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K-SUBDIRECT SUMS OF NEKRASOV MATRICES*

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Abstract. In this paper, we give a sufficient and necessary condition for the subdirect sum of a Nekrasov matrix and a strictly diagonally dominant matrix being still a Nekrasov matrix. Adopting this sufficient and necessary condition, we present several simple sufficient conditions ensuring that the subdirect sum of Nekrasov matrices is in the same class. Examples are reported to illustrate the theoretical results.

Key words. Subdirect sum, Nekrasov matrices, Overlapping blocks.

AMS subject classifications. 15A06, 15A42, 15A48.

1. Introduction. Subdirect sums of matrices are generalizations of the usual sums of matrices, which have applications in several contexts such as matrix completion problems, overlapping subdomains in domain decomposition methods, global stiffness matrices in finite elements, etc., see [1, 2]. This concept was introduced by Fallat and Johnson in [3], where many of their properties were analyzed. They also showed that the subdirect sum of two H-matrices may not be an H-matrix.

Following the steps of Fallat and Johnson, Bru et al. [2] presented sufficient conditions guaranteeing that the k-subdirect sums of two nonsingular M-matrices are in the same class. Later, the subdirect sums of two matrices in a subclass of nonsingular H-matrix, for example, strictly diagonally dominant matrices, doubly diagonally dominant matrices, S-strictly diagonally dominant matrices, α_1-matrices, B-matrices and doubly B-matrices, Nekrasov matrices, have been studied extensively in [4, 5, 6, 7, 8, 9, 10, 11] and the references therein. In this paper, we are concerned with the subdirect sum of Nekrasov matrices.

Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$. Further, let A and B be partitioned into 2×2 blocks as follows.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1.1}$$

where A_{22} and B_{11} are square matrices of order k. The k-subdirect sum of A and B, denoted by $C = A \bigoplus_k B$, is defined to be

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix}.$$

Let $n = n_1 + n_2 - k$ and let us define the following set of indices:

$$S_1 = \{1, \ldots, n_1 - k\}, \quad S_2 = \{n_1 - k + 1, \ldots, n_1\}, \quad S_3 = \{n_1 + 1, \ldots, n\}. \tag{1.2}$$

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It is clear that

$$c_{ij} = \begin{cases} a_{ij}, & i \in S_1, j \in S_1 \cup S_2, \\ 0, & i \in S_1, j \in S_3, \\ a_{ij}, & i \in S_2, j \in S_1, \\ a_{ij} + b_{i-t,j-t}, & i \in S_2, j \in S_2, \\ b_{i-t,j-t}, & i \in S_2, j \in S_3, \\ 0, & i \in S_3, j \in S_1, \\ b_{i-t,j-t}, & i \in S_3, j \in S_2 \cup S_3, \end{cases} \tag{1.3}$$

where $t = n_1 - k$.

Nekrasov matrix is a well-known subclass of H-matrices; for more properties of Nekrasov matrix, see [12, 13, 14, 15]. Let us recall the definitions of strictly diagonally dominant matrix and Nekrasov matrix.

Definition 1.1. *A matrix A of order n is said to be a strictly diagonally dominant matrix if*

$$|a_{ii}| > r_i(A), \quad \forall\, i \in \{1, 2, \dots, n\}, \tag{1.4}$$

where $r_i(A) := \sum\limits_{j \neq i} |a_{ij}|$ *for* $i \in \{1, 2, \dots, n\}$.

Definition 1.2. *A matrix A of order n is said to be a Nekrasov matrix if*

$$|a_{ii}| > h_i(A), \quad \forall\, i \in \{1, 2, \dots, n\}, \tag{1.5}$$

where $h_1(A) = \sum\limits_{j=2}^{n} |a_{1j}|$, *and* $h_i(A) = \sum\limits_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum\limits_{j=i+1}^{n} |a_{ij}|$ *for* $i = \{2, 3, \dots, n\}$.

Throughout the paper, we always let $A = [a_{ij}]$ and $B = [b_{ij}]$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively, and let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$ which defines the sets S_1, S_2, S_3 in (1.2).

2. Main results. In this section, we first give a sufficient and necessary condition for the subdirect sum of a Nekrasov matrix and a strictly diagonally dominant matrix being still a Nekrasov matrix. Then we present several simple sufficient conditions ensuring that the subdirect sum of Nekrasov matrices is in the same class. Let us recall the existing results presented in [10] and [11].

Theorem 2.1 ([10]). *Let A be a Nekrasov matrix and let B be strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), and*

$$a_{i+n_1-k, j+n_1-k} = 0 \quad \text{for } i, j \in \{1, 2, \dots, k\} \text{ and } i > j, \tag{2.1}$$

then $A \bigoplus_k B$ is a Nekrasov matrix.

