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BOUNDS VIA SPECTRAL RADIUS-PRESERVING ROW SUM EXPANSIONS*

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Abstract. We show a simple method for constructing larger dimension nonnegative matrices with somewhat arbitrary entries which can be irreducible or reducible but preserving the spectral radius via row sum expansions. This yields a sufficient criteria for two square nonnegative matrices of arbitrary dimension to have the same spectral radius, a way to compare spectral radii of two arbitrary square nonnegative matrices, and a way to derive new upper and lower bounds on the spectral radius which give the standard row sum bounds as a special case.

Key words. Spectral radius, Nonnegative matrix.

AMS subject classifications. 15A18, 15A42.

1. Introduction. For nonnegative square matrices, it is well known that the spectral radius is between the minimum and maximum row sums. If the row sums are constant, then that sum is the spectral radius. It is also known that equality between the spectral radius and either the minimal or maximal row sum for an irreducible matrix implies its row sums are constant (see Theorem 1.1 in [3]). Here, we will take a square matrix and increase its dimension by expanding one of its diagonal elements (or a square principal submatrix) into a larger square block while still preserving the spectral radius of the original matrix. Then, we use this to derive new bounds on the spectral radius and show that the standard row sum bounds are a special case.

Let $M \geq 0$ be a nonnegative square matrix (each component is nonnegative) with spectral radius $\rho(M) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } M\}$. Consider a diagonal element $M_{ii} = d$. Via a permutation similarity, since the spectral radius is invariant under such transformations, we can simply assume d is the lower right most diagonal component of the matrix, that is,

$$M = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^\top & d \end{bmatrix},$$

where A is an $(n-1) \times (n-1)$ block, and $\mathbf{b}$ and $\mathbf{c}$ are vectors with $n-1$ components.

We now create a new matrix:

$$\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where D is a nonnegative square matrix of arbitrary dimension $s \times s$ with constant row sums equal to d (d is row-sum-expanded into an $s \times s$ block). The components of nonnegative matrix B simply need to satisfy $\sum_j B_{ij} = b_i$ but are arbitrary otherwise (the i^{th} row sum equals the i^{th} element of $\mathbf{b}$). We refer to this as a row sum expansion for each component b_i. The nonnegative matrix C has each row identical to $\mathbf{c}^\top$, denoted $C_{i-} = \mathbf{c}^\top$. We refer to $\tilde{M}$ as a *row sum expansion of* M *on* d. The overall dimensions of $\tilde{M}$ are now $(n-1+s) \times (n-1+s)$. It is in fact the case that $\rho(M) = \rho(\tilde{M})$ (which we prove in Theorem 2.1 below).

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Similarly, one can perform *column sum expansions* instead (with the rules on B and C symmetrically adjusted, since a column sum expansion for M is a row sum expansion for $M^\top$). It is not hard to come up with rules which allow one to expand multiple diagonal elements simultaneously, applying row sum expansions to some and column sum expansions to others, as long as the adjacent blocks are created appropriately. As an example, if, simultaneously, M_{ii} is row-sum-expanded into an $s \times s$ block and M_{jj} column-sum-expanded into an $r \times r$ block, then:

1. For all $k \neq i, j$, M_{ik} is expanded into an $s \times 1$ block with each entry identical to M_{ik},
2. For all $k \neq i, j$, M_{ki} is expanded into a $1 \times s$ block whose components are arbitrary but sum to M_{ki},
3. For all $k \neq i, j$, M_{jk} is expanded into a $1 \times r$ block whose components are arbitrary but sum to M_{jk},
4. For all $k \neq i, j$, M_{kj} is expanded into a $r \times 1$ block whose entries are all identical to M_{kj},
5. M_{ij} is expanded into an $s \times r$ block with arbitrary entries whose sum equals M_{ij}, and
6. M_{ji} is expanded into an $r \times s$ block with each component equal to M_{ji}.

Similar rules can be written down for multiple simultaneous row sum expansions or for any number of simultaneous row and column sum expansions. In particular, when row-sum-expanding multiple diagonal components (hence we are concerned with the corresponding principal submatrix), we row-sum-expand each component of the principal submatrix into an appropriately-sized block and follow rules 1 and 2 above.

