Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

ELA

PATHS OF MATRICES WITH THE STRONG PERRON-FROBENIUS PROPERTY CONVERGING TO A GIVEN MATRIX WITH THE PERRON-FROBENIUS PROPERTY*

ABED ELHASHASH†, URIEL G. ROTHBLUM‡, AND DANIEL B. SZYLD§

Abstract. A matrix is said to have the Perron-Frobenius property (strong Perron-Frobenius property) if its spectral radius is an eigenvalue (a simple positive and strictly dominant eigenvalue) with a corresponding semipositive (positive) eigenvector. It is known that a matrix A with the Perron-Frobenius property can always be the limit of a sequence of matrices $A(\varepsilon)$ with the strong Perron-Frobenius property such that $\|A - A(\varepsilon)\| \leq \varepsilon$. In this note, the form that the parameterized matrices $A(\varepsilon)$ and their spectral characteristics can take are studied. It is shown to be possible to have $A(\varepsilon)$ cubic, its spectral radius quadratic and the corresponding positive eigenvector linear (all as functions of ε); further, if the spectral radius of A is simple, positive and strictly dominant, then $A(\varepsilon)$ can be taken to be quadratic and its spectral radius linear (in ε). Two other cases are discussed: when A is normal it is shown that the sequence of approximating matrices $A(\varepsilon)$ can be written as a quadratic polynomial in trigonometric functions, and when A has semipositive left and right Perron-Frobenius eigenvectors and $\rho(A)$ is simple, the sequence $A(\varepsilon)$ can be represented as a polynomial in trigonometric functions of degree at most six.

Key words. Perron-Frobenius property, Generalization of nonnegative matrices, Eventually nonnegative matrices, Eventually positive matrices, Perturbation.

AMS subject classifications. 15A48.

1. Introduction.

A real matrix A is called nonnegative (respectively, positive) if it is entry-wise nonnegative (respectively, positive) and we write $A \geq 0$ (respectively, $A > 0$). This notation and nomenclature is also used for vectors. A column or a row vector v is called *semipositive* if v is nonzero and nonnegative. Likewise, if v is nonzero and entry-wise nonpositive then v is called *seminegative*. We denote the spectral radius of a matrix A by $\rho(A)$. We say that a real square matrix A has the *Perron-Frobenius (P-F) property* if $Av = \rho(A)v$ for some semipositive vector v, called a right P-F eigenvector, or simply a P-F eigenvector. We call a semipositive vector w a left P-F eigenvector if $A^T w = \rho(A)w$. Moreover, we say that A possesses the

*Received by the editors August 16, 2009. Accepted for publication December 8, 2009. Handling Editor: Michael J. Tsatsomeros.

†Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104-2816, USA (abed@drexel.edu).

‡Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel (rothblum@technion.ac.il).

§Department of Mathematics, Temple University (038-16), 1805 N. Broad Street, Philadelphia, Pennsylvania 19122-6094, USA (szyld@temple.edu).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

strong P-F property if A has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector. Further, define the sets WPFn (respectively, PFn) of $n \times n$ real matrices A such that A and A^T have the P-F property (respectively, the strong P-F property); see, e.g., [2], [3], [11], [14], where these concepts are studied and used. The P-F property is historically associated with nonnegative matrices; see the seminal papers by Perron [12] and Frobenius [5] or the classic books [1], [7], [15], for many applications.

In [2, Theorem 6.15], it is shown that for any matrix A with the P-F property, and any $\varepsilon > 0$, there exists a matrix $A(\varepsilon)$ with the strong P-F property such that $\|A - A(\varepsilon)\| \leq \varepsilon$. In the same paper it is shown that the closure of PFn is not necessarily WPFn. Nevertheless, there are two situations in which we can identify paths of matrices $A(\varepsilon)$ in PFn converging to any given matrix A in WPFn. One of these cases is that of normal matrices in WPFn, and the other is when the spectral radius is a simple eigenvalue.