Theorem 2.2 ([11]). *Let A be a Nekrasov matrix and let B be strictly diagonally dominant. If the diagonal entries of A_{22} and B_{11} have the same sign pattern, and*

$$\frac{h_{n_1-k+i}(A)}{|a_{n_1-k+i, n_1-k+i}|} \geq \frac{r_i(B)}{|b_{ii}|} \quad \text{for all } i \in \{1, 2, \dots, k\}, \tag{2.2}$$

then $A \bigoplus_k B$ is a Nekrasov matrix.

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THEOREM 2.3 ([11]). *Let A and B be Nekrasov matrices. If the diagonal entries of A_{22} and B_{11} have the same sign pattern, $A_{21} = O$, and*

$$|a_{i+n_1-k,j+n_1-k} + b_{i,j}| \leq |b_{i,j}| \quad \text{for } i,j \in \{1,2,\dots,k\} \text{ and } i > j, \tag{2.3}$$

then $A \bigoplus_k B$ is a Nekrasov matrix.

Remark that Theorem 2.2 still holds when the condition $\frac{h_{n_1}(A)}{|a_{n_1n_1}|} \geq \frac{r_k(B)}{|b_{kk}|}$ is omitted. Hence, $A \bigoplus_1 B$ is always a Nekrasov matrix, which is exactly the result of [10, Theorem 1]. It is clear that the conditions ' the diagonal entries of A_{22} and B_{11} have the same sign pattern' and 'all diagonal entries of A_{22} and B_{11} are positive (or all negative)' could be extended to the condition as follows.

$$|a_{ii} + b_{i-n_1+k,i-n_1+k}| = |a_{ii}| + |b_{i-n_1+k,i-n_1+k}| \quad for\ i \in S_2. \tag{2.4}$$

Now we present a sufficient and necessary condition for the subdirect sum of a Nekrasov matrix and a diagonally dominant matrix being still a Nekrasov matrix.

THEOREM 2.4. *Let A be a Nekrasov matrix and let B be strictly diagonally dominant satisfying (2.4). Then $C = A \bigoplus_k B$ is a Nekrasov matrix if and only if*

$$|a_{n_1-k+i,n_1-k+i}| + |b_{ii}| > h_{n_1-k+i}(A \oplus_k B) \quad for\ all\ i \in \{2,3,\dots,k\}, \tag{2.5}$$

Proof. The necessity part is clear. It is sufficient to prove that for $i \in S_1 \cup S_2 \cup S_3$, we have

$$h_i(C) < |c_{ii}|. \tag{2.6}$$

From the structure of C, $c_{ii} = a_{ii}$ and $h_i(C) = h_i(A)$ for $i \in S_1$. Since A is a Nekrasov matrix, (2.6) holds for $i \in S_1$. For $i = n_1 - k + 1$, we have

$$\begin{aligned}
h_i(C) &= \sum_{j=1}^{n_1-k} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+2}^{n} |c_{ij}| \\
&= \sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=n_1-k+2}^{n_1} |a_{ij} + b_{1,j-n_1+k}| + \sum_{j=n_1+1}^{n} |b_{1,j-n_1+k}| \\
&\leq \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| + \sum_{j\neq 1} |b_{1j}| \\
&= h_i(A) + r_1(B) < |a_{ii}| + |b_{11}|.
\end{aligned}$$

The last inequality is due to A is a Nekrasov matrix and B is strictly diagonally dominant. Combining this with (2.5), we obtain that (2.6) holds for $i \in S_2$.

Recalling that $c_{ij} = 0$ for $i \in S_3$ and $j \in S_1$; $c_{ij} = b_{i-n_1+k,j-n_1+k}$ for $i \in S_3$ and $j \in S_2 \cup S_3$. For $i = n_1 + 1$, since B is strictly diagonally dominant, we have

$$\begin{aligned}
\frac{h_i(C)}{|c_{ii}|} &= \frac{\sum_{j=1}^{n_1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^{n} |c_{ij}|}{|c_{ii}|} \\
&= \frac{\sum_{j=1}^{k} |b_{k+1,j}| \frac{h_{j+n_1-k}(C)}{|c_{j+n_1-k,j+n_1-k}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}|}{|b_{k+1,k+1}|}
\end{aligned}$$

