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SIGN PATTERNS THAT REQUIRE EVENTUAL POSITIVITY OR REQUIRE EVENTUAL NONNEGATIVITY*

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Abstract. It is shown that a square sign pattern $\mathcal{A}$ requires eventual positivity if and only if it is nonnegative and primitive. Let the set of vertices in the digraph of $\mathcal{A}$ that have access to a vertex s be denoted by $\mathrm{In}(s)$ and the set of vertices to which t has access denoted by $\mathrm{Out}(t)$. It is shown that $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity if and only if for every s, t such that $\alpha_{st} = -$, the two principal submatrices of $\mathcal{A}$ indexed by $\mathrm{In}(s)$ and $\mathrm{Out}(t)$ require nilpotence. It is shown that $\mathcal{A}$ requires eventual exponential positivity if and only if it requires exponential positivity, i.e., $\mathcal{A}$ is irreducible and its off-diagonal entries are nonnegative.

Key words. Eventually nonnegative matrix, Eventually positive matrix, Eventually exponentially positive matrix, Exponentially positive matrix, Sign pattern, Perron-Frobenius.

AMS subject classifications. 15A48, 05C50, 15A18.

1. Introduction.

A real square matrix A is called eventually positive (nonnegative) if A^k is an entrywise positive (nonnegative) matrix for all sufficiently large k. The problem of characterizing all sign patterns that require eventual nonnegativity (positivity) and the following conjecture were posed at the AIM workshop "Nonnegative Matrix Theory: Generalizations and Applications" [1].

The sign pattern $\mathcal{A}$ requires eventual positivity if and only if $\mathcal{A} \geq 0$ and $\mathcal{A}$ is primitive.

In this paper, we will establish the above conjecture (Theorem 2.3) and also characterize all sign patterns that require eventual nonnegativity (Theorem 2.6) in terms of access relationships to and from the vertices corresponding to the negative entries of $\mathcal{A}$. As an application, we will characterize sign patterns that require eventual exponential positivity (Theorem 2.9).

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1.1. Definitions and notation. Given an n-by-n matrix A, the *spectrum* of A is denoted by $\sigma(A)$ and its spectral radius by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. Given nonempty, increasingly ordered sets $W, U \subseteq \{1, 2, \ldots, n\}$, we let $A[W]$ denote the *principal submatrix* of A indexed by W and $A[W, U]$ denote the submatrix of A whose rows are indexed by W and columns by U. A matrix A is nilpotent if there is a positive integer k such that $A^k = 0$; the least such positive integer is called the *index of nilpotence* of A.

The following classes of matrices are considered. $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is

- *nonnegative* (*positive*), denoted by $A \geq 0$ $(A > 0)$, if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for all i and j;
- *eventually nonnegative* (*positive*) if there exists a nonnegative integer m such that $A^k \geq 0$ $(A^k > 0)$ for all $k \geq m$;
- *exponentially nonnegative* (*positive*) if $\forall\, t > 0$, $e^{tA} = \displaystyle\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ $(e^{tA} > 0)$;
- *eventually exponentially nonnegative* (*positive*) if $\exists\, t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tA} \geq 0$ $(e^{tA} > 0)$.

A *sign pattern matrix* (or *sign pattern* for short) is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\operatorname{sgn}(A)$ is the sign pattern whose entries are the signs of the corresponding entries in A. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *sign pattern class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\operatorname{sgn}(A) = \mathcal{A}$.

We say that a sign pattern $\mathcal{A}$ *requires* a property P if every real matrix $A \in \mathcal{Q}(\mathcal{A})$ has property P. Thus $\mathcal{A}$ *requires nilpotence* if for every $A \in \mathcal{Q}(\mathcal{A})$, A requires nilpotence. The *index of nilpotence* of a sign pattern $\mathcal{A}$ that requires nilpotence is the maximum of the indices of nilpotence of $A \in \mathcal{Q}(\mathcal{A})$ (note that such a maximum must exist because if an $n \times n$ matrix is nilpotent, its index of nilpotence is at most n).

