GD} &= AA^{GD} + AB^{GD} \\ AA^{GD}BB^{GD} &= AA^{GD}, \end{aligned}$$

respectively. Again, post-multiply by B^{k+1} and B in equation (5.11), we get

$$\begin{aligned} A^{GD}(A+B)B^{GD}B^{k+1} &= A^{GD}B^{k+1} + B^{GD}B^{k+1} \\ A^{GD}(A+B)B^k &= A^{GD}B^{k+1} + B^k \\ A^{GD}AB^k + A^{GD}B^{k+1} &= A^{GD}B^{k+1} + B^k \\ A^{GD}AB^k &= B^k \end{aligned}$$

and

$$\begin{aligned} A^{GD}(A+B)B^{GD}B &= A^{GD}B + B^{GD}B \\ A^{GD}AB^{GD}B + A^{GD}BB^{GD}B &= A^{GD}B + B^{GD}B \\ A^{GD}AB^{GD}B + A^{GD}B &= A^{GD}B + B^{GD}B \\ A^{GD}AB^{GD}B &= B^{GD}B, \end{aligned}$$

respectively. ∎

An immediate consequence of the above result is shown next as a corollary.

COROLLARY 5.5. *Let $A, B \in \mathbb{C}^{m\times m}$ and $k = max\{ind(A), ind(B)\}$. If $A^{GD}(A+B)B^{GD} = A^{GD}+B^{GD}$, then $R(B^k) \subseteq R(A^{GD}AB^{GD}B)$ and $R(A^k) = R(A^kB)$.*

Proof. From Theorem 5.4, we get

$$R(B^k) = R(A^{GD}AB^k) = R(A^{GD}AB^{GD}B^{k+1}) \subseteq R(A^{GD}AB^{GD}B).$$

Again, from Theorem 5.4, we get $R(A^k) = R(A^kBB^{GD}) \subseteq R(A^kB) \subseteq R(A^k)$ which implies that $R(A^k) = R(A^kB)$. ∎

Baksalary *et al.* [2] proved that $(A+B)^\dagger = A^\dagger + B^\dagger$ under certain assumptions. Sufficient conditions for $(A+B)^{GD} = A^{GD} + B^{GD}$ are obtained next.

THEOREM 5.6. *Let $A, B \in \mathbb{C}^{m\times m}$ with $AB = BA = O$ and $k = max\{ind(A), ind(B)\}$. If $A^{GD}B = BA^{GD} = O$ and $B^{GD}A = AB^{GD} = O$, then $(A+B)^{GD} = A^{GD} + B^{GD}$.*

Proof. We have $AB = BA = O$, so by the binomial expansion $(A+B)^n = A^n + B^n$ for every positive integer n. Further, we have

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$$
\begin{aligned}
(A+B)(A^{GD}+B^{GD})(A+B) &= (AA^{GD}+AB^{GD}+BA^{GD}+BB^{GD})(A+B)\\
&= AA^{GD}A+AB^{GD}A+BA^{GD}A+BB^{GD}A+AA^{GD}B\\
&\quad + AB^{GD}B+BA^{GD}B+BB^{GD}B\\
&= AA^{GD}A+BB^{GD}B\\
&= A+B, \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
(A^{GD}+B^{GD})(A+B)^{k+1} &= (A^{GD}+B^{GD})(A^{k+1}+B^{k+1})\\
&= A^{GD}A^{k+1}+A^{GD}B^{k+1}+B^{GD}A^{k+1}+B^{GD}B^{k+1}\\
&= A^{k}+A^{GD}B^{k+1}+B^{GD}A^{k+1}+B^{k}\\
&= A^{k}+B^{k}\\
&= (A+B)^{k}, \qquad (5.13)
\end{aligned}
$$

and

$$
\begin{aligned}
(A+B)^{k+1}(A^{GD}+B^{GD}) &= (A^{k+1}+B^{k+1})(A^{GD}+B^{GD})\\
&= A^{k+1}A^{GD}+A^{k+1}B^{GD}+B^{k+1}A^{GD}+B^{k+1}B^{GD}\\
&= A^{k}+B^{k}\\
&= (A+B)^{k}. \qquad (5.14)
\end{aligned}
$$

From (5.12), (5.13), and (5.14), we get $(A+B)^{GD}=A^{GD}+B^{GD}$. ∎

5.2. W-Weighted GD inverse. If W is involutory, then the reverse-order law holds for W-weighted GD inverse under some conditions. Throughout this subsection, we consider $k=max\{ind(WA), ind(AW), ind(WB), ind(BW)\}$.