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$$\leq \frac{\sum_{j=1}^{k}|b_{k+1,j}|+\sum_{j=k+2}^{n_2}|b_{k+1,j}|}{|b_{k+1,k+1}|}$$

$$\leq \frac{r_{k+1}(B)}{|b_{k+1,k+1}|}<1.$$

We assume that (2.6) holds for $i < n_1+t \leq n$. Now consider the case $i = n_1+t$.

$$\begin{aligned}
\frac{R_i(C)}{|c_{ii}|} &= \frac{\sum_{j=1}^{n_1}|c_{ij}|\frac{h_j(C)}{|c_{jj}|}+\sum_{j=i+1}^{n}|c_{ij}|}{|c_{ii}|}\\
&\leq \frac{\sum_{j=1}^{k+t-1}|b_{k+t,j}|\frac{h_{j+n_1-k}(C)}{|c_{j+n_1-k,j+n_1-k}|}+\sum_{j=k+t+1}^{n_2}|b_{k+t,j}|}{|b_{k+t,k+t}|}\\
&\leq \frac{\sum_{j=1}^{k+t-1}|b_{k+t,j}|+\sum_{j=k+t+1}^{n_2}|b_{k+t,j}|}{|b_{k+t,k+t}|}\\
&\leq \frac{r_{k+t}(B)}{|b_{k+t,k+t}|}<1.
\end{aligned}$$

Hence, (2.6) holds for $i \in S_3$. Therefore, C is a Nekrasov matrix. This completes the proof. ∎

It is hard to check out (2.5) in general. Hence, in the following, we use Theorem 2.4 to present some simple sufficient conditions.

THEOREM 2.5. *Let A be a Nekrasov matrix and let B be strictly diagonally dominant satisfying* (2.4). *If*

$$|a_{i+n_1-k,j+n_1-k}+b_{ij}| \leq |b_{ij}| \quad \text{for } i,j \in \{1,2,\ldots,k\} \text{ and } i>j, \tag{2.7}$$

then $C = A \bigoplus_k B$ is a Nekrasov matrix.

Proof. We can easily get that $h_i(C) = h_i(A)$ for $i \in S_1$ by induction. For $i = n_1-k+1$, we have

$$\begin{aligned}
h_i(C) &= \sum_{j=1}^{n_1-k}|c_{ij}|\frac{h_j(C)}{|c_{jj}|}+\sum_{j=n_1-k+2}^{n}|c_{ij}|\\
&\leq \sum_{j=1}^{n_1-k}|a_{ij}|\frac{h_j(A)}{|a_{jj}|}+\sum_{j=n_1-k+2}^{n_1}|a_{ij}+b_{1,j-n_1+k}|+\sum_{j=n_1+1}^{n}|b_{1,j-n_1+k}|\\
&\leq \sum_{j=1}^{i-1}|a_{ij}|\frac{h_j(A)}{|a_{jj}|}+\sum_{j=i+1}^{n_1}|a_{ij}|+\sum_{j\neq 1}|b_{1j}|\\
&= h_i(A)+r_1(B) < |a_{ii}|+|b_{11}|.
\end{aligned}$$

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The last inequality is due to A is a Nekrasov matrix and B is strictly diagonally dominant. Assume that (2.5) holds for $i < n_1 - k + t$ with $2 \le t < k$. Now consider $i = n_1 - k + t$. We have

$$\begin{aligned}
h_i(C) &= \sum_{j=1}^{n_1-k} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^{n} |c_{ij}| \\
&\le \sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=n_1-k+1}^{i-1} |a_{ij} + b_{t,j-n_1+k}| + \sum_{j=i+1}^{n_1} |a_{ij}| + \sum_{j=i+1}^{n} |b_{t,j-n_1+k}| \\
&\le \sum_{j=1}^{n_1-k} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=n_1-k+1}^{i-1} |b_{t,j-n_1+k}| + \sum_{j=i+1}^{n_1} |a_{ij}| + \sum_{j=i+1}^{n} |b_{t,j-n_1+k}| \\
&\le \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| + \sum_{j \ne t} |b_{tj}| \\
&= h_i(A) + r_t(B) < |a_{ii}| + |b_{tt}|.
\end{aligned}$$

Hence, (2.5) holds for $i \in S_2$. By Theorem 2.4, C is a Nekrasov matrix. This completes the proof. ∎

Remark 2.6. Since (2.1) leads to (2.5), the condition of Theorem 2.5 is weaker than the condition of Theorem 2.1.