Consider a digraph $\Gamma = (V, E)$ with vertex set V and directed edge set E. A *walk of length k* from vertex u to vertex v in Γ is a sequence of k edges

$$u = i_1 \to i_2 \to \cdots \to i_{k+1} = v.$$

When the edges are distinct, we refer to the walk as a *path* from u to v. A *cycle of length k* in Γ is a path

$$i_1 \to i_2 \to \cdots \to i_k \to i_1,$$

where the vertices $i_1, i_2, \ldots, i_k$ are distinct. We say that vertex $u \in E$ *has access to* vertex $v \in E$ if there exists a path in Γ from u to v or $u = v$. For any vertex v of Γ, define $\operatorname{In}(v)$ to be the set of vertices that have access to v and define $\operatorname{Out}(v)$ to be the set of vertices to which v has access. If u has access to v and v has access to u, then u and v are *access equivalent*; an equivalence class under the relation of access is

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called an *access class*. We say access class W *has access to* vertex u (u *has access to* W) if there is a path from some vertex $w \in W$ to u (there is a path to u from some vertex $w \in W$). The subdigraphs induced by each of the access classes of Γ are the *strongly connected components* of Γ. A strongly connected component of Γ is *trivial* if it has no edges (i.e., it comprises a single vertex that does not have a loop). Γ is called *strongly connected* if it has exactly one strongly connected component and that component is not trivial.

A digraph Γ is called *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. Equivalently, Γ is primitive if and only if there exists $k \in \mathbb{N}$ such that for all vertices i, j of Γ, there is a walk of length k from i to j.

Given an $n \times n$ sign pattern $\mathcal{A}$ and $A = [a_{ij}] \in Q(\mathcal{A})$, the digraph of A (or $\mathcal{A}$), denoted by $\Gamma(A)$ (or $\Gamma(\mathcal{A})$), is

$$(\{1, 2, \ldots, n\}, \{(i, j) : a_{ij} \neq 0\}).$$

We denote by $A_{\mathcal{A}}$ the $(0, 1, -1)$-matrix obtained from $\mathcal{A}$ by replacing $+$ by 1 and $-$ by -1. We write $\mathcal{A} \geq 0$ if $A_{\mathcal{A}} \geq 0$.

If $A_{\mathcal{A}} = [a_{ij}]$ and $\gamma = i_1 \to i_2 \to \cdots \to i_k \to i_1$ is a cycle in $\Gamma(\mathcal{A})$, define the *sign* of γ to be $\operatorname{sgn}(\gamma) = a_{i_1,i_2} \cdots a_{i_{k-1},i_k} a_{i_k,i_1}$; we refer to the entries $a_{i_1,i_2}, \cdots, a_{i_{k-1},i_k}, a_{i_k,i_1}$ as the *cycle entries* associated with γ.

For a nonnegative sign pattern $\mathcal{A}$, the digraph $\Gamma(\mathcal{A})$ is primitive if and only if for some (and hence for all) $A \in \mathcal{Q}(\mathcal{A})$, there exists m such that $A^k > 0$ for all $k \geq m$. We refer to this condition as the *Frobenius test for primitivity*; see e.g., [2, Chapter 2], where this condition serves as the definition of a nonnegative primitive matrix. We say $\mathcal{A}$ is *primitive* if $\Gamma(\mathcal{A})$ is primitive.

Next, recall that an n-by-n matrix A is called *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square, non-vacuous matrices. Otherwise, A is called *irreducible*. Note that $[0]$ is considered reducible. It is well known that A is irreducible if and only if $\Gamma(A)$ is strongly connected.

In the theory of (eventually) nonnegative matrices, the following notions play an important role. An eigenvalue λ of $A \in \mathbb{R}^{n \times n}$ is called *dominant* if $|\lambda| = \rho(A)$. We say A *has the Perron-Frobenius property* if A has a dominant eigenvalue that is positive and there exists a nonnegative eigenvector for this eigenvalue. We say A *has the strong Perron-Frobenius property* if A has exactly one dominant eigenvalue, the

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dominant eigenvalue is positive, and the corresponding eigenvector is positive. The definition of the Perron-Frobenius property given above is the one used in [7] and [8], which is slightly different from that used in [3]. The definition in [3] allows $\rho(A) \geq 0$ (i.e., A can be nilpotent).

1.2. Results cited.

It is well known that for any matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of A are continuous functions of the entries of A. For a simple eigenvalue, the same is true of the eigenvector.