We now give some examples of row and column sum expansions to familiarize the reader with the process. See (1.1) for an example below where a column sum and row sum expansion are performed simultaneously.

1.1. Examples of matrix expansions. Let A and B be nonnegative square matrices, we write $A \sim B$ if and only if $\rho(A) = \rho(B)$.

When we perform an expansion of a diagonal element into a block, we sometimes add additional parentheses to illustrate the new block which was previously a single component. Here is an introductory example:

$$[5] \sim \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \end{pmatrix} & 1 \end{bmatrix},$$

where in the first step we simply created a 2×2 matrix with constant row sums of 5 and then transposed the result in the next step. Concerning the final step, we row-sum-expanded the first diagonal element. This requires us to repeat the horizontal off-diagonal component 4 (for a row sum expansion, we always replicate exactly any components to the left or right) and we can expand the vertical off-diagonal element 3 into an arbitrary 1×2 block as long as its row sum is 3 (for a row sum expansion, we are allowed to replicate any above or below matrix components into arbitrary rows that each sum to the original component they are expanded from).

Here is an example with a simultaneous row sum expansion on both diagonal elements into different sized diagonal blocks. In this example, we simply require the row sums in each block to be constant and equal to the original matrix component being expanded:

$$\begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix} & \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1.5 & 2.5 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 1.5 & 0.5 \end{pmatrix} & \begin{pmatrix} 3 & 0.5 & 0.5 \\ 1 & 1 & 2 \\ 0 & 4 & 0 \end{pmatrix} \end{bmatrix}.$$

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Here is an example where we perform a column sum expansion on the first diagonal element and a row sum expansion on the second. In this case, the lower left block must be the lower left matrix component repeated in all positions (rule 6 above), but the upper right block just needs to sum to the original upper right component (rule 5 above):

$$\begin{bmatrix} 5 & 8 \\ 7 & 3 \end{bmatrix} \sim \begin{bmatrix} \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix} \\ \begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix} & \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \end{bmatrix}. \tag{1.1}$$

Again, we could go on concocting complicated mixes of transposes, permutation similarity transforms, and row and column sum expansions. This allows the overall list of row and column sums to become almost any collection of nonnegative numbers; thus, the standard bounds on spectral radius do not make it obvious that the spectral radius is preserved.

In Section 2, we prove that the row sum expansion procedure preserves the spectral radius. Then in Section 3, we show how to get bounds on the spectral radius of a matrix based on the idea of row sum contractions and give a way to compare spectral radii of two matrices with different dimensions.

2. Row sum expansions preserve the spectral radius. We now prove that row sum expansions preserve the spectral radius. Here, it is important that we are expanding diagonal elements into square blocks. In general, a row sum expansion that does not expand diagonal elements into square blocks does not preserve the spectral radius.

THEOREM 2.1. *Let M be a nonnegative matrix of dimension $n \times n$, and let $\tilde{M}$ be a row sum expansion of M on any one of its diagonal elements. Then $\rho(M) = \rho(\tilde{M})$.*

Proof. Without loss of generality by permutational similarity, we prove the theorem for the last diagonal component of M. Let

$$M = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^\top & d \end{bmatrix},$$

where A is a $(n-1) \times (n-1)$ block, $\mathbf{b}$ and $\mathbf{c}$ are vectors with $n-1$ components, and d is a single diagonal component.

Since $M \geq 0$, we know that its spectral radius $\rho \geq 0$ is an eigenvalue with nonnegative left eigenvector $(\mathbf{x}^\top, y)$ where $\mathbf{x}$ has $n-1$ components and y is a single component (see Theorem 8.3.1 in [1]). Assume the eigenvector components sum to one $y + \sum_{i=1}^{n-1} x_i = 1$. Note that we do not require any uniqueness, so there may be several such eigenvectors.

