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POSITIVE LINEAR MAPS AND SPREADS OF NORMAL MATRICES*

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Abstract. We obtain some inequalities involving positive linear maps on matrix algebra. The special cases provide bounds for the spreads of normal matrices.

Key words. Normal matrices, Positive unital linear maps, Eigenvalues, Spread.

AMS subject classifications. 15A42, 15A45.

1. Introduction. Let $\mathbb{M}(n)$ denote the algebra of all complex $n \times n$ matrices. A linear map $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ and unital if $\Phi(I_n) = I_k$. A linear functional $\varphi : \mathbb{M}(n) \to \mathbb{C}$ is a special case of such maps, see [4]. Beginning with Kadison [18], several authors have studied the inequalities involving positive unital linear maps. An inequality of interest in the present context is due to Bhatia and Davis [3]: if A is any Hermitian element of $\mathbb{M}(n)$ whose spectrum is contained in the interval $[m, M]$, then

$$\Phi\left(A^2\right) - \Phi(A)^2 \leq (MI_k - \Phi(A))(\Phi(A) - mI_k) \leq \left(\frac{M-m}{2}\right)^2 I_k. \tag{1.1}$$

Bhatia and Sharma [5] extended this inequality for arbitrary matrices and have shown that for any matrix $A \in \mathbb{M}(n)$,

$$\Phi(A^*A) - \Phi(A^*)\Phi(A) \leq \Delta(A)^2 I_k, \tag{1.2}$$

where $\Delta(A) = \inf_{z \in \mathbb{C}} \|A - zI\|$ and $\|\cdot\|$ denotes the operator norm. The inequality (1.2) also holds good if we replace $\Phi(A^*A)$ by $\Phi(AA^*)$ in the left-hand side of (1.2). Bhatia and Sharma [5] obtained several lower bounds for $\Delta(A)$ on choosing different linear maps in (1.2). They showed that $\Delta(A) \geq r$, where r is the radius of the smallest disk containing the eigenvalues of A and for normal matrices $\Delta(A) = r$. Also, by a classical theorem of Jung [14], $\operatorname{spd}(A) \geq \sqrt{3}r$, where $\operatorname{spd}(A) = \max_{i,j} |\lambda_i(A) - \lambda_j(A)|$. Beginning with Mirsky [19], several authors have investigated bounds for the spreads of matrices. The inequality (1.2) also provides several lower bounds for the spreads of normal matrices. Jiang and Zhan [12] have discussed some stronger lower bounds for the spreads of Hermitian matrices. Bhatia and Sharma [7] have shown that these lower bounds also follow from the inequality (1.2). For some related complementary inequalities involving positive unital linear maps, see Kian et al. [15].

Bhatia and Sharma [7] also discussed a variant of (1.2) in the special case when A is normal and φ is a positive unital linear functional,

$$\varphi(A^*A) - |\varphi(A)|^2 + |\varphi\left(A^2\right) - \varphi(A)^2| \leq \frac{\operatorname{spd}(A)^2}{2}. \tag{1.3}$$

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It is also shown that the lower bounds for the spread derived by Johnson et al. [13] and Merikosky and Kumar [17] follow as the special cases of (1.3). See [6].

Sharma et al. [21, 22] extended the work of Bhatia and Davis [3] and showed that for any positive unital linear functional $\varphi : \mathbb{M}(n) \to \mathbb{C}$ and for any Hermitian element A of $\mathbb{M}(n)$, we have

$$\varphi\left(B^4\right) \leq \frac{\operatorname{spd}(A)^4}{12} \tag{1.4}$$

and

$$\varphi\left(B^4\right) + 3\varphi\left(B^2\right)^2 \leq \frac{\operatorname{spd}(A)^4}{4}, \tag{1.5}$$

where $B = A - \varphi(A)\, I$.