In this note we address the following question: Can we determine a simple expression for the aforementioned matrices $A(\varepsilon)$ as a function of ε? and in particular, can we write $A(\varepsilon)$ and its corresponding spectral characteristics (spectral radius and corresponding eigenvector) as a polynomial in ε of low degree (with matrix coefficients)? In other words, what can we say about a path of matrices $A(\varepsilon)$ with the strong P-F property that converges to a given matrix A possessing the P-F property as $\varepsilon \to 0$? We show that it is possible to have $A(\varepsilon)$ cubic, $\rho[A(\varepsilon)]$ quadratic and the corresponding positive P-F eigenvector linear in ε (where the constant term of the corresponding polynomials are the true characteristics of A). In particular, the result demonstrates that it is possible to construct a matrix satisfying the strong P-F property such that the matrix, its spectral radius and its positive P-F eigenvector approximate the unperturbed values with order $O(\varepsilon)$. For normal matrices in WPFn and those in WPFn for which the spectral radius is simple, we present approximating matrices that satisfy the strong P-F property and are polynomials of small degree in trigonometric functions of ε; the corresponding positive P-F eigenvectors have a similar expansion whereas the corresponding spectral radii are linear in ε.

2. Polynomial representation of approximating sequences. To answer the questions posed in the introduction on the form of the sequence $A(\varepsilon)$ with the strong P-F property converging to A with the P-F property, we begin with the following result.

THEOREM 2.1. *Let A be an $n \times n$ real matrix with the P-F property and let v be a semipositive eigenvector of A corresponding to $\rho(A)$. Then, there exist $n \times n$ real matrices A_1, A_2 and A_3, an n-vector v_1 and scalars ρ_1 and ρ_2 such that for all*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

sufficiently small positive scalars ε, the matrix

$$A(\varepsilon) = A + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 \tag{2.1}$$

has the strong P-F property,

$$\rho[(A(\varepsilon)] = \rho(A) + \varepsilon\rho_1 + \varepsilon^2\rho_2, \tag{2.2}$$

and a corresponding positive eigenvector of $A(\varepsilon)$ is $v(\varepsilon) = v + \varepsilon v_1$. Furthermore, if $\rho(A)$ is a simple, positive and strictly dominant eigenvalue of A, then $A_3 = 0$ and $\rho_2 = 0$.

Proof. The proof is constructive. Let v be a P-F eigenvector of A, that is, v is semi-positive and $Av = \rho(A)v$. Let P be an $n \times n$ nonsingular real matrix such that $P^{-1}AP = J(A)$ is in real Jordan canonical form. We may assume that v is the first column of P and that the first diagonal block of $P^{-1}AP$ corresponds to $\rho(A)$. Consider the vector w given by

$$w^T e_i = \begin{cases} 1 & \text{if} \quad v^T e_i = 0 \\ 0 & \text{if} \quad v^T e_i \neq 0, \end{cases} \tag{2.3}$$

where e_i $(i = 1, \ldots, n)$ denotes the i^{th} canonical vector of $\mathbb{R}^n$, i.e., the i^{th} entry of e_i is 1 while all the other entries are 0's. The vector w satisfies $w^T v = 0$ and its nonzero coordinates are all ones, in particular, $w = 0$ if and only if $v > 0$. For any $\varepsilon > 0$, let $P(\varepsilon) = P + \varepsilon w e_1^T$ and let $J(\varepsilon) = J(A) + \delta\varepsilon e_1 e_1^T$, where $\delta = 0$ if $\rho(A)$ is a simple positive and strictly dominant eigenvalue of A, or else $\delta = 1$. Note that for all scalars $\varepsilon > 0$, we have that $\rho[J(\varepsilon)] = \rho(A) + \delta\varepsilon$ is a simple positive and strictly dominant eigenvalue of $J(\varepsilon)$ with a corresponding eigenvector e_1. Furthermore, for a sufficiently small $\varepsilon > 0$, it holds that $P(\varepsilon)$ is nonsingular and for any such ε, the matrix $B(\varepsilon) = P(\varepsilon)J(\varepsilon)P(\varepsilon)^{-1}$ has $\rho[B(\varepsilon)] = \rho[J(\varepsilon)] = \rho(A) + \delta\varepsilon$ as a simple positive and strictly dominant eigenvalue of $B(\varepsilon)$ with a corresponding eigenvector $v(\varepsilon) = P(\varepsilon)e_1 = v + \varepsilon w > 0$. Therefore, $B(\varepsilon)$ has the strong P-F property. In order to build $A(\varepsilon)$ from $B(\varepsilon)$, we observe that $P(\varepsilon)^{-1}$ can be expressed explicitly by