THEOREM 5.7. *Let $A,B\in\mathbb{C}^{m\times m}$, and $W\in\mathbb{C}^{m\times m}$ be an involutory matrix with $WA=AW$ and $WB=BW$. If $AB=BA$, $WBWA^{GD,W}=A^{GD,W}WBW$ and $WAWB^{GD,W}=B^{GD,W}WAW$, then $(AB)^{GD,W}=B^{GD,W}A^{GD,W}$.*

Proof. We have $W^2=I$ and $AB=BA$. Now, pre and post-multiplying by W, we get

$$
\begin{aligned}
WABW &= WBAW\\
WAW^2BW &= WBW^2AW\\
(WAW)(WBW) &= (WBW)(WAW). \qquad (5.15)
\end{aligned}
$$

Now, we check $B^{GD,W}A^{GD,W}$ is a W-weighted GD inverse of AB.

$$
\begin{aligned}
WABWB^{GD,W}A^{GD,W}WABW &= WBWWAWB^{GD,W}A^{GD,W}WBWWAW\\
&= WBWB^{GD,W}WAWWBWA^{GD,W}WAW\\
&= WBWB^{GD,W}WBWWAWA^{GD,W}WAW\\
&= WBWWAW\\
&= WBAW\\
&= WABW. \qquad (5.16)
\end{aligned}
$$

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If $W^{k+1} = I$, then

$$\begin{aligned}
(ABW)^{k+1}B^{GD,W}A^{GD,W}W &= A^{k+1}(BW)^{k+1}B^{GD,W}A^{GD,W}W \\
&= A^{k+1}W^{k+1}(BW)^{k+1}B^{GD,W}A^{GD,W}W \\
&= A^{k+1}W^{k+1}(BW)^{k+1}B^{GD,W}WWA^{GD,W}W \\
&= (AW)^{k+1}(BW)^{k}WA^{GD,W}W \\
&= (BW)^{k}W(AW)^{k+1}A^{GD,W}W \\
&= (BW)^{k}W(AW)^{k} \\
&= (ABW)^{k}, \qquad (5.17)
\end{aligned}$$

and

$$\begin{aligned}
WB^{GD,W}A^{GD,W}(WAB)^{k+1} &= WB^{GD,W}A^{GD,W}(WA)^{k+1}B^{k+1} \\
&= WB^{GD,W}A^{GD,W}(WA)^{k+1}W^{k+1}B^{k+1} \\
&= WB^{GD,W}WWA^{GD,W}(WA)^{k+1}W^{k+1}B^{k+1} \\
&= WB^{GD,W}W(WA)^{k}(WB)^{k+1} \\
&= WB^{GD,W}(WB)^{k+1}W(WA)^{k} \\
&= (WB)^{k}W(WA)^{k} \\
&= (WAB)^{k}. \qquad (5.18)
\end{aligned}$$

If $W^{k} = I$, then

$$\begin{aligned}
(ABW)^{k+1}B^{GD,W}A^{GD,W}W &= A^{k+1}(BW)^{k+1}B^{GD,W}A^{GD,W}W \\
&= A^{k+1}W^{k+2}(BW)^{k+1}B^{GD,W}WWA^{GD,W}W \\
&= A^{k+1}W^{k+1}W(BW)^{k+1}B^{GD,W}WWA^{GD,W}W \\
&= W(AW)^{k+1}(BW)^{k}WA^{GD,W}W \\
&= (BW)^{k}(AW)^{k+1}A^{GD,W}W \\
&= (BW)^{k}(AW)^{k} \\
&= W^{2k}(AB)^{k} \\
&= W^{k}(AB)^{k} \\
&= (ABW)^{k}, \qquad (5.19)
\end{aligned}$$