We here mainly consider normal matrices and for the simplicity of notations denote A^*A by $|A|^2$ where A^* is the conjugate transpose of A, see [2]. Our first theorem provides a refinement of the inequality (1.2) for normal matrices. This inequality (Theorem 2.1) is used to derive some further results. The upper bounds for $\varphi\left(|A|^4\right)$ and $\varphi\left(|B|^4\right)$ are obtained in terms of $\varphi\left(|A|^2A\right)$, $\varphi\left(|A|^2\right)$, $\varphi(A)$ when all the eigenvalues of A lie in $|z-c| = r$ (Theorems 2.2 and 2.3). The upper bounds for $\varphi(|A|^4)$ and $\varphi(|B|^4)$ in terms of $\varphi(|A|^2)$ and $\varphi(A)$ derived in Theorems 2.4 and 2.5 yield several lower bounds for the radius r of the smallest disk containing all the eigenvalues of a normal matrix in terms of $\varphi\left(|B|^2\right)$ and $\varphi\left(|B|^4\right)$ (Corollaries 2.6–2.8). We discuss special cases and obtain lower bounds for the spread of normal matrices in terms of Frobenius norm of $A - \frac{\operatorname{tr}A}{n} I$ and $\left(A - \frac{\operatorname{tr}A}{n} I\right)^2$ (Corollaries 3.1–3.2). Corollary 2.7 also provides a lower bound for r in terms of $\varphi\left(B|B|^2\right)$ (Corollary 3.2). We discuss upper bounds for the ratios $\frac{\varphi\left(|B|^2\right)}{|\varphi(A)|^2}$ and $\frac{\varphi\left(|B|^4\right)}{|\varphi(A)|^2}$ when A is normal and a lower bound for the condition number of the Cauchy matrix is given (Corollaries 3.3–3.4).

2. Main results.

THEOREM 2.1. *Let $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ be a positive unital linear map and let A be any normal element of $\mathbb{M}(n)$ whose spectrum is contained in the disk $|z-c| \leq r$. Then*

$$\Phi\left(|A|^2\right) - |\Phi(A)|^2 + |\Phi(A) - cI|^2 \leq r^2 I_k. \tag{2.1}$$

Proof. A simple computation shows that for any complex number c, we have

$$\Phi\left(|A|^2\right) - |\Phi(A)|^2 + |\Phi(A) - cI|^2 = \Phi\left(|A - cI|^2\right). \tag{2.2}$$

Further, by the Spectral Theorem, we have

$$A = \sum_{i=1}^{n} \lambda_i P_i,\, A^* = \sum_{i=1}^{n} \overline{\lambda_i} P_i \text{ and } |A|^2 = \sum_{i=1}^{n} |\lambda_i|^2 P_i,$$

where λ_i's are the eigenvalues of A and P_i's are corresponding orthogonal projections. Using this, we have

$$\Phi\left(|A - cI|^2\right) = \Phi\left(\sum_{i=1}^{n} |\lambda_i|^2 P_i - \overline{c}\sum_{i=1}^{n} \lambda_i P_i - c\sum_{i=1}^{n} \overline{\lambda_i} P_i + |c|^2\right)$$

$$= \sum_{i=1}^{n} |\lambda_i - c|^2 \Phi(P_i) \leq r^2 I_k. \tag{2.3}$$

From (2.2) and (2.3), we immediately get (2.1). ∎

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The inequality (2.1) becomes equality when all the eigenvalues of A lie on the circle $|z-c|=r$. Also, for a positive unital linear functional $\varphi:\mathbb{M}(n)\to\mathbb{C}$, we have $\varphi\left(|A|^2\right)-|\varphi\left(A\right)|^2=\varphi\left(|A-\varphi\left(A\right)I|^2\right)$. Denote $A-\varphi\left(A\right)I$ by B. Then, (2.1) yields

$$\varphi\left(|B|^2\right)+|\varphi\left(A\right)-c|^2\le r^2. \tag{2.4}$$

THEOREM 2.2. *Let $\varphi:\mathbb{M}(n)\to\mathbb{C}$ be a positive unital linear functional and let A be any normal element of $\mathbb{M}(n)$ whose spectrum is contained in the disk $|z-c|\le r$. Then*

$$\begin{aligned}\varphi\left(|A|^4\right)\le{}& 2\mathrm{Re}\ \bar{c}\varphi\left(|A|^2A\right)+\left(r^2-|c|^2\right)\varphi\left(|A|^2\right)\\ &-\frac{\left(\mathrm{Re}\ \varphi\left(|A|^2A\right)-2\mathrm{Re}\ \bar{c}\varphi\left(HA\right)+\left(|c|^2-r^2\right)\varphi\left(H\right)\right)^2}{r^2-|c|^2-\varphi\left(|A|^2\right)+2\mathrm{Re}\ \bar{c}\varphi\left(A\right)},\end{aligned} \tag{2.5}$$

where $H=\frac{A+A^*}{2}$ and $2\mathrm{Re}\ \bar{c}\varphi\left(A\right)-\varphi\left(|A|^2\right)+r^2-|c|^2\neq 0$.