$$P(\varepsilon)^{-1} = P^{-1} - \frac{\varepsilon P^{-1} w e_1^T P^{-1}}{1 + \varepsilon e_1^T P^{-1} w};$$

the above is easy to verify directly (in fact, it is an instance of the Sherman-Morrison-Woodbury formula; see, e.g., [6]). We continue by considering $\varepsilon > 0$ sufficiently small so that, in addition to satisfying the aforementioned properties, $1 + \varepsilon e_1^T P^{-1} w > 0$. Letting $A(\varepsilon) := (1 + \varepsilon e_1^T P^{-1} w)B(\varepsilon)$, we then have that $A(\varepsilon)$ has the strong P-F property. Furthermore,

$$\begin{aligned} A(\varepsilon) &= (P + \varepsilon w e_1^T)[J(A) + \delta\varepsilon e_1 e_1^T][(1 + \varepsilon e_1^T P^{-1} w)P^{-1} - \varepsilon P^{-1} w e_1^T P^{-1}] \\ &= (P + \varepsilon w e_1^T)[J(A) + \delta\varepsilon e_1 e_1^T]\{P^{-1} + \varepsilon[(e_1^T P^{-1} w)P^{-1} - P^{-1} w e_1^T P^{-1}]\}. \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

Thus, $A(\varepsilon)$ has a representation $A_0+\varepsilon A_1+\varepsilon^2 A_2+\varepsilon^3 A_3$ with $A_0 = PJ(A)P^{-1} = A$, $A_3 = \delta w e_1^T[(e_1^T P^{-1}w)P^{-1} - P^{-1}we_1^T P^{-1}]$ and corresponding A_1 and A_2. Further,

$$\rho[A(\varepsilon)] = (1+\varepsilon e_1^T P^{-1}w)[\rho(A)+\varepsilon\delta] = \rho(A)+\varepsilon\rho_1+\varepsilon^2\rho_2$$

with $\rho_1 = \delta+\rho(A)e_1^T P^{-1}w$ and $\rho_2 = \delta e_1^T P^{-1}w$, and $v(\varepsilon) = v+\varepsilon w$ is a corresponding positive eigenvector. In particular, if $\rho(A)$ is a simple, positive and strictly dominant eigenvalue of A, then $\delta = 0$ which implies $A_3 = 0$ and $\rho_2 = 0$. □

Remark 2.2. Let $\tau \equiv e_1^T P^{-1}w$. We note that this quantity may be negative. In this case, in (2.2), ρ_1 and/or ρ_2 may take negative values, and $\rho[A(\varepsilon)]$ may be smaller than $\rho(A)$ for sufficiently small $\varepsilon > 0$; the proof of Theorem 2.1 still verifies (2.2) with the corresponding expressions for ρ_1 and ρ_2. We note also that when $\delta = 1$, the eigenvalues of $A(\varepsilon)$ (with the exception of $\rho[A(\varepsilon)]$) are multiples of eigenvalues of A by the scalar $1+\tau\varepsilon$, while $\rho[A(\varepsilon)] = (1+\tau\varepsilon)(\rho(A)+\varepsilon)$. Thus, when $\tau < 0$, and $\delta = 1$, for sufficiently small $\varepsilon > 0$, we have that $0 < 1+\tau\varepsilon < 1$ and thus the order of the eigenvalues of $A(\varepsilon)$ in absolute value is maintained, i.e., it is the same order (in absolute value) as that of the eigenvalues of A.