and

$$\begin{aligned}
WB^{GD,W}A^{GD,W}(WAB)^{k+1} &= WB^{GD,W}A^{GD,W}(WA)^{k+1}B^{k+1} \\
&= WB^{GD,W}WWA^{GD,W}(WA)^{k+1}W^{k+2}B^{k+1} \\
&= WB^{GD,W}WWA^{GD,W}(WA)^{k+1}WW^{k+1}B^{k+1} \\
&= WB^{GD,W}W(WA)^{k}W(WB)^{k+1} \\
&= WB^{GD,W}(WB)^{k+1}(WA)^{k} \\
&= (WB)^{k}(WA)^{k} \\
&= (W)^{2k}(BA)^{k} \\
&= (W)^{k}(BA)^{k} \\
&= (WAB)^{k}. \qquad (5.20)
\end{aligned}$$

Hence, $(AB)^{GD,W} = B^{GD,W}A^{GD,W}$. ∎

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

The proof for the forward-order law of W-weighted GD inverse is similar to the proof of Theorem 5.7 and is stated next.

THEOREM 5.8. *Let $A, B \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{m\times m}$ be an involutory matrix with $WA = AW$ and $WB = BW$. If $AB = BA$, $WBWA^{GD,W} = A^{GD,W}WBW$ and $WAWB^{GD,W} = B^{GD,W}WAW$, then $(AB)^{GD,W} = A^{GD,W}B^{GD,W}$.*

The following result collects a set of sufficient conditions for the additive property of W-weighted GD inverse.

THEOREM 5.9. *Let $A, B, W \in \mathbb{C}^{m\times m}$ with $AWBW = BWAW = WBWA = WAWB = O$. If $B^{GD,W}WA = AWB^{GD,W} = O$ and $A^{GD,W}WB = BWA^{GD,W} = O$, then $(A+B)^{GD,W} = A^{GD,W}+B^{GD,W}$.*

Proof. Putting $A^{GD,W} + B^{GD,W}$ in W-weighted GD inverse definition, we get

$$\begin{aligned}
&W(A+B)W(A^{GD,W}+B^{GD,W})W(A+B)W\\
&\quad = (WAW+WBW)(A^{GD,W}+B^{GD,W})(WAW+WBW)\\
&\quad = (WAW+WBW)(A^{GD,W}WAW + A^{GD,W}WBW\\
&\qquad + B^{GD,W}WAW + B^{GD,W}WBW)\\
&\quad = (WAW+WBW)(A^{GD,W}WAW + B^{GD,W}WBW)\\
&\quad = WAWA^{GD,W}WAW + WAWB^{GD,W}WBW\\
&\qquad + WBWA^{GD,W}WAW + WBWB^{GD,W}WBW\\
&\quad = WAW + WBW\\
&\quad = W(A+B)W.
\end{aligned}$$

We have $BWAW = AWBW = O$, so by the binomial expansion $(AW+BW)^{k+1} = (AW)^{k+1} + (BW)^{k+1}$. Further, we obtain

$$\begin{aligned}
(AW+BW)^{k+1}(A^{GD,W}+B^{GD,W})W &= ((AW)^{k+1}+(BW)^{k+1})(A^{GD,W}W + B^{GD,W}W)\\
&= (AW)^{k+1}A^{GD,W}W + (AW)^{k+1}B^{GD,W}W\\
&\quad + (BW)^{k+1}A^{GD,W}W + (BW)^{k+1}B^{GD,W}W\\
&= (AW)^k + (BW)^k\\
&= (AW+BW)^k\\
&= ((A+B)W)^k.
\end{aligned}$$