Proof. The eigenvalues λ_i's of A all lie in the disk $|z-c|\le r$. Therefore, for any real number α, we have

$$|\lambda_i-\alpha|^2\left(|\lambda_i-c|^2-r^2\right)\le 0. \tag{2.6}$$

This gives

$$\begin{aligned}|\lambda_i|^4\le{}& 2\mathrm{Re}\ \left(\alpha+\bar{c}\right)|\lambda_i|^2\lambda_i-\left(|c|^2-r^2+\alpha^2+2\alpha\mathrm{Re}\ c\right)|\lambda_i|^2\\ &-2\alpha\mathrm{Re}\ \bar{c}\lambda_i^2+2\mathrm{Re}\ (\alpha^2\bar{c}+\alpha(|c|^2-r^2))\lambda_i+\alpha^2(r^2-|c|^2).\end{aligned}$$

Multiplying both sides by $\varphi\left(P_i\right)$ and adding n resulting inequalities, we get

$$\varphi\left(|A|^4\right)\le 2\mathrm{Re}\ \bar{c}\varphi\left(|A|^2A\right)+(r^2-|c|^2)\varphi\left(|A|^2\right)+f(\alpha) \tag{2.7}$$

where

$$\begin{aligned}f(\alpha)={}&(r^2-|c|^2+2\mathrm{Re}\ \bar{c}\varphi\left(A\right)-\varphi\left(|A|^2\right))\alpha^2+2((|c|^2-r^2)\varphi\left(H\right)\\ &+\mathrm{Re}\ \varphi\left(|A|^2A\right)-\mathrm{Re}\ c\varphi\left(|A|^2\right)-\mathrm{Re}\ \bar{c}\varphi\left(A^2\right))\alpha.\end{aligned} \tag{2.8}$$

We note that $f(x)=ax^2+2bx$ with $a>0$ attains its greatest lower bound at $x=-\frac{b}{a}$ as the derivative $f'(x)=2(ax+b)$ vanishes there and $f''(x)=2a>0$. Also, by (2.4), the coefficient of α^2 in (2.8) is positive. It follows that $f(\alpha)$ attains its minimum at

$$\alpha=\frac{(|c|^2-r^2)\varphi\left(H\right)+\mathrm{Re}\ \varphi\left(|A|^2A\right)-\mathrm{Re}\ c\varphi\left(|A|^2\right)-\mathrm{Re}\ \bar{c}\varphi\left(A^2\right)}{|c|^2-r^2-2\mathrm{Re}\ \bar{c}\varphi\left(A\right)+\varphi\left(|A|^2\right)}, \tag{2.9}$$

where

$$f(\alpha)=\frac{\left(\left(|c|^2-r^2\right)\varphi\left(H\right)+\mathrm{Re}\ \varphi\left(|A|^2A\right)-\mathrm{Re}\ c\varphi\left(|A|^2\right)-\mathrm{Re}\ \bar{c}\varphi\left(A^2\right)\right)^2}{|c|^2-r^2-2\mathrm{Re}\ \bar{c}\varphi\left(A\right)+\varphi\left(|A|^2\right)}. \tag{2.10}$$

Combining (2.7) and (2.10), we immediately get (2.5). ∎

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Theorem 2.3. *With notations and conditions as in Theorem 2.2,*

$$\varphi(|B|^4) \leq 2\text{Re }(\bar{c} - \overline{\varphi(A)})\varphi\left(|B|^2 B\right) + (r^2 - |\varphi(A) - c|^2)\varphi\left(|B|^2\right) - \frac{(\text{Re }\varphi\left(|B|^2 B\right) - 2\text{Re }(\bar{c} - \overline{\varphi(A)})\varphi(H_1 B))^2}{r^2 - |\varphi(A) - c|^2 - \varphi(|B|^2)}, \tag{2.11}$$

where $H_1 = \frac{B+B^*}{2}$ *and* $r^2 - |\varphi(A) - c|^2 - \varphi\left(|B|^2\right) \neq 0$.