We have that

$$\begin{aligned} (\mathbf{x}^\top, y) \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^\top & d \end{bmatrix} &= (\mathbf{x}^\top A + y\mathbf{c}^\top, \mathbf{x}^\top \mathbf{b} + yd) \\ &= (\rho \mathbf{x}^\top, \rho y), \end{aligned}$$

thus giving

$$\mathbf{x}^\top \mathbf{b} = y(\rho - d), \tag{2.2}$$

$$\mathbf{x}^\top (\rho I_{n-1} - A) = y\mathbf{c}^\top. \tag{2.3}$$

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It is already known that $\rho \geq d$. If $\rho = d$, then $\mathbf{x}$ is orthogonal to $\mathbf{b}$ and thus has zeros wherever $\mathbf{b}$ has positive components (recall that everything is nonnegative). If ρ is also an eigenvalue of the principle submatrix A, then we can find infinitely many $\mathbf{x}$, but, again, we do not require uniqueness here. We just require some nonnegative eigenvector to exist.

We now create a new $(n-1+s) \times (n-1+s)$ matrix $\tilde{M} \geq 0$ by expanding d into a $s \times s$ matrix D with constant row sums $\sum_{j=1}^{s} D_{ij} = d$ for all $i = 1, 2, \ldots, s$,

$$\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B is an $(n-1) \times s$ block whose row sums equal the elements of vector $\mathbf{b}$: $\sum_{j=1}^{s} B_{ij} = b_i$ for $i = 1, 2, \ldots, n-1$, and C is an $s \times (n-1)$ block where row C_{i-} is identical to $\mathbf{c}^\top$ for each $i = 1, 2, \ldots, s$.

Similarly, this expanded matrix has a nonnegative spectral radius $\tilde{\rho} \geq 0$ which is an eigenvalue with nonnegative eigenvector $(\tilde{\mathbf{x}}^\top, \mathbf{y}^\top)$. Again, we assume that $\sum_{i=1}^{s} y_i + \sum_{i=1}^{n-1} \tilde{x}_i = 1$. We will show that $\tilde{\mathbf{x}} = \mathbf{x}$ is a possible choice, with $\sum y_i = y$, and $\tilde{\rho} = \rho$.

First, we show that $\rho, (\mathbf{x}^\top, \mathbf{y}^\top)$ is an eigenpair for $\tilde{M}$ with $\mathbf{y}$ being a vector that solves

$$\mathbf{x}^\top B = \mathbf{y}^\top (\rho I_s - D). \tag{2.4}$$

There may be several such vectors $\mathbf{y}$ (if $\rho = d$ the constant row sum for matrix D). Note that summing each side of (2.4) gives (2.2), hence $y = \sum_{i=1}^{s} y_i = 1 - \sum_{i=1}^{n-1} x_i$.

Since block C simply has vector $\mathbf{c}^\top$ repeated s times for its rows, we have

$$\mathbf{y}^\top C = \left(\sum_{i=1}^{s} y_i \right) \mathbf{c}^\top = y\mathbf{c}^\top. \tag{2.5}$$

Using (2.3) and (2.5), we see that

$$\mathbf{x}^\top A + \mathbf{y}^\top C = \mathbf{x}^\top A + y\mathbf{c}^\top = \rho \mathbf{x}^\top. \tag{2.6}$$

Using (2.4), we see that

$$\mathbf{x}^\top B + \mathbf{y}^\top D = \rho \mathbf{y}^\top. \tag{2.7}$$

Now we have by using (2.6) and (2.7) that

$$\begin{aligned} (\mathbf{x}^\top, \mathbf{y}^\top) \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= (\mathbf{x}^\top A + \mathbf{y}^\top C, \mathbf{x}^\top B + \mathbf{y}^\top D) \\ &= (\rho \mathbf{x}^\top, \rho \mathbf{y}^\top). \end{aligned}$$

This shows that $\rho, (\mathbf{x}^\top, \mathbf{y}^\top)$ is an eigenpair for $\tilde{M}$. This establishes that $\tilde{\rho} \geq \rho$.