Theorem 2.3. *Let A be a normal $n\times n$ real matrix with the P-F property, and let v be its P-F eigenvector. Then, for all sufficiently small positive scalars ε, there exists an approximating sequence of normal matrices $A(\varepsilon)$ in PFn (and hence having the strong P-F property) having the form*

$$A(\varepsilon) = A+\varepsilon A_1+\sum_{1\le j+k\le 2}\sin^j\varepsilon\ (\cos\varepsilon-1)^k\ A_{jk}+\ \varepsilon\sum_{1\le j+k\le 2}\sin^j\varepsilon\ (\cos\varepsilon-1)^k\ B_{jk}$$

where the matrices A_1, A_{jk}, and B_{jk} are real $n\times n$ matrices, their spectral radius has the form $\rho[A(\varepsilon)] = \rho(A)+\varepsilon$, and the corresponding eigenvector is $v(\varepsilon) = (\cos\varepsilon)v + (\sin\varepsilon)u$, where $u^T v = 0$. Furthermore, if $\rho(A)$ is simple positive and strictly dominant eigenvalue, then $B_{jk} = 0$ and $\rho[A(\varepsilon)] = \rho(A)$, and if A has a positive P-F vector, then $A_{jk} = B_{jk} = 0$, and $v(\varepsilon) = v$.

Proof. This proof is also constructive. Let A be a normal $n\times n$ real matrix with the P-F property and let v be a P-F eigenvector of A. Then, $A = PMP^T$, where P is an $n\times n$ real orthogonal matrix and M is a direct sum of 1×1 real blocks or positive scalar multiples of 2×2 real orthogonal blocks; see, e.g., [7, Theorem 2.5.8]. We may assume that the first diagonal block of M is the 1×1 block $[\rho(A)]$ and that v is the first column of P. Note that v is in this case a unit vector and that it is both a right and a left P-F eigenvector of A. Let ε be any given nonnegative real number. We define a matrix Q_ε as follows: If v is a positive vector then define the matrix Q_ε to be the $n\times n$ identity matrix for all $\varepsilon\in[0,\infty)$. Otherwise consider the vector w given by (2.3), then the vector $u := w/||w||$ is a semipositive vector of unit length which is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

orthogonal to v. For all $\varepsilon \in \left[0, \frac{\pi}{2}\right)$ define the orthogonal matrix

$$Q_\varepsilon := I + (\cos\varepsilon - 1)(vv^T + uu^T) + \sin\varepsilon(uv^T - vu^T), \tag{2.4}$$

and then define the matrix

$$A(\varepsilon) := Q_\varepsilon P(M + \varepsilon\delta e_1 e_1^T)P^T Q_\varepsilon^T, \tag{2.5}$$

where $\delta = 0$ if $\rho(A)$ is a simple positive and strictly dominant eigenvalue of A or else $\delta = 1$. Observe that the spectral radius of the matrix $A(\varepsilon)$ (which is $\rho(A) + \varepsilon\delta$) is a simple positive and strictly dominant eigenvalue of $A(\varepsilon)$ for all $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ and that the vector $Q_\varepsilon v = (\cos\varepsilon)v + (\sin\varepsilon)u$ is a positive right and left P-F eigenvector of $A(\varepsilon)$ for all $\varepsilon \in \left(0, \frac{\pi}{2}\right)$. Hence, $A(\varepsilon)$ is in PFn and thus $A(\varepsilon)$ has the strong P-F property. Moreover, it follows from (2.5) that $A(\varepsilon)$ is unitarilly diagonalizable and therefore is normal. Taking into consideration the explicit form of Q_ε from (2.4), we can write the matrix $A(\varepsilon)$ as follows:

$$\begin{aligned} A(\varepsilon) &= Q_\varepsilon(PMP^T + \varepsilon\delta P e_1 e_1^T P^T)Q_\varepsilon^T \\ &= Q_\varepsilon(A + \varepsilon\delta vv^T)Q_\varepsilon^T \\ &= A + \varepsilon vv^T + \sum_{1 \le j+k \le 2} \sin^j\varepsilon\ (\cos\varepsilon - 1)^k\ A_{jk} \\ &\qquad + \ \varepsilon\ \delta \sum_{1 \le j+k \le 2} \sin^j\varepsilon\ (\cos\varepsilon - 1)^k\ B_{jk}, \end{aligned}$$

where A_{jk} and B_{jk} are real $n \times n$ matrices. Furthermore, it follows from (2.5) that $A(\varepsilon) \to A$ as $\varepsilon \to 0$ and that if v is a positive vector then $Q_\varepsilon = I$ and thus $A(\varepsilon) = A + \varepsilon vv^T$. $\square$