We have $WBWA = WAWB = O$, so by the binomial expansion $(WA+WB)^{k+1} = (WA)^{k+1} + (WB)^{k+1}$. Further, we get

$$\begin{aligned}
W(A^{GD,W}+B^{GD,W})(WA+WB)^{k+1} &= W(A^{GD,W}+B^{GD,W})((WA)^{k+1}+(WB)^{k+1})\\
&= WA^{GD,W}(WA)^{k+1} + WB^{GD,W}(WA)^{k+1}\\
&\quad + WA^{GD,W}(WB)^{k+1} + WB^{GD,W}(WB)^{k+1}\\
&= (WA)^k + (WB)^k\\
&= (WA+WB)^k\\
&= (W(A+B))^k.
\end{aligned}$$

Hence, $(A+B)^{GD,W} = A^{GD,W} + B^{GD,W}$. ∎

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A publication of the International Linear Algebra Society
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Next example demonstrates Theorems 5.7 and 5.8.

EXAMPLE 5.10. *Let* $A = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}, B = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \in \mathbb{C}^{3\times 3}$, *and* $W = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1} \in \mathbb{C}^{3\times 3}$ *be an involutory matrix. Then* $AB = P \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}$ *and from Theorem 2.2, we have* $A^{GD,W} = P \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X_1 \end{bmatrix} P^{-1}$ *and* $B^{GD,W} = P \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & X_2 \end{bmatrix} P^{-1}$, *where* $X_1, X_2 \in \mathbb{C}$ *are arbitrary. Now,* $(AB)^{GD,W} = P \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & X_3 \end{bmatrix} P^{-1}$, *where* X_3 *is arbitrary and we can always choose* $X_3 = X_1X_2$. *Hence,* $(AB)^{GD,W} = A^{GD,W}B^{GD,W} = B^{GD,W}A^{GD,W}$.

5.3. GDMP inverse. When A and B both are orthogonal projections, then the reverse-order law for GDMP inverse holds under a few conditions as follows:

THEOREM 5.11. *Let* $A, B \in \mathbb{C}^{m\times m}$ *be two orthogonal projections and* $k = max\{ind(A), ind(B)\}$. *If* $AB = BA$, $A^{GD}AB = BA^{GD}A$, *and* $BB^{GD}A^{GD} = A^{GD}BB^{GD}$, *then* $(AB)^{GD\dagger} = B^{GD\dagger}A^{GD\dagger}$.

Proof. We have $A^\dagger = A$ and $B^\dagger = B$ as A and B both are orthogonal projections, and from orthogonal properties and $AB = BA$, we get $(AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger$. Now, from Theorem 5.3, we get

$$\begin{aligned} (AB)^{GD\dagger} &= (AB)^{GD}AB(AB)^\dagger \\ &= B^{GD}A^{GD}ABB^\dagger A^\dagger \\ &= B^{GD}A^{GD}ABBA \\ &= B^{GD}A^{GD}AB \\ &= B^{GD}BA^{GD}A \\ &= B^{GD}BB^\dagger A^{GD}AA^\dagger \\ &= B^{GD\dagger}A^{GD\dagger}. \end{aligned}$$

Hence, $(AB)^{GD\dagger} = B^{GD\dagger}A^{GD\dagger}$. ∎

When A and B both are orthogonal projections, then the forward-order law for GDMP inverse holds under a few conditions as follows:

THEOREM 5.12. *Let* $A, B \in \mathbb{C}^{m\times m}$ *be two orthogonal projections and* $k = max\{ind(A), ind(B)\}$. *If* $AB = BA$ *and* $B^{GD}BA = AB^{GD}B$, *then* $(AB)^{GD\dagger} = A^{GD\dagger}B^{GD\dagger}$.

Proof. Since A and B both are orthogonal projections, so $A^\dagger = A$ and $B^\dagger = B$, and from orthogonal properties and $AB = BA$, we get $(AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger$. Now, from Theorem 5.3, we get

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$$
\begin{aligned}
(AB)^{GD\dagger} &= (AB)^{GD}AB(AB)^{\dagger} \\
&= A^{GD}B^{GD}ABB^{\dagger}A^{\dagger} \\
&= A^{GD}B^{GD}ABBA \\
&= A^{GD}B^{GD}BA \\
&= A^{GD}AB^{GD}B \\
&= A^{GD}AA^{\dagger}B^{GD}BB^{\dagger} \\
&= A^{GD\dagger}B^{GD\dagger}.
\end{aligned}
$$

Hence, $(AB)^{GD\dagger} = A^{GD\dagger}B^{GD\dagger}$. ☐

The next result is in the direction of Theorem 5.6.