Now, we show that $\tilde{\rho} = \rho$ by contradiction. Assume that $\tilde{\rho} > \rho$ and that we have eigenpair $\tilde{\rho}, (\tilde{\mathbf{x}}^\top, \tilde{\mathbf{y}}^\top)$ for $\tilde{M}$. Then, we have

$$\tilde{\mathbf{x}}^\top B = \tilde{\mathbf{y}}^\top (\tilde{\rho} I_s - D),$$

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which, after checking the sum of components on each side, yields

$$\tilde{\mathbf{x}}^\top \mathbf{b} = \left(\sum_{i=1}^{s} \tilde{y}_i\right)(\tilde{\rho} - d). \tag{2.8}$$

And of course, we also have that

$$\tilde{\mathbf{x}}^\top (\tilde{\rho} I_{n-1} - A) = \tilde{\mathbf{y}}^\top C = \left(\sum_{i=1}^{s} \tilde{y}_i\right)\mathbf{c}^\top. \tag{2.9}$$

Using (2.2) and (2.3) with $\tilde{\rho}, \tilde{\mathbf{x}}, \tilde{y} = \sum_{i=1}^{s} \tilde{y}_i$ instead of $\rho, \mathbf{x}, y$, together with (2.8) and (2.9) shows that $\tilde{\rho}, (\tilde{\mathbf{x}}^\top, \tilde{y})$ is an eigenpair for M which contradicts the fact that ρ is its spectral radius. Hence, $\tilde{\rho} = \rho$. ◻

This theorem applies to column sum expansions as well (since this is equivalent to row sum expansions on the transpose), and hence it applies to any sequence of row and column sum expansions, permutation similarity transforms, and transposes, since each of these operations preserves the spectral radius. This theorem only establishes the result for expanding a single diagonal component though; when expanding multiple components simultaneously, a row and column reduction argument easily establishes the result.

3. Spectral radius bounds via row sum contractions. If we were to perform the reverse procedure, contracting our matrix to lower its dimension, then unless the row or column sums in all blocks are constant or otherwise allowing the reverse procedure to proceed properly without affecting spectral radius (maybe this is achievable with some permutation similarity transform or transpose), we would have to break the preservation of spectral radius. However, we use this to derive bounds on the spectral radius.

Again, letting M be an $n \times n$ nonnegative square matrix, we define upward and downward row and column sum contractions. We use the standard ordering of matrices: $A \leq B$ means $A_{ij} \leq B_{ij}$ for all i, j.

Definition 3.1. *Let M be a nonnegative square matrix. A* downward row sum contraction *of M is denoted $M^{\downarrow}$ and is a matrix created as follows. First, we partition M into a block matrix with k^2 blocks denoted $M = (A_{ij})$ for i, j in $\{1, 2, ..., k\}$ requiring that all diagonal blocks A_{ii} are square matrices. In each block, we take the minimum row sum: if block A_{ij} has size $s \times t$, then $a_{ij} = \min_u\{\sum_{v=1}^{t} (A_{ij})_{uv}\}$. Then, we set $M^{\downarrow}_{ij} = a_{ij}$.*

Similarly, we define an *upward row sum contraction* of M denoted by $M^{\uparrow}$ which uses the maximum row sum in each block: $M^{\uparrow} = (b_{ij})$ where $b_{ij} = \max_u\{\sum_{v=1}^{t} (B_{ij})_{uv}\}$ (with blocks B_{ij} the blocks of M and square diagonal blocks still required). Naturally, we also define upward and downward column sum contractions (which are identical to applying row sum contractions to $M^\top$). We could also perform any sequence of permutation similarity tranforms, transposes, and row and column sum contractions to arrive at a new matrix of lower dimension.

Example 3.2. Consider the matrix M partitioned as follows:

$$M = \left[\begin{array}{c|ccc|cc} 1 & 2 & 0 & 3 & 7 & 1 \\ \hline 4 & 1 & 0 & 5 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 5 \\ 5 & 2 & 0 & 1 & 3 & 1 \\ \hline 0 & 0 & 3 & 5 & 0 & 2 \\ 4 & 1 & 3 & 0 & 1 & 3 \end{array}\right].$$

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Then taking the minimum and maximum row sum in each block gives downward and upward row sum contractions:

$$M^{\downarrow} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \quad M^{\uparrow} = \begin{bmatrix} 1 & 5 & 8 \\ 5 & 6 & 5 \\ 4 & 8 & 4 \end{bmatrix}.$$