THEOREM 1.1. [6, p. 323] *Let $A, B \in \mathbb{R}^{n\times n}$. If λ is a simple eigenvalue of A and $A(\varepsilon) = A + \varepsilon B$, then in a neighborhood of the origin there exist differentiable (and thus continuous) functions $\lambda(\varepsilon)$ and $\mathbf{x}(\varepsilon)$ such that $A(\varepsilon)\mathbf{x}(\varepsilon) = \lambda(\varepsilon)\mathbf{x}(\varepsilon)$.*

It is well known that a positive matrix has the strong Perron-Frobenius property, and if A is an irreducible nonnegative matrix, then $\rho(A)$ is a simple eigenvalue that has a positive eigenvector.

THEOREM 1.2. [7, Theorem 2.2] *The matrix $A \in \mathbb{R}^{n\times n}$ is eventually positive if and only if both the matrices A and A^T possess the strong Perron-Frobenius property.*

The next result appeared in [7, Theorem 2.3] but without the necessary hypothesis that A is not nilpotent. The need for this assumption was observed in [3, Lemma 2.1], and the corrected form below then appeared in [8].

THEOREM 1.3. [8, Theorem 3.12] *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both matrices A and A^T possess the Perron-Frobenius property.*

THEOREM 1.4. [8, Theorem 3.3] *Let $A \in \mathbb{R}^{n\times n}$. Then A is eventually exponentially positive if and only if there exists $a \geq 0$ such that $A + aI$ is eventually positive.*

2. Sign patterns requiring eventual positivity or eventual nonnegativity.

Let $\mathcal{A}$ be a sign pattern that requires eventual nonnegativity. By Theorem 1.3, it follows that any $A \in \mathcal{Q}(\mathcal{A})$ is nilpotent or has the Perron-Frobenius property, and thus $\mathcal{A}$ requires that the spectral radius be an eigenvalue. As shown in [4], a sign pattern $\mathcal{A}$ requires that the spectral radius be an eigenvalue if and only if all cycles in $\Gamma(\mathcal{A})$ are positively signed. Thus, sign patterns that require eventual nonnegativity can have no negatively signed cycles. However, the Perron-Frobenius property for A and A^T is not sufficient to imply eventual nonnegativity; see [7]. As a consequence, there are further restrictions on the cycles of sign patterns that require eventual nonnegativity.

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Indeed, the following theorem shows that no entry associated with a cycle of such a sign pattern can be negative.

THEOREM 2.1. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If $\alpha_{st} = -$ and there is a cycle $\gamma = i_1 \to i_2 \to \cdots \to i_k \to i_1$ with $i_1 = s, i_2 = t$ in $\Gamma(\mathcal{A})$, then $\mathcal{A}$ does not require eventual nonnegativity.*

Proof. Let $\mathcal{A}$ be as prescribed. If $s = t$ then $\mathcal{A}$ does not require eventual nonnegativity since it has a negatively signed cycle. So assume that the length of γ is $k \geq 2$. Consider the matrix $C = [c_{ij}]$ obtained from $A_{\mathcal{A}}$ by setting all entries but those associated with the cycle γ equal to zero. For $\varepsilon > 0$, consider

$$B = C + \varepsilon A_{\mathcal{A}} \in \mathcal{Q}(\mathcal{A}).$$

The characteristic polynomial of C is $x^{n-k}(x^k \pm 1)$, so all of the nonzero eigenvalues of C are simple and of modulus 1. Assume that C has the Perron-Frobenius property; i.e., C has an eigenvalue equal to 1 and $C\mathbf{x} = \mathbf{x}$, where $\mathbf{x} \geq 0$ and $\mathbf{x} \neq 0$. Thus

$$\begin{aligned} c_{i_1,i_2} x_{i_2} &= x_{i_1} \\ c_{i_2,i_3} x_{i_3} &= x_{i_2} \\ &\vdots \\ c_{i_k,i_1} x_{i_1} &= x_{i_k} \end{aligned}$$

and since $c_{i_1,i_2} = c_{s,t} = -1$, this implies $x_{i_j} = 0$ for $j = 1, 2, \ldots, k$. Therefore, $C\mathbf{x} = 0$, since row j of C is equal to 0 if $j \notin \{i_1, i_2, \ldots, i_k\}$; this contradicts $C\mathbf{x} = \mathbf{x}$. Thus C cannot have a nonnegative eigenvector corresponding to a positive eigenvalue. Consequently, by Theorem 1.1, we have that for sufficiently small $\varepsilon > 0$, B cannot have a nonnegative eigenvector corresponding to a positive eigenvalue. It is also clear that B is not nilpotent for sufficiently small $\varepsilon > 0$. Since B is not nilpotent and does not possess the Perron-Frobenius property, by Theorem 1.3, B is not eventually nonnegative and thus $\mathcal{A}$ does not require eventual nonnegativity. □