Proof. For any real number α, we have

$$|\lambda_i - \varphi(A) - \alpha|^2 \left(|\lambda_i - \varphi(A) + \varphi(A) - c|^2 - r^2\right) \leq 0. \tag{2.12}$$

Beginning with the inequality (2.12) and using the arguments similar to those used in the proof of Theorem 2.2, we easily get (2.11). ∎

It may be noted here that $r^2 - |\varphi(A) - c|^2 - \varphi\left(|B|^2\right) = 0$ when all the eigenvalues of A lie on the circle $|z - c| = r$ and therefore (2.5) and (2.11) are not applicable. In this case, we have

$$\varphi\left(|A|^4\right) = 2\text{Re }\bar{c}\varphi\left(|A|^2 A\right) + (r^2 - |c|^2)\varphi\left(|A|^2\right)$$

and

$$\varphi\left(|B|^4\right) = 2\text{Re }(\bar{c} - \overline{\varphi(A)})\varphi\left(|B|^2 B\right) + (r^2 - |\varphi(A) - c|^2)\varphi\left(|B|^2\right).$$

The inequalities (2.5) and (2.11) become equalities when A has three distinct eigenvalues such that two of them lie on the circle $|z - c| = r$ and one inside it. The eigenvalues are then $\lambda_1 = c - r$, $\lambda_3 = c + r$ and λ_2 satisfies $|\lambda_2 - c| < r$.

Let $A \in \mathbb{M}(n)$ be Hermitian. Then, all its eigenvalues are real and lies in the disk with centre $c = \frac{\lambda_1 + \lambda_n}{2}$ and radius $r = \frac{\lambda_n - \lambda_1}{2}$. So, the inequality (2.5) yields the following result, see Sharma et al. [22],

$$\varphi\left(A^4\right) \leq (\lambda_1 + \lambda_n)\varphi\left(A^3\right) - \lambda_1\lambda_n\varphi\left(A^2\right) - \frac{\left(\varphi\left(A^3\right) - (\lambda_1 + \lambda_n)\varphi\left(A^2\right) + \lambda_1\lambda_n\varphi(A)\right)^2}{(\lambda_1 + \lambda_n)\varphi(A) - \varphi(A^2) - \lambda_1\lambda_n}, \tag{2.13}$$

where $(\lambda_1 + \lambda_n)\varphi(A) - \lambda_1\lambda_n \neq \varphi\left(A^2\right)$.

We now find an upper bound for $\varphi\left(|A|^4\right)$ in the following theorem which is independent of $\text{Re }\bar{c}\varphi\left(|A|^2 A\right)$. The inequality (2.7) suggests that to achieve the desired result we must choose $\alpha = -c$ in (2.6) and in this case the resulting inequality is also valid for linear maps.

Theorem 2.4. *Let* $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ *be a positive unital linear map and let* A *be any normal element of* $\mathbb{M}(n)$ *whose spectrum is contained in the disk* $|z - c| \leq r$. *Then*

$$\Phi\left(|A|^4\right) \leq r^2\Phi\left(|A|^2\right) + r^2\Phi(X) + \Phi(Y) + |c|^2(r^2 - |c|^2), \tag{2.14}$$

where $X = \bar{c}A + cA^*$ *and* $Y = (\bar{c}A)^2 + (cA^*)^2$.

Proof. The eigenvalues λ_i's of A all lie in the disk $|z - c| \leq r$. Therefore,

$$|\lambda_i + c|^2 \left(|\lambda_i - c|^2 - r^2\right) \leq 0, \tag{2.15}$$

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for all $i = 1, 2, \ldots, n$. From (2.15), we get

$$|\lambda_i|^4 - r^2 |\lambda_i|^2 + |c|^4 - |c|^2 r^2 - (\bar{c}^2 \lambda_i^2 + c^2 \overline{\lambda_i}^2) - (\bar{c}\lambda_i + c\overline{\lambda_i}) r^2 \leq 0. \tag{2.16}$$

This gives

$$|\lambda_i|^4 \leq (\bar{c}^2 \lambda_i^2 + c^2 \overline{\lambda_i}^2) + r^2 (|\lambda_i|^2 + |c|^2 + \bar{c}\lambda_i + c\overline{\lambda_i}) - |c|^4 . \tag{2.17}$$

Multiplying both sides by $\Phi(P_i)$ and adding n resulting inequalities we immediately get (2.14). ∎

For the linear functional $\varphi : \mathbb{M}(n) \to \mathbb{C}$, the inequality (2.14) can be written as

$$\varphi\left(|A|^4\right) \leq r^2 \varphi\left(|A|^2\right) + 2\mathrm{Re}\ \bar{c}^2 \varphi\left(A^2\right) + 2r^2 \mathrm{Re}\ c\varphi\left(A^*\right) + |c|^2 (r^2 - |c|^2). \tag{2.18}$$