Notice that before reducing the dimensions of the matrix, we can increase or decrease components of M in order to create an ordering. For this example, here is one possibility with the changed components in bold:

$$\tilde{M}^{\downarrow} = \left[\begin{array}{c|ccc|cc} 1 & 2 & 0 & 3 & 7 & 1 \\ \hline \mathbf{0} & 1 & 0 & \mathbf{2} & 1 & 0 \\ 0 & 2 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ \mathbf{0} & 2 & 0 & 1 & \mathbf{0} & 1 \\ \hline 0 & 0 & 3 & \mathbf{1} & 0 & 2 \\ \mathbf{0} & 1 & 3 & 0 & 1 & \mathbf{1} \end{array}\right] \leq \left[\begin{array}{c|ccc|cc} 1 & 2 & 0 & 3 & 7 & 1 \\ \hline 4 & 1 & 0 & 5 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 5 \\ 5 & 2 & 0 & 1 & 3 & 1 \\ \hline 0 & 0 & 3 & 5 & 0 & 2 \\ 4 & 1 & 3 & 0 & 1 & 3 \end{array}\right] \leq \left[\begin{array}{c|ccc|cc} 1 & 2 & 0 & 3 & 7 & 1 \\ \hline \mathbf{5} & 1 & 0 & 5 & \mathbf{5} & 0 \\ \mathbf{5} & 2 & 3 & \mathbf{1} & 0 & 5 \\ 5 & 2 & \mathbf{3} & 1 & 3 & \mathbf{2} \\ \hline \mathbf{4} & 0 & 3 & 5 & 0 & \mathbf{4} \\ 4 & 1 & 3 & \mathbf{4} & 1 & 3 \end{array}\right] = \tilde{M}^{\uparrow}.$$

Note that $\tilde{M}^{\downarrow}$ is a row sum expansion of $M^{\downarrow}$ (on multiple components) and similarly for $\tilde{M}^{\uparrow}$ and $M^{\uparrow}$. This requires understanding how to perform simultaneous row sum expansions on the diagonal of a principal submatrix, but this just means that each component of the corresponding principal submatrix is row-sum-expanded into an appropriately-sized block, and all other components follow the rules of Theorem 2.1.

It should be intuitively clear that this will always be possible, to arbitrarily decrease or increase some components to make all row sums in each block constant and equal to the minimum or maximum row sum in that block. It is important that we keep the diagonal blocks as square principle submatrices in order to use Theorem 2.1. As mentioned above already, a row sum expansion that does not expand diagonal elements into square blocks does not generally preserve the spectral radius. Hence, if our diagonal blocks are not square (after adjusting matrix entries up or down to get constant row sums in each block), then reducing the dimension can change the spectral radius, which we wish to avoid here.

It is a standard result for square nonnegative matrices that if $A \leq B$, then the spectral radii are also ordered $\rho(A) \leq \rho(B)$ (see Corollary 8.1.20 in [1]). Thus, we get

$$\rho(M^{\downarrow}) = \rho(\tilde{M}^{\downarrow}) \leq \rho(M) \leq \rho(\tilde{M}^{\uparrow}) = \rho(M^{\uparrow}),$$

for any arbitrary downward and upward adjustment of M to make constant row sums in each block. We formalize this as a theorem in the general case, for which the proof is now essentially illustrated in the previous example. This is somewhat related to the idea of using the Perron complement to derive spectral radius bounds (see [4], [2], [5]) which also involves dimension reduction, but our method is comparatively simple and does not require M to be irreducible.