COROLLARY 2.2. *If the sign pattern $\mathcal{A}$ requires eventual nonnegativity, then every irreducible principal submatrix of $\mathcal{A}$ is nonnegative.*

Proof. We will prove the contrapositive. Let $\mathcal{B}$ be an irreducible principal submatrix of a sign pattern $\mathcal{A}$. Then any negative entry of $\mathcal{B}$ is a cycle entry associated with some cycle in $\Gamma(\mathcal{B})$ and thus in $\Gamma(\mathcal{A})$. By Theorem 2.1, $\mathcal{A}$ does not require eventual nonnegativity. □

THEOREM 2.3. *The sign pattern $\mathcal{A}$ requires eventual positivity if and only if $\mathcal{A}$ is nonnegative and primitive.*

Proof. If $\mathcal{A}$ is nonnegative and primitive, then by the Frobenius test for primitivity it is clear that $\mathcal{A}$ requires eventual positivity. For the converse, suppose that $\mathcal{A} = [\alpha_{ij}]$

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requires eventual positivity. We first show that $\mathcal{A} \geq 0$. By way of contradiction, if $\alpha_{st} = -$, then by Corollary 2.2, α_{st} is not in any irreducible principal submatrix of $\mathcal{A}$. Hence $\mathcal{A}$ is reducible and so all powers of any $A \in Q(\mathcal{A})$ contain at least one zero entry. Thus $\mathcal{A}$ does not require eventually positivity; this is a contradiction, proving that $\mathcal{A} \geq 0$. By the Frobenius test for primitivity, it now follows that $\mathcal{A}$ must also be primitive. □

Next, we consider sign patterns that require eventual nonnegativity. It is well known that sign pattern $\mathcal{A}$ requires nilpotence if and only if $\mathcal{A}$ is permutationally similar to a strictly upper triangular sign pattern [5, p. 82]. Clearly, if $\mathcal{A}$ is permutationally similar to the direct sum of sign patterns that are either nonnegative or require nilpotence, then $\mathcal{A}$ requires eventual nonnegativity. One might think that the converse is also true, namely, that if $\mathcal{A}$ requires eventual nonnegativity, then $\mathcal{A}$ is permutationally similar to $\mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_1 \geq 0$ and $\mathcal{A}_2$ requires nilpotence. This is not the case, however, as the next example shows.

Example 2.4. Let

$$\mathcal{A} = \begin{bmatrix} 0 & - & + & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix}.$$

Note that $\Gamma(\mathcal{A})$ has three access classes, namely, $W_1 = \{1\}$, $W_2 = \{2\}$ and $W_3 = \{3, 4\}$, and W_1 has access to W_2 and W_3. Thus $\mathcal{A}$ is not permutationally similar to the direct sum of sign patterns that are nonnegative or require nilpotence. Since row 2 and column 1 of $\mathcal{A}$ are zero, $\mathcal{A}^2 \geq 0$ and row 2, as well as column 1, of $\mathcal{A}^2$ are zero. It follows that $\mathcal{A}$ requires eventual nonnegativity.

As the next result shows, the key to $\mathcal{A}$ requiring eventual nonnegativity is the access to and from each vertex associated with a negative edge in $\Gamma(\mathcal{A})$.