THEOREM 5.13. *Let $A, B \in \mathbb{C}^{m\times m}$ be two orthogonal projections with $AB = BA = O$ and $k = max\{ind(A), ind(B)\}$. If $A^{GD}B = BA^{GD} = O$ and $B^{GD}A = AB^{GD} = O$, then $(A+B)^{GD\dagger} = A^{GD\dagger}+B^{GD\dagger}$.*

Proof. From Theorems 5.6 and 2.3, we have $(A+B)^{GD} = A^{GD} + B^{GD}$ and $(A+B)^{\dagger} = A^{\dagger} + B^{\dagger}$. Now,

$$
\begin{aligned}
(A+B)^{GD\dagger} &= (A+B)^{GD}(A+B)(A+B)^{\dagger} \\
&= (A^{GD} + B^{GD})(A+B)(A^{\dagger} + B^{\dagger}) \\
&= (A^{GD}A + A^{GD}B + B^{GD}A + B^{GD}B)(A^{\dagger} + B^{\dagger}) \\
&= (A^{GD}A + B^{GD}B)(A^{\dagger} + B^{\dagger}) \\
&= A^{GD}AA^{\dagger} + A^{GD}AB^{\dagger} + B^{GD}BA^{\dagger} + B^{GD}BB^{\dagger} \\
&= A^{GD\dagger} + A^{GD}AB + B^{GD}BA + B^{GD\dagger} \\
&= A^{GD\dagger} + B^{GD\dagger}.
\end{aligned}
$$

Hence, $(A+B)^{GD\dagger} = A^{GD\dagger} + B^{GD\dagger}$. ☐

5.4. W-weighted GDMP inverse. Throughout this subsection, we consider $k = max\{ind(WA), ind(AW), ind(WB), ind(BW)\}$. With the help of Theorem 5.7 and some additional conditions, the reverse-order law for W-weighted GDMP inverse can be proved. The first result of this subsection is in this direction.

THEOREM 5.14. *Let $A, B \in \mathbb{C}^{m\times m}$ be two orthogonal projections, and $W \in \mathbb{C}^{m\times m}$ be involutory matrix with $WA = AW$ and $WB = BW$. If $AB = BA$, $WBWA^{GD,W} = A^{GD,W}WBW$ and $WAWB^{GD,W} = B^{GD,W}WAW$, then $(AB)^{GD\dagger,W} = B^{GD\dagger,W}A^{GD\dagger,W}$.*

Proof. From Theorem 5.7, we have $(AB)^{GD,W} = B^{GD,W}A^{GD,W}$. Further, we have

$$
\begin{aligned}
(AB)^{GD\dagger,W} &= W(AB)^{GD,W}WAB(AB)^{\dagger} \\
&= WB^{GD,W}A^{GD,W}WAB \\
&= WB^{GD,W}A^{GD,W}WBWWA \\
&= WB^{GD,W}WBWA^{GD,W}WA \\
&= WB^{GD,W}WBB^{\dagger}WA^{GD,W}WAA^{\dagger} \\
&= B^{GD\dagger,W}A^{GD\dagger,W}.
\end{aligned}
$$

Hence, $(AB)^{GD\dagger,W} = B^{GD\dagger,W}A^{GD\dagger,W}$. ☐

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

The forward-order law for W-weighted GDMP inverse can also be proved using Theorem 5.8 and proceeding similarly as in the proof of Theorem 5.14. The statement is recorded below.