It is clear that the inequality (2.14) becomes equality when the spectrum of A lies on the circle $|z - c| = r$. Further, if A is Hermitian, $c = \frac{\lambda_1 + \lambda_n}{2}$ and $r = \frac{\lambda_n - \lambda_1}{2}$, and from (2.14), we get

$$\Phi\left(A^4\right) \leq \left(\frac{3}{4}(\lambda_1 + \lambda_n)^2 - \lambda_1\lambda_n\right) \Phi\left(A^2\right) + (\lambda_1 + \lambda_n)\left(\frac{\lambda_n - \lambda_1}{2}\right)^2 \Phi(A) - \lambda_1\lambda_n \left(\frac{\lambda_1 + \lambda_n}{2}\right)^2 I_k. \tag{2.19}$$

The inequality (2.19) becomes equality when A has at most two distinct eigenvalues. So, for a 2×2 Hermitian matrix A, we have

$$\Phi\left(A^4\right) = \left(\frac{3}{4}(\mathrm{tr}A)^2 - \mathrm{det}A\right) \Phi\left(A^2\right) + \frac{1}{4}\mathrm{tr}A\left(\mathrm{tr}A^2 - 2\mathrm{det}A\right) \Phi(A) - \frac{1}{4}\mathrm{det}A(\mathrm{tr}A)^2 I_k.$$

Our goal now is to find the lower bounds for the spreads of normal matrices. For this, we find the upper bound for $\varphi\left(|B|^2\right)$ analogous to upper bound for $\varphi\left(|A|^2\right)$ in (2.18).

THEOREM 2.5. *Let $\varphi : \mathbb{M}(n) \to \mathbb{C}$ be a positive unital linear functional and let A be a normal element of $\mathbb{M}(n)$ whose spectrum is contained in the disk $|z - c| \leq r$. Then*

$$\varphi\left(|B|^4\right) \leq r^2 \varphi\left(|B|^2\right) + 2\mathrm{Re}\ (\overline{\varphi(A)} - \bar{c})^2 \varphi\left(B^2\right) + |\varphi(A) - c|^2 (r^2 - |\varphi(A) - c|^2), \tag{2.20}$$

where $B = A - \varphi(A) I$.

Proof. The eigenvalues λ_i's of A all lie in the disk $|z - c| \leq r$. Therefore,

$$|\lambda_i - \varphi(A) + c - \varphi(A)|^2 \left(|\lambda_i - \varphi(A) - (c - \varphi(A))|^2 - r^2\right) \leq 0. \tag{2.21}$$

From (2.21), we find that

$$\begin{aligned} |\lambda_i - \varphi(A)|^4 \leq{}& r^2 |\lambda_i - \varphi(A)|^2 + 2\mathrm{Re}\ (\overline{\varphi(A)} - \bar{c})^2 (\lambda_i - \varphi(A))^2 \\ &+ 2r^2 \mathrm{Re}\ (\bar{c} - \overline{\varphi(A)})(\lambda_i - \varphi(A)) + |\varphi(A) - c|^2 (r^2 - |\varphi(A) - c|^2). \end{aligned} \tag{2.22}$$

Multiplying both sides by $\varphi(P_i)$ and using $\sum_{i=1}^n (\lambda_i - \varphi(A))\varphi(P_i) = 0$, we immediately get (2.20). ∎

COROLLARY 2.6. *With notations and conditions as in Theorem 2.5,*

$$\varphi\left(|B|^4\right) + 3\left(\varphi(|B|^2)\right)^2 \leq 4r^2 (r^2 - |\varphi(A) - c|^2) \leq 4r^4. \tag{2.23}$$

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Proof. Since Re $z \leq |z|$, we have

$$\text{Re } (\overline{\varphi(A)} - \bar{c})^2 \varphi\left(B^2\right) \leq |\varphi(A) - c|^2 \left|\varphi\left(B^2\right)\right|. \tag{2.24}$$

Using the triangle inequality, we have

$$\left|\varphi\left(B^2\right)\right| = \left|\sum_{i=1}^{n} (\lambda_i - \varphi(A))^2 \varphi(P_i)\right| \leq \sum_{i=1}^{n} |\lambda_i - \varphi(A)|^2 \varphi(P_i) = \varphi\left(|B|^2\right). \tag{2.25}$$