Let $C_s^{\downarrow}(M)$ be the set of all $s \times s$ downward contractions of matrix M and $C_s^{\uparrow}(M)$ be the set of all $s \times s$ upward contractions. Each $M^{\downarrow} \in C_s^{\downarrow}(M)$ is an $s \times s$ matrix whose components are equal to the minimum row sum of each block for some partition of M, and similarly for any $M^{\uparrow} \in C_s^{\uparrow}(M)$. Note that these are distinct from $\tilde{M}^{\downarrow}$ and $\tilde{M}^{\uparrow}$ which have the same dimensions as M but with some components adjusted downward or upward, respectively, but resulting in the ordering: $\tilde{M}^{\downarrow} \leq M \leq \tilde{M}^{\uparrow}$. By Theorem 2.1, we have that $\rho(M^{\downarrow}) = \rho(\tilde{M}^{\downarrow})$ even though the matrices have different dimensions and also that $\rho(M^{\uparrow}) = \rho(\tilde{M}^{\uparrow})$.

Note that trivially, $[\min_i\{\sum_{j=1}^{n} M_{ij}\}] \in C_1^{\downarrow}(M)$ and $[\max_i\{\sum_{j=1}^{n} M_{ij}\}] \in C_1^{\uparrow}(M)$ for the minimal and maximal row sums of M viewed as 1×1 matrices and trivially that $M \in C_n^{\downarrow}(M) \cap C_n^{\uparrow}(M)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Let $C^{\downarrow}(M) = \cup_{j=1}^{n-1} C_j^{\downarrow}(M)$ and $C^{\uparrow}(M) = \cup_{j=1}^{n-1} C_j^{\uparrow}(M)$ be the sets of all possible downward and upward contractions of M (with dimensions less than M). It should be clear that each of these sets is finite. The process of creating a downward or upward contraction involves some finite sequence of permutation similarity transforms, transposes, and dimension decreases. Eventually, including more permutation similarity transforms or transposes creates no new matrices, and obviously the dimension can only be decreased to 1 at the lowest. Transpose and permutation similarity transforms also preserve the spectral radius, so it is only the increase or decrease of components to achieve constant row sums in each block (before performing dimension reductions) that can make the spectral radius smaller or larger. Note that a matrix in $C^{\downarrow}(M)$ may include several dimension decrease steps in its creation, which could mean the spectral radius undergoes several decreases relative to M, and similar for $C^{\uparrow}(M)$.

Theorem 3.3. *Let $C^{\downarrow}(M)$ and $C^{\uparrow}(M)$ be the set of all downward and upward contractions of nonnegative square matrix M. Then,*

$$\max_{M^{\downarrow} \in C^{\downarrow}(M)} \{\rho(M^{\downarrow})\} \leq \rho(M) \leq \min_{M^{\uparrow} \in C^{\uparrow}(M)} \{\rho(M^{\uparrow})\}.$$

Proof. For any $M^{\downarrow} \in C^{\downarrow}(M)$, we can create a matrix $\tilde{M}^{\downarrow}$ by adjusting individual components of M downward and have $\rho(M^{\downarrow}) = \rho(\tilde{M}^{\downarrow})$ by Theorem 2.1. We also have that $\tilde{M}^{\downarrow} \leq M$, and similarly for any $M^{\uparrow} \in C^{\uparrow}(M)$. So that we always have $\rho(M^{\downarrow}) = \rho(\tilde{M}^{\downarrow}) \leq \rho(M) \leq \rho(\tilde{M}^{\uparrow}) = \rho(M^{\uparrow})$. The result follows by taking the maximum on the left over all such downward contractions and minimum on the right over all such upward contractions. ∎

Corollary 3.4. *The standard bounds on $\rho(M)$ using its row sums is a special case of Theorem 3.3.*

Proof. We have that $[\min_i\{\sum_{j=1}^{n} M_{ij}\}] \in C_1^{\downarrow}(M)$ and $[\max_i\{\sum_{j=1}^{n} M_{ij}\}] \in C_1^{\uparrow}(M)$ which gives

$$\min_i \left\{ \sum_{j=1}^{n} M_{ij} \right\} \leq \rho(M) \leq \max_i \left\{ \sum_{j=1}^{n} M_{ij} \right\},$$

as a direct consequence of Theorem 3.3. ∎

We leave as an open question on whether Theorem 3.3 is guaranteed to give better bounds on $\rho(M)$ than the standard row sum bounds, though it does improve the bounds in at least some cases (some examples are given below).