THEOREM 5.15. *Let $A, B \in \mathbb{C}^{m\times m}$ be two orthogonal projections, and $W \in \mathbb{C}^{m\times m}$ be an involutory matrix with $WA = AW$ and $WB = BW$. If $AB = BA$, $WBWA^{GD,W} = A^{GD,W}WBW$, and $WAWB^{GD,W} = B^{GD,W}WAW$, then $(AB)^{GD\dagger,W} = A^{GD\dagger,W}B^{GD\dagger,W}$.*

The next result shows that $(A+B)^{GD\dagger,W} = A^{GD\dagger,W} + B^{GD\dagger,W}$, using Theorem 5.9 and Theorem 2.3.

THEOREM 5.16. *Let $A, B, W \in \mathbb{C}^{m\times m}$ with $AWBW = BWAW = BWAW = WAWB = O$. Let A and B both be orthogonal projections. If $B^{GD,W}WA = AWB^{GD,W} = O$ and $A^{GD,W}WB = BWA^{GD,W} = O$, then $(A+B)^{GD\dagger,W} = A^{GD\dagger,W} + B^{GD\dagger,W}$.*

Proof. From Theorem 5.9, we have $(A+B)^{GD,W} = A^{GD,W} + B^{GD,W}$ and from Theorem 2.3 $(A+B)^\dagger = A^\dagger + B^\dagger = A + B$. Now,

$$\begin{aligned}
(A+B)^{GD\dagger,W} &= W(A+B)^{GD,W}W(A+B)(A+B)^\dagger \\
&= W(A^{GD,W} + B^{GD,W})W(A^2 + B^2) \\
&= (WA^{GD,W}W + WB^{GD,W}W)(A^2 + B^2) \\
&= WA^{GD,W}WA^2 + WA^{GD,W}WB^2 + WB^{GD,W}WB^2 + WB^{GD,W}WA^2 \\
&= WA^{GD,W}WA^2 + WB^{GD,W}WB^2 \\
&= WA^{GD,W}AA^\dagger + WB^{GD,W}WBB^\dagger \\
&= A^{GD\dagger,W} + B^{GD\dagger,W}.
\end{aligned}$$

Hence, $(A+B)^{GD\dagger,W} = A^{GD\dagger,W} + B^{GD\dagger,W}$. ∎

Next example shows that the given conditions in Theorem 5.14 and Theorem 5.15 are sufficient but not necessary.

EXAMPLE 5.17. *Let $A = P\begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix}P^{-1}, B = P\begin{bmatrix}3&0&0\\0&2&0\\0&0&0\end{bmatrix}P^{-1} \in \mathbb{C}^{3\times 3}$, both are not orthogonal projections and $W = P\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}P^{-1} \in \mathbb{C}^{3\times 3}$ be an involutory matrix. Then $AB = P\begin{bmatrix}6&0&0\\0&2&0\\0&0&0\end{bmatrix}P^{-1}$ and from Theorem 3.15, we have $A^{GD\dagger,W} = P\begin{bmatrix}1/2&0&0\\0&1&0\\0&0&0\end{bmatrix}P^{-1}$, $B^{GD\dagger,W} = P\begin{bmatrix}1/3&0&0\\0&1/2&0\\0&0&0\end{bmatrix}P^{-1}$, $(AB)^{GD\dagger,W} = P\begin{bmatrix}1/6&0&0\\0&1/2&0\\0&0&0\end{bmatrix}P^{-1}$. Hence, $(AB)^{GD\dagger,W} = A^{GD\dagger,W}B^{GD\dagger,W} = B^{GD\dagger,W}A^{GD\dagger,W}$.*

6. Conclusions. The notion of GDMP inverse of a square matrix has been extended to a rectangular matrix using a weight W. Some properties of this inverse, along with its representation, have been obtained. We have also presented sufficient conditions such that the reverse and forward-order laws for GD, W-weighted GD, GDMP, and W-weighted GDMP generalized inverses hold. The problems of computing GD, W-weighted

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A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

GD, GDMP, and W-weighted GDMP generalized inverse of a sum of matrices have also been illustrated. These theories can also be studied in a ring with involution and a tensor setting.

Acknowledgements. The authors thank the anonymous referees for their insightful comments that have significantly improved the quality of the article. The first author acknowledges the support of the Council of Scientific and Industrial Research, India.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 632-654, September 2022.

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