Theorem 2.5. *If the $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity and $\alpha_{st} = -$, then any access class that has access to s or to which t has access must be trivial.*

Proof. Let $\mathcal{A}$ be as hypothesized. By Corollary 2.2, s and t must belong to separate access classes of $\Gamma(\mathcal{A})$. Suppose t has access to a nontrivial access class. Without loss of generality, we may assume that

$$s = 1 \to t = 2 \to 3 \to \cdots \to r$$

and $U = \{r, \ldots, \ell\}$ is the first nontrivial access class to which t has access (note it is possible that $r = 2$ and/or $\ell = r$). Consider the matrix C obtained from $A_{\mathcal{A}} = [a_{ij}]$ by setting all entries to zero except $a_{i,i+1}, i = 1, \ldots, r-1$ and those in $A_{\mathcal{A}}[U]$; the

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excepted entries remain unchanged. If we let $W = \{1, \ldots, r-1\}$, then C has the block form

$$C = \begin{bmatrix} C[W] & C[W,U] & 0 \\ 0 & C[U] & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because $C[W]$ is nilpotent, the eigenvalues of C are precisely the eigenvalues of $C[U]$ together with $n - \ell + r - 1$ additional zero eigenvalues. Thus $\rho(C) = \rho(C[U])$; let ρ denote this common value. Suppose $C\mathbf{v} = \rho\mathbf{v}$. Then $C[U]\,\mathbf{v}[U] = \rho\,\mathbf{v}[U]$. Because U is a nontrivial access class, $C[U]$ is irreducible, and by Corollary 2.2, $C[U] \geq 0$. By Perron-Frobenius, $\rho > 0$ and without loss of generality $\mathbf{v}[U] > 0$. Working backwards from r, we have that $v_i \neq 0$ for $i = 1, \ldots, r-1$, and $v_k < 0$, where k is the greatest index in $\{1, \ldots, r-1\}$ such that $\alpha_{k,k+1} = -$ (there is some such index since $\alpha_{12} = -$). Note that ρ is a simple eigenvalue of C. So, as in the proof of Theorem 2.1, for all sufficiently small $\varepsilon > 0$, $B = C + \varepsilon A_{\mathcal{A}} \in Q(\mathcal{A})$ is not nilpotent and does not have $\rho(B) > 0$ as an eigenvalue with a nonnegative eigenvector. That is, by Theorem 1.3, B is not eventually nonnegative. This contradiction proves that t cannot have access to a nontrivial class. The case of access to s is similar; it involves considering C^T rather than C. ∎

Theorem 2.6. *The sign pattern $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity if and only if for every s, t such that $\alpha_{st} = -$, $\mathcal{A}[\mathrm{In}(s)]$ and $\mathcal{A}[\mathrm{Out}(t)]$ require nilpotence.*

Proof. Assume that for every s, t such that $\alpha_{st} = -$, $\mathcal{A}[\mathrm{In}(s)]$ and $\mathcal{A}[\mathrm{Out}(t)]$ require nilpotence. Then no negative entry of $\mathcal{A}$ is associated with a cycle of $\Gamma(\mathcal{A})$. Let k_0 be the maximum of the indices of nilpotence among all $\mathcal{A}[\mathrm{In}(s)]$ and $\mathcal{A}[\mathrm{Out}(t)]$ such that $\alpha_{st} = -$. Then for any $A \in \mathcal{Q}(\mathcal{A})$ and any $k > k_0$, $A^k \geq 0$, because any walk in $\Gamma(\mathcal{A})$ that includes a $-$ must have length at most k_0. The converse follows from Theorem 2.5. ∎

The next two examples illustrate Theorem 2.6.

Example 2.7. Let

$$\mathcal{A}_1 = \begin{bmatrix} 0 & + & 0 & + & - & 0 \\ 0 & + & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\mathrm{In}(1) = \{1\}$ and $\mathrm{Out}(5) = \{5, 6\}$, so $\mathcal{A}_1[\mathrm{In}(1)] = \begin{bmatrix} 0 \end{bmatrix}$ and $\mathcal{A}_1[\mathrm{Out}(5)] = \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}$. Both of these sign patterns require nilpotence, with indices of nilpotence 1

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and 2, respectively. Thus for any $A \in \mathcal{Q}(\mathcal{A}_1)$, $A^3 \geq 0$.