A normal $n \times n$ real matrix A has the P-F property if and only if A is in WPFn. Hence, Theorem 2.3 gives the form of a normal approximating sequence of matrices $A(\varepsilon)$ in PFn that converges to a given normal matrix A in WPFn as $\varepsilon \to 0$ (even though it is not true that WPFn is the closure of PFn; see [2]). However, if we consider matrices A in WPFn for which $\rho(A)$ is a simple eigenvalue then we obtain the next result.

THEOREM 2.4. *Let A be a matrix in WPFn such that $\rho(A)$ is a simple eigenvalue, and let u and v be the corresponding right and left P-F eigenvectors, respectively. Then, there is an approximating sequence of matrices $A(\varepsilon)$ in PFn of the form:*

$$A(\varepsilon) = \sum_{k,m} (\cos^k\varepsilon \sin^m\varepsilon) B_{km} + \sum_{k,m} (\varepsilon \cos^k\varepsilon \sin^m\varepsilon) C_{km},$$

where k and m are integers such that $0 \le k \le 6$ and $0 \le m \le 2$; B_{km} and C_{km} are real $n \times n$ matrices, their spectral radius have the form $\rho[A(\varepsilon)] = \rho(A) + \varepsilon$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

and the corresponding P-F eigenvectors have the form

$$u(\varepsilon) = u + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} u, \tag{2.6}$$

$$v(\varepsilon) = v + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} v, \tag{2.7}$$

where $0 \le k \le 3$*,* $0 \le m \le 2$*, and* $\hat{B}_{km}$ *are real* $n \times n$ *matrices. Thus* $A(\varepsilon) \to A$ *as* $\varepsilon \to 0$*. Furthermore, if* $\rho(A)$ *is a strictly dominant eigenvalue then* $C_{km} = 0$ *for all* k *and* m*.*

Proof. Consider a matrix A in WPFn for which $\rho(A)$ is a simple eigenvalue. Let $A = P\,[[\rho(A)] \oplus J_2]\,P^{-1}$ be the real Jordan decomposition of matrix A, where J_2 is the direct sum of all the real Jordan blocks that correspond to eigenvalues other than $\rho(A)$ and suppose that u and v are respectively the first column of P and the transpose of the first row of P^{-1}. Thus, u and v are, respectively, right and left eigenvectors of A corresponding to $\rho(A)$. Moreover, $u^T v = v^T u = 1 > 0$ since $P^{-1}P = I$.

Let a nonnegative scalar ε be given. We begin by finding an orthogonal matrix Q_ε that converges to the identity matrix as $\varepsilon \to 0$ and that maps the two semipositive vectors u and v simultaneously to a pair of positive vectors for all sufficiently small positive values of ε. Most of the proof that follows is dedicated to constructing Q_ε. The orthogonal matrix Q_ε will be defined as the product of three orthogonal matrices $Q_{(j,\varepsilon)}$ $(j = 1, 2, 3)$ which are rotations.

Partition the set $\langle n\rangle$ by writing $\langle n\rangle = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ where $\alpha_1 = \{j \mid u_j = v_j = 0\}$, $\alpha_2 = \{j \mid u_j > 0 \text{ and } v_j = 0\}$, $\alpha_3 = \{j \mid u_j > 0 \text{ and } v_j > 0\}$, and $\alpha_4 = \{j \mid u_j = 0 \text{ and } v_j > 0\}$. Let k_j denote the cardinality of α_j for $j = 1, 2, 3, 4$, and note that $k_3 \neq 0$ because $u^T v > 0$. We may assume that the elements of α_1 are the first k_1 integers in $\langle n\rangle$, the elements of α_2 are the following k_2 integers in $\langle n\rangle$, the elements of α_3 are the following k_3 integers in $\langle n\rangle$, and the elements of α_4 are the last k_4 integers in $\langle n\rangle$, i.e., $\alpha_1 = \{1, 2, \ldots, k_1\}$, $\alpha_2 = \{k_1+1, k_1+2, \ldots, k_1+k_2\}$, $\alpha_3 = \{k_1+k_2+1, k_1+k_2+2, \ldots, k_1+k_2+k_3\}$, and $\alpha_4 = \{k_1+k_2+k_3+1, k_1+k_2+k_3+2, \ldots, n\}$. Let w_1 denote the vector $e_{k_1+k_2+1}$. If the cardinality of α_2 is zero, i.e., $k_2 = 0$ then let $Q_{(1,\varepsilon)} = I$ otherwise define the vector