From (2.24) and (2.25),

$$\text{Re}(\overline{\varphi(A)} - \bar{c})^2 \varphi\left(B^2\right) \leq |\varphi(A) - c|^2 \varphi\left(|B|^2\right). \tag{2.26}$$

Combining (2.20) and (2.26), we get

$$\varphi\left(|B|^4\right) \leq (r^2 + 2|\varphi(A) - c|^2)\varphi(|B|^2) + |\varphi(A) - c|^2 (r^2 - |\varphi(A) - c|^2). \tag{2.27}$$

It follows from (2.27) that

$$\begin{aligned} \varphi\left(|B|^4\right) + 3\left(\varphi\left(|B|^2\right)\right)^2 \leq\ & (r^2 + 2|\varphi(A) - c|^2)\varphi\left(|B|^2\right) \\ & + |\varphi(A) - c|^2 (r^2 - |\varphi(A) - c|^2) + 3\left(\varphi\left(|B|^2\right)\right)^2. \end{aligned} \tag{2.28}$$

Using (2.4) in the right-hand side of (2.28) and simplifying the resulting expression, we easily get the first inequality (2.23). The second inequality (2.23) is self-evident. ∎

We note that $\varphi\left(|B|^4\right) \geq \left(\varphi\left(|B|^2\right)\right)^2$ and therefore the inequality (2.23) provides a refinement of the inequality (1.2) in the special case when A is normal and Φ is functional,

$$\varphi\left(|B|^2\right) \leq \frac{1}{2}\sqrt{\varphi\left(|B|^4\right) + 3\left(\varphi\left(|B|^2\right)\right)^2} \leq r^2.$$

This also implies (1.1) when A is Hermitian. Likewise, we obtain an extension of (1.4) for normal matrices in the following corollary.

Corollary 2.7. *Under the conditions of Theorem 2.5, we have*

$$\varphi\left(|B|^4\right) \leq (r^2 - |\varphi(A) - c|^2)(r^2 + 3|\varphi(A) - c|^2) \leq \frac{4}{3}r^4. \tag{2.29}$$

Proof. The first inequality (2.29) follows by using (2.4) in (2.27). The second inequality (2.29) is immediate;

$$(r^2 - |\varphi(A) - c|^2)(r^2 + 3|\varphi(A) - c|^2) = \frac{4}{3}r^4 - \frac{1}{3}\left(3|\varphi(A) - c|^2 - r^2\right) \leq \frac{4}{3}r^4.$$

∎

Corollary 2.8. *Under the conditions of Theorem 2.5, we have*

$$\varphi\left(|B|^2\right)\varphi\left(|B|^4\right) \leq \frac{256}{243}r^6. \tag{2.30}$$

Proof. It follows from the inequality (2.27) that

$$\begin{aligned} \varphi\left(|B|^2\right)\varphi\left(|B|^4\right) \leq\ & (r^2 + 2|\varphi(A) - c|^2)\varphi\left(|B|^2\right)^2 \\ & + |\varphi(A) - c|^2 (r^2 - |\varphi(A) - c|^2)\varphi\left(|B|^2\right). \end{aligned} \tag{2.31}$$

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Combining (2.4) and (2.31), we get

$$\varphi\left(|B|^2\right)\varphi\left(|B|^4\right) \leq (r^2+3\,|\varphi\,(A)-c|^2)(r^2-|\varphi\,(A)-c|^2)^2. \tag{2.32}$$

The function $f(x)=\left(r^2+3x^2\right)\left(r^2-x^2\right)^2$ with derivative $f^{'}(x)=18x\left(x^2-r^2\right)\left(x^2-\frac{r^2}{9}\right)$ attains its maximum at $x=\frac{r}{3}$ and therefore, $f(x)\leq\frac{256}{243}r^6$. It follows that

$$(r^2+3\,|\varphi\,(A)-c|^2)(r^2-|\varphi\,(A)-c|^2)^2 \leq \frac{256}{243}r^6. \tag{2.33}$$

Combining (2.32) and (2.33), we immediately get (2.30). ∎

3. Special cases.

We here demonstrate some important consequences of the above results.