EXAMPLE 2.8. Let

$$\mathcal{A}_2 = \begin{bmatrix} 0 & + & 0 & + & - & 0 \\ 0 & + & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix}.$$

Again $\text{In}(1) = \{1\}$ and $\text{Out}(5) = \{5, 6\}$, but now $\mathcal{A}_2[\text{Out}(5)] = \begin{bmatrix} 0 & + \\ 0 & + \end{bmatrix}$, so $\mathcal{A}_2[\text{Out}(5)]$ does not require nilpotence. The matrix

$$A = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 & -1 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{Q}(\mathcal{A})$$

is not eventually nonnegative, because A is not nilpotent and A does not have the Perron-Frobenius property.

The strictly upper triangular sign pattern, all of whose entries above the diagonal are $-$ has $\frac{n(n-1)}{2}$ negative entries and this pattern requires nilpotence. The largest number of negative entries possible in an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ that requires eventual nonnegativity is $\frac{n(n-1)}{2}$, since no diagonal entries can be negative and if $\alpha_{st} = -$, then $\alpha_{ts} = 0$.

As an application of our results, we can characterize the sign patterns that require eventual exponential positivity.

THEOREM 2.9. *Let $\mathcal{A} = [\alpha_{ij}]$ be a square sign pattern. Then the following are equivalent.*
(i) *$\mathcal{A}$ requires eventual exponential positivity.*
(ii) *$\mathcal{A}$ is irreducible and its off-diagonal entries are nonnegative.*
(iii) *$\mathcal{A}$ requires exponential positivity.*

Proof.

(i)$\Rightarrow$(ii) Assume $\mathcal{A}$ requires eventual exponential positivity and let $A \in \mathcal{Q}(A)$. Considering the power series expansion of e^{tA}, it is clear that $\mathcal{A}$ must be irreducible. Next, we show by contradiction that the off-diagonal entries of $\mathcal{A}$ are nonnegative. Suppose that for some $s \neq t$, $\alpha_{st} = -$. Since $\mathcal{A}$ is irreducible, α_{st} is associated with

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some cycle γ in $\Gamma(\mathcal{A})$. Consider the matrix C and the matrix $B = C + \varepsilon A_{\mathcal{A}} \in Q(\mathcal{A})$ as defined in the proof of Theorem 2.1.

On one hand, as in the proof of Theorem 2.1, we have that all nonzero eigenvalues of C are simple of modulus 1, and if 1 is an eigenvalue, it cannot have a positive eigenvector. Thus, for all sufficiently small $\varepsilon > 0$, the eigenvalues of B are partitioned in two disjoint sets S and T. The subset S belongs to an ε-neighborhood of zero and the subset T contains the eigenvalues of B that are perturbations of the nonzero eigenvalues of C. It follows that for some sufficiently small and fixed $\varepsilon > 0$, the eigenvalues of B in T satisfy the following:

- They are simple and nonzero.
- At most one eigenvalue is positive and at most one is negative.
- If $\mu > 0$ is an eigenvalue of B in T, then μ does not have a positive eigenvector.
- If $\nu < 0$ is an eigenvalue of B in T, then $\nu < \mathrm{Re}(\lambda)$ for every eigenvalue $\lambda \notin \mathbb{R}$.

Thus for such choice of ε, $B + aI$ does not have the strong Perron-Frobenius property for any $a \geq 0$.

On the other hand, as $B \in Q(\mathcal{A})$, B is eventually exponentially positive and so, by Theorem 1.4, there exists $a \geq 0$ such that $B + aI$ is eventually positive. Thus, by Theorem 1.2, $B + aI$ has the strong Perron-Frobenius property. This a contradiction, proving that the off-diagonal entries of $\mathcal{A}$ are nonnegative.

(ii)⇒(iii) Assume $\mathcal{A}$ is irreducible and its off-diagonal entries are nonnegative. Let $A \in Q(\mathcal{A})$. Since A has nonnegative off-diagonal entries, there exists sufficiently large $a \geq 0$ such that $A + aI$ is nonnegative and has at least one positive diagonal entry, so the greatest common divisor of the lengths of its cycles is 1. Since A is irreducible, so is $A + aI$, and thus $\Gamma(A + aI)$ is strongly connected. Thus $A + aI$ is primitive and so $A + aI$ is eventually positive. Therefore, $e^{tA} = e^{-ta}\, e^{t(A+aI)} > 0$ for all $t > 0$, i.e., $\mathcal{A}$ requires exponential positivity.

(iii)⇒(i) This implication follows readily from the definitions. □

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