$$w_4[\alpha_j] = \begin{cases} 0 & \text{if } j = 1, 3, 4 \\ \frac{1}{||u[\alpha_2]||} u[\alpha_2] & \text{if } j = 2 \end{cases}$$

and let

$$Q_{(1,\varepsilon)} = I + (\cos\varepsilon - 1)(w_1 w_1^T + w_4 w_4^T) - \sin\varepsilon (w_1 w_4^T - w_4 w_1^T)$$

where $\varepsilon \in [0, \delta_1]$ and δ_1 is a sufficiently small positive scalar. Similarly, if the cardi-

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

nality of α_4 is zero, i.e., $k_4 = 0$ then let $Q_{(2,\varepsilon)} = I$ otherwise define the vector

$$w_3[\alpha_j] = \begin{cases} 0 & \text{if } j = 1,2,3 \\ \frac{1}{||v[\alpha_4]||} v[\alpha_4] & \text{if } j = 4 \end{cases}$$

and let

$$Q_{(2,\varepsilon)} = I + (\cos\varepsilon - 1)(w_1 w_1^T + w_3 w_3^T) - \sin\varepsilon (w_1 w_3^T - w_3 w_1^T)$$

where $\varepsilon \in [0, \delta_2]$ and δ_2 is a sufficiently small positive scalar. Furthermore, if the cardinality of α_1 is zero, i.e., $k_1 = 0$ then let $Q_{(3,\varepsilon)} = I$ otherwise define the vector $w_2 = (k_1)^{-1/2} \sum_{j=1}^{k_1} e_j$ and let

$$Q_{(3,\varepsilon)} = I + (\cos\varepsilon - 1)(w_1 w_1^T + w_2 w_2^T) - \sin\varepsilon (w_1 w_2^T - w_2 w_1^T)$$

where $\varepsilon \in [0, \delta_3]$ and δ_3 is any scalar in the open interval $\left(0, \frac{\pi}{2}\right)$. Define the rotation $Q_\varepsilon := Q_{(3,\varepsilon)} Q_{(2,\varepsilon)} Q_{(1,\varepsilon)}$ for all $\varepsilon \in [0, \delta]$ where $\delta = min\{\delta_1, \delta_2, \delta_3\}$. Thus,

$$\begin{aligned} Q_\varepsilon = I &+ (\cos^3\varepsilon - 1) w_1 w_1^T - (\sin\varepsilon) w_1 w_2^T - (\cos\varepsilon \sin\varepsilon) w_1 w_3^T - (\cos^2\varepsilon \sin\varepsilon) w_1 w_4^T \\ &+ (\cos^2\varepsilon \sin\varepsilon) w_2 w_1^T + (\cos\varepsilon - 1) w_2 w_2^T - (\sin^2\varepsilon) w_2 w_3^T - (\cos\varepsilon \sin^2\varepsilon) w_2 w_4^T \\ &+ (\cos\varepsilon \sin\varepsilon) w_3 w_1^T + (\cos\varepsilon - 1) w_3 w_3^T - (\sin^2\varepsilon) w_3 w_4^T + (\sin\varepsilon) w_4 w_1^T \\ &+ (\cos\varepsilon - 1) w_4 w_4^T. \end{aligned}$$