Let $\langle x,y\rangle=y^*x$ denote the standard inner product on $\mathbb{C}^n$ and $\sqrt{\langle x,x\rangle}=\|x\|$ be the associated norm. Let $B=A-\langle Ax,x\rangle I$, where $x\in\mathbb{C}^n$ and $\|x\|=1$. Bjorck and Thomee [11] proved that for normal operators in a Hilbert space,

$$\|Bx\|^2\leq r^2. \tag{3.1}$$

Let $\varphi(A)=\langle Ax,x\rangle$, where $x\in\mathbb{C}^n$ and $\|x\|=1$. Then $\varphi\,(A)$ is a positive unital linear functional and for this choice of φ the inequality (2.4) provides a refinement of the inequality (3.1),

$$\|Bx\|^2\leq r^2-|\langle Ax,x\rangle-c|^2\,.$$

Also, see Audenaert [1], Barnes [8, 9], Brujin [10].

Likewise, the inequality (2.23) provides some further refinements of (3.1), and we have

$$\|Bx\|^2\leq\sqrt{\frac{\|B^2x\|^2+3\|Bx\|^4}{4}}\leq\sqrt{r^2(r^2-|\langle Ax,x\rangle-c|^2)}\leq r^2.$$

An independent and related inequality

$$\|B^2x\|^2\leq\frac{4}{3}r^4$$

follows from (2.29) by using similar arguments.

We now give some related results for Frobenius norm.

Corollary 3.1. *Let $A\in\mathbb{M}(n)$ be normal. Then*

$$\frac{1}{n}\|B^2\|_F^2+3\left(\frac{1}{n}\|B\|_F^2\right)^2 \leq\; 4r^4\leq\frac{4}{9}\mathrm{spd}\,(A)^4\,, \tag{3.2}$$

$$\frac{1}{n}\|B^2\|_F^2\leq\frac{4}{3}r^4\leq\frac{4}{27}\mathrm{spd}\,(A)^4\,, \tag{3.3}$$

and

$$\frac{1}{n}\|B\|_F\|B^2\|_F\leq\frac{16}{9\sqrt{3}}r^3\leq\frac{16}{81}\mathrm{spd}\,(A)^3\,, \tag{3.4}$$

where $\|A\|_F^2=\mathrm{tr}A^*A$ is the Frobenius norm of A.

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Proof. The linear functional $\varphi(A) = \frac{\operatorname{tr} A}{n}$ is positive and unital. On using $\varphi(A) = \frac{\operatorname{tr} A}{n}$ in (2.23), (2.29) and (2.30), we immediately get (3.2), (3.3) and (3.4), respectively. ∎

Sharma et al. [20] proved that

$$\varphi\left(|B|^4\right) \geq \frac{\left|\varphi\left(B|B|^2\right)\right|^2}{\varphi\left(|B|^2\right)} + \varphi\left(|B|^2\right)^2. \tag{3.5}$$

We use (3.5) to derive our next result.

Corollary 3.2. *Let $\varphi : \mathbb{M}(n) \to \mathbb{C}$ be a positive unital linear functional and let A be any normal element of $\mathbb{M}(n)$. Then*

$$\left|\varphi\left(B|B|^2\right)^2\right| \leq \varphi\left(|B|^2\right)\varphi\left(|B|^4\right) - \varphi\left(|B|^2\right)^3 \leq \frac{16}{27}r^6. \tag{3.6}$$

Proof. The first inequality (3.6) follows from (3.5). Further, $f(x) = ax - x^3 \leq \frac{2}{3\sqrt{3}}a^{\frac{3}{2}}$ for $a > 0$. So

$$\varphi\left(|B|^2\right)\varphi\left(|B|^4\right) - \varphi\left(|B|^2\right)^3 \leq \frac{2}{3\sqrt{3}}\varphi\left(|B|^4\right)^{\frac{3}{2}}. \tag{3.7}$$

From (2.29), $\varphi\left(|B|^4\right)^{\frac{3}{2}} \leq \left(\frac{4}{3}\right)^{\frac{3}{2}} r^6$. Then, (3.7) yields the second inequality (3.6). ∎

For a positive definite matrix $A \in \mathbb{M}(n)$,

$$\varphi\left(B^2\right) \leq \frac{\left(\lambda_n - \lambda_1\right)^2}{4\lambda_n\lambda_1}\varphi(A)^2. \tag{3.8}$$

See Krasnoselski and Krien [16]. A related result for normal matrices is given in the following corollary.