Remark 3.5. Considering 2×2 row sum contractions, for which calculating the spectral radius is straightforward, gives an easy way of potentially refining the spectral radius bounds over the standard minimum and maximum row sums. Let A be a 2×2 downward row sum contraction of M, and let B be a 2×2 upward row sum contraction of M. Then,

$$\frac{\mathrm{tr}(A)}{2} + \sqrt{\left(\frac{\mathrm{tr}(A)}{2}\right)^2 - \det(A)} \leq \rho(M) \leq \frac{\mathrm{tr}(B)}{2} + \sqrt{\left(\frac{\mathrm{tr}(B)}{2}\right)^2 - \det(B)}.$$

One could also optimize over all possible 2×2 contractions as a general improvement over Corollary 3.4.

Again we call attention to the fact that we have required the diagonal blocks to be square submatrices as it is not true in general when the diagonal blocks are allowed to have arbitrary dimensions. In the general case, when not requiring diagonal blocks to be square, taking the minimum row sums of each block does not necessarily give a lower spectral radius. This is because we are relying on Theorem 2.1 to preserve the spectral radius when contracting the (downward and upward component-adjusted) matrices to a smaller size.

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Note that rather than performing contractions, one could perform expansions and then adjust elements up or down appropriately to get different bounds on the spectral radius, but generally calculating spectral radius for a larger matrix is more difficult or computationally intensive, so we do not state that result separately. The benefit of this method is in that it allows one to consider lower dimensional matrices.

Here is an example application of Remark 3.5. We give matrix M and the bounds on its spectral radius from row sums alone:

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 1 \\ 2 & 4 & 3 \end{bmatrix}, \quad 6 \leq \rho(M) \leq 9.$$

Now we row-sum-contract the upper left 2×2 block of M to produce downward and upward contractions with spectral radii given below:

$$M^{\downarrow} = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix},\ M^{\uparrow} = \begin{bmatrix} 6 & 2 \\ 6 & 3 \end{bmatrix} \Rightarrow \rho(M^{\downarrow}) = 6 \leq \rho(M) \leq \rho(M^{\uparrow}) = \frac{9}{2} + \frac{1}{2}\sqrt{57} \approx 8.3.$$

Thus, our method gives a slight refinement over considering the row sums of M alone. If we considered all possible 2×2 contractions of this matrix, the best bounds are approximately

$$6.3 \approx 2 + \sqrt{19} \leq \rho(M) \leq (9 + \sqrt{57})/2 \approx 8.3.$$

By considering column sums of M, we get $6 \leq \rho(M) \leq 8$ and by considering all 2×2 column sum contractions we get approximately $6.5 \approx (5 + \sqrt{65})/2 \leq \rho(M) \leq 8$, still a slight refinement over the basic method.

Here is an example where there is a slight refinement of the lower bound but a more significant refinement on the upper bound (relative to row or column sums alone):

$$M = \begin{bmatrix} \boxed{1} & 3 & \boxed{2} & 1 & \boxed{2} \\ 7 & \underline{1} & 1 & \underline{3} & 3 \\ \boxed{2} & 4 & \boxed{3} & 1 & \boxed{0} \\ 1 & \underline{1} & 5 & \underline{2} & 2 \\ \boxed{4} & 3 & \boxed{0} & 2 & \boxed{1} \end{bmatrix}, \quad 8 \leq \rho(M) \leq 15.$$

The best bounds are found with contracting the principle submatrices on indices $\{1, 3, 5\}$ (boxed above) and on $\{2, 4\}$ (underlined above) each into a single component (i.e. using index permutation $(1, 2, 3, 4, 5) \to (1, 3, 5, 2, 4)$ and partitioning into a 3×3 upper left diagonal block and 2×2 lower right diagonal block) which results in the downward and upward row sum contraction matrices:

$$M^{\downarrow} = \begin{bmatrix} 5 & 4 \\ 8 & 3 \end{bmatrix},\ M^{\uparrow} = \begin{bmatrix} 5 & 5 \\ 11 & 4 \end{bmatrix},$$

giving the bounds:

$$9.74 \approx 4 + \sqrt{33} = \rho(M^{\downarrow}) \leq \rho(M) \leq \rho(M^{\uparrow}) = \frac{9 + \sqrt{153}}{2} \approx 11.93,$$

with the actual value being approximately $\rho(M) \approx 10.995$.