Define the approximating matrix $A(\varepsilon)$ as follows:

$$A(\varepsilon) := Q_\varepsilon P \left[[\rho(A) + \varepsilon] \oplus J_2 \right] (Q_\varepsilon P)^{-1} \tag{2.8}$$

for all ε in $[0, \delta]$. The matrix $A(\varepsilon)$ is in PFn for all ε in $(0, \delta]$ and the right and left P-F eigenvectors of $A(\varepsilon)$ are $Q_\varepsilon u$ and $Q_\varepsilon v$ respectively, which have the form (2.6) and (2.7). Moreover, it is clear from the form of Q_ε and (2.8) that $A(\varepsilon)$ can be written as follows:

$$\begin{aligned} A(\varepsilon) &= Q_\varepsilon P \left[[\rho(A) + \varepsilon] \oplus J_2 \right] P^{-1} Q_\varepsilon^T \\ &= \sum_{k,m} (\cos^k\varepsilon \sin^m\varepsilon) B_{km} + \sum_{k,m} (\varepsilon \cos^k\varepsilon \sin^m\varepsilon) C_{km} \end{aligned}$$

where k and m are integers such that $0 \le k \le 6$ and $0 \le m \le 2$; B_{km} and C_{km} are real $n \times n$ matrices; and $A(\varepsilon) \to A$ as $\varepsilon \to 0$. Furthermore, if $\rho(A)$ is a strictly dominant eigenvalue then $C_{km} = 0$ for all k and m. ☐

Remark 2.5. We note that Theorem 2.4 holds for more general matrices. The spectral radius in this theorem does not need to be a simple eigenvalue. It suffices that a 1×1 Jordan block corresponding to the spectral radius exists and that to this block there correspond right and left P-F eigenvectors u and v, respectively. Furthermore, *the approximating matrices in Theorems 2.3 and 2.4 can be written as power series in* ε *after replacing* $\cos\varepsilon$ *and* $\sin\varepsilon$ *with their corresponding Taylor series.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 19, pp. 90-97, December 2009

Acknowledgments. The authors would like to thank an anonymous referee for helpful comments and remarks. Research leading to this note was commenced during a Workshop on Nonnegative Matrices held at the American Institute of Mathematics (AIM), 1–5 December 2008. Support of AIM to the participants of the workshop and the atmosphere of collaboration that the Institute fosters is greatly appreciated. The work of the second author was also supported in part by a grant for the promotion of research at the Technion. The work of the third author was also supported in part by the U.S. Department of Energy under grant DE-FG02-05ER25672.

REFERENCES

[1] A. Berman and R. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Second edition. Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
[2] A. Elhashash and D. B. Szyld. On general matrices having the Perron-Frobenius property. *Electronic Journal of Linear Algebra*, 17:389–413, 2008.
[3] ———. Two characterizations of matrices with the Perron-Frobenius property. *Numerical Linear Algebra with Applications*, 16:863–869, 2009.
[4] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29:43–60, 1978.
[5] G. Frobenius. Über Matrizen aus nicht negativen Elementen. Preussiche Akademie der Wissenschaften zu Berlin, 1912:456–477, 1912.
[6] G.H. Golub and C.F. Van Loan. *Matrix Computations.* Third Edition, The Johns Hopkins University Press, Baltimore, Maryland 1996.
[7] R. Horn and C.R. Johnson. *Matrix Analysis* Cambridge University Press, Cambridge, 1985.
[8] ———. *Topics in Matrix Analysis* Cambridge University Press, Cambridge, 1991.
[9] C. R. Johnson and P. Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8:327–338, 2004.
[10] ———. A characterization of positive matrices. *Positivity*, 9:149–152, 2005.
[11] D. Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Linear Algebra and its Applications*, 412:132–153, 2006.
[12] O. Perron. Zur Theorie Der Matrizen. *Mathematische Annalen*, 64:248–263, 1907.
[13] G. W. Stewart and J.-g. Sun. *Matrix Perturbation Theory.* Academic Press, San Diego, 1990.
[14] P. Tarazaga, M. Raydan, and A. Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Linear Algebra and its Applications*, 328:57–68, 2001.
[15] R. S. Varga. *Matrix Iterative Analysis.* Second edition, Springer-Verlag, Berlin, 2000.