Corollary 3.3. *Under the conditions of the Theorem 2.5 and with $|c| > r$, we have*

$$\frac{\varphi\left(|B|^2\right)}{|\varphi(A)|^2} \leq \frac{r^2}{|c|^2 - r^2} \tag{3.9}$$

and

$$\frac{\varphi\left(|B|^4\right)}{|\varphi(A)|^2} \leq \frac{r^4}{|c|^2 - r^2}. \tag{3.10}$$

Proof. We first note that $\varphi(A) \neq 0$ for $|c| > r$. This follows from the fact that $|\varphi(A) - c| = \left|\sum_{i=1}^n (\lambda_i - c)\, p_i\right| \leq r$ for $|\lambda_i - c| \leq r$, $i = 1, 2, ..., n$ and therefore by the triangle inequality, we have $|\varphi(A)| \geq |c| - r > 0$ for $|c| > r$.

Using the triangle inequality, $|\varphi(A) - c|^2 \geq (|\varphi(A)| - |c|)^2$. Therefore,

$$\frac{r^2 - |\varphi(A) - c|^2}{|\varphi(A)|^2} \leq \frac{2\,|\varphi(A)|\,|c| - \left(|c|^2 - r^2\right)}{|\varphi(A)|^2} - 1. \tag{3.11}$$

The function $f(x) = \frac{2ax - b}{x^2}$ attains its maximum at $x = \frac{b}{a}$ and $f\left(\frac{b}{a}\right) = \frac{a^2}{b}$. So (3.11) implies that

$$\frac{r^2 - |\varphi(A) - c|^2}{|\varphi(A)|^2} \leq \frac{r^2}{|c|^2 - r^2}. \tag{3.12}$$

Using (3.12) in (2.4) and (2.23) we immediately get the inequalities (3.9) and (3.10), respectively. ∎

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It may be noted here that for a positive definite matrix $A \in \mathbb{M}(n)$, $c = \frac{\lambda_1+\lambda_n}{2}$, $r = \frac{\lambda_n-\lambda_1}{2}$ and $|c| > r$. Therefore, from the inequality (3.10), we have

$$\varphi\left(B^4\right) \leq \frac{\left(\lambda_n-\lambda_1\right)^4}{4\lambda_1\lambda_n}\varphi\left(A^2\right). \tag{3.13}$$

The ratio spread $c(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ is also important in the study of positive definite matrices. The inequality

$$\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \geq \left(\sqrt{\frac{\varphi\left(A^2\right)-\varphi\left(A\right)^2}{\varphi\left(A\right)^2}} + \sqrt{1+\frac{\varphi\left(A^2\right)-\varphi\left(A\right)^2}{\varphi\left(A\right)^2}}\right)^2 \tag{3.14}$$

due to Bhatia and Sharma [5] also follows from (3.8).

A matrix $A \in \mathbb{M}(n)$ with entries, $a_{ij} = \frac{1}{x_i+x_j}$, x_i's > 0, is called the Cauchy matrix; see Bhatia [4]. An interesting consequence of (3.14) gives a lower bound for the ratio spread of the Cauchy matrix, which is independent of the entries and depends only on the order of the matrix.

Corollary 3.4. *Let $A = (a_{ij}) \in \mathbb{M}(n)$ be the Cauchy matrix. Then*

$$\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \geq \left(\sqrt{n-1}+\sqrt{n}\right)^2. \tag{3.15}$$

Proof. Without restricting generality suppose that $x_1 \geq x_2 \geq ... \geq x_n$. Let $\varphi(A) = a_{11}$. Then, φ is a positive unital linear functional and

$$\frac{\varphi\left(A^2\right)-\varphi\left(A\right)^2}{\varphi\left(A\right)^2} = \sum_{j=1}^{n}\frac{a_{1j}^2-a_{11}^2}{a_{11}^2} = \sum_{j=2}^{n}\left(\frac{a_{1j}}{a_{11}}\right)^2. \tag{3.16}$$

Now, for $x_1 \geq x_j$, we have $\frac{a_{1j}}{a_{11}} = \frac{\frac{1}{x_1+x_j}}{\frac{1}{2x_1}} = \frac{2x_1}{x_1+x_j} \geq 1$. So, from (3.16),

$$\frac{\varphi\left(A^2\right)-\varphi\left(A\right)^2}{\varphi\left(A\right)^2} \geq \sum_{j=2}^{n} 1 = n-1. \tag{3.17}$$

Combining (3.14) and (3.17), we immediately get (3.15). ∎

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