Here is another straightforward corollary which discusses comparing the spectral radii of two matrices of arbitrarily dimensions.

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Corollary 3.6. *Let A and B be two square nonnegative matrices of arbitrary and possibly distinct dimension. If we can apply some sequence of permutation similarity transforms, transposes, and upward row or column sum contractions or expansions to A and also some sequence of these operations to B but with downward row or column sum contractions or expansions to create matrices satisfying $A^{\uparrow} \leq B^{\downarrow}$ (with both being of the same dimension) or more generally satisfying $\rho(A^{\uparrow}) \leq \rho(B^{\downarrow})$ (with neither $A^{\uparrow} \not\leq B^{\downarrow}$ nor their being of the same dimension required), then $\rho(A) \leq \rho(B)$.*

For example, consider square matrices of possibly distinct arbitrary dimensions written in block form with blocks of arbitrary dimensions (as long as the diagonal blocks are square submatrices):

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Let a_{ij} represent the maximum row sum of block A_{ij}, $A^{\uparrow} = (a_{ij})$, and b_{ij} the minimum row sum of block B_{ij}, $B^{\downarrow} = (b_{ij})$. If we have that

$$\frac{\operatorname{tr}(A^{\uparrow})}{2} + \sqrt{\left(\frac{\operatorname{tr}(A^{\uparrow})}{2}\right)^2 - \det(A^{\uparrow})} \leq \frac{\operatorname{tr}(B^{\downarrow})}{2} + \sqrt{\left(\frac{\operatorname{tr}(B^{\downarrow})}{2}\right)^2 - \det(B^{\downarrow})},$$

then $\rho(A) \leq \rho(B)$.

Example 3.7. In this example, we compare two matrices using Corollary 3.6. We define matrix A, perform a permutation similarity transform, and construct a 2×2 upward row sum contraction:

$$A = \left[\begin{array}{c|cc|c} \boxed{2} & 1 & 1 & \boxed{2} \\ \hline 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & 1 \\ \hline \boxed{1} & 2 & 0 & \boxed{4} \end{array}\right] \rightarrow \tilde{A} = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 1 & \boxed{2} & \boxed{2} \\ 2 & 0 & \boxed{1} & \boxed{4} \end{array}\right] \rightarrow A^{\uparrow} = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}.$$

The permutation similarity transform applied to A is indicated by the boxed entries, with indices permuted according to $(1, 2, 3, 4) \rightarrow (3, 1, 2, 4)$. Now, we define matrix B and perform a downward row sum contraction to it:

$$B = \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & 1 & 3 \\ \hline 1 & 1 & 5 \end{array}\right] \rightarrow B^{\downarrow} = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}.$$

In this case, it is clear that $\rho(A^{\uparrow}) = 6$ due to constant column sums and that $\rho(B^{\downarrow}) = 4 + \sqrt{5} > 6$ implying that $\rho(A) < \rho(B)$, even though there is no obvious relationship between their spectral radii by simply looking at row or column sums or comparing matrix components. Note that simply partitioning A without any permutation similarity transform does not give the same spectral radius comparison and thus is not useful.

REFERENCES

[1] R.A. Horn and C.R. Johnson. *Matrix Analysis*, 2nd edition. Cambridge, 2012.
[2] G.-X. Huang, F. Yin, and K. Guo. The lower and upper bounds on Perron root of nonnegative irreducible matrices. *Journal of Computational and Applied Mathematics*, 217(1):259–267, 2008.

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[3] H. Minc. *Nonnegative Matrices*, 1st edition. Wiley, 1988.

[4] S.-M. Yang and T.-Z. Huang. A note on estimates for the spectral radius of a nonnegative matrix. *The Electronic Journal of Linear Algebra*, 13, January 2005.

[5] Z.-M. Yang. Some closer bounds of Perron root basing on generalized Perron complement. *Journal of Computational and Applied Mathematics*, 235(1):315–324, 2010.