Example 2.7. Let $k = 3$. We consider the following matrices:

$$A = \begin{bmatrix} 8 & 1 & 1 & 2 \\ 4 & 10 & 2 & 5 \\ 0 & -20 & 19 & 0 \\ 2 & 10 & 0 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 20 & 4 & 5 & 10 \\ 10 & 19 & 7 & 1 \\ -5 & 2 & 12 & 4 \\ 3 & 10 & 6 & 20 \end{bmatrix}.$$

Since A and B satisfy (2.5), $A \bigoplus_3 B$ is a Nekrasov matrix by Theorem 2.5. However, Theorem 2.1 is not valid since $a_{32} \neq 0$; Theorem 2.2 is not valid since $h_2(A)/|a_{22}| = 9/10 < 19/20 = r_1(B)/|b_{11}|$; Theorem 2.3 is not valid since $A_{21} \neq O$. We will see that Theorem 2.8 cannot verify $A \bigoplus_3 B$ is a Nekrasov matrix or not because A does not satisfy (2.8).

To present a new result, we need the following notations. Given a matrix A of order n and two positive integers i, t, denote by

$$l_{i,t}(A) = \sum_{j=1}^{t} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} \text{ and } u_{i,t}(A) = \sum_{j=t+1, j \neq i}^{n} |a_{ij}|.$$

Theorem 2.8. *Let A be a Nekrasov matrix and let B be strictly diagonally dominant satisfying (2.4). If*

$$|a_{n_1-k+i,n_1-k+i}| \ge l_{n_1-k+i,n_1-k}(A) + u_{n_1-k+i,n_1-k}(A) \quad \text{for all } i \in \{2, \ldots, k\}, \tag{2.8}$$

then $C = A \bigoplus_k B$ is a Nekrasov matrix.

Proof. It is sufficient to prove (2.5). Since A is a Nekrasov matrix and B is a strictly diagonally dominant matrix, we have

$$\frac{h_{n_1-k+1}(C)}{|c_{n_1-k+1,n_1-k+1}|} = \frac{h_{n_1-k+1}(A) + r_1(B)}{|a_{n_1-k+1,n_1-k+1}| + |b_{11}|} < 1.$$

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Assume that $h_i(C)/|c_{ii}| \leq 1$ for $i = n_1 - k + 1, n_1 - k + 2, \ldots, n_1 - k + t$, where $t \leq k - 1$. Now consider the case $i = n_1 - k + t + 1$. Let $s = n_1 - k$. We have

$$\begin{aligned}
\frac{h_i(C)}{|c_{ii}|} &= \frac{l_{i,s}(C) + \sum\limits_{j=s+1}^{s+t} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum\limits_{j=i+1}^{n} |c_{ij}|}{|b_{t+1,t+1}|} \\
&\leq \frac{l_{i,s}(A) + \sum\limits_{j=s+1}^{s+t} |c_{ij}| + \sum\limits_{j=i+1}^{n} |c_{ij}|}{|b_{t+1,t+1}|} \\
&\leq \frac{l_{i,s}(A) + \sum\limits_{j=s+1}^{s+t} (|a_{ij}| + |b_{t+1,j-s}|) + \sum\limits_{j=i+1}^{n} |c_{ij}|}{|b_{t+1,t+1}|} \\
&\leq \frac{l_{i,s}(A) + \sum\limits_{j=s+1}^{s+t} (|a_{ij}| + |b_{t+1,j-s}|) + \sum\limits_{j=i+1}^{n_1} |a_{ij}| + \sum\limits_{j=t+2}^{n} |b_{t+1,j}|}{|a_{i,i}| + |b_{t+1,t+1}|} \\
&= \frac{l_{i,s}(A) + u_{i,s}(A) + r_{t+1}(B)}{|a_{i,i}| + |b_{t+1,t+1}|} \\
&< 1 \quad \text{(by (2.8) and B is strictly diagonally dominant).}
\end{aligned}$$

Hence, we obtain (2.5). By Theorem 2.4, C is a Nekrasov matrix. This completes the proof. ∎

Remark 2.9. If A satisfies (2.8), then the k-subdirect sum of A and every strictly diagonally dominant matrix is a Nekrasov matrix.

Example 2.10. Let $k = 3$. We consider the following matrices:

$$A = \begin{bmatrix} 8 & 1 & 1 & 2 \\ 4 & 10 & 5 & 2 \\ 0 & 10 & 19 & 8 \\ 2 & 10 & 0 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 20 & 4 & 5 & 10 \\ 1 & 19 & 7 & 10 \\ 0 & 2 & 12 & 4 \\ 3 & 10 & 6 & 20 \end{bmatrix}.$$

Since A and B satisfy (2.8), $A \bigoplus_3 B$ is a Nekrasov matrix by Theorem 2.8. However, Theorem 2.1 is not valid since $a_{32} \neq 0$; Theorem 2.2 is not valid since $h_2(A)/|a_{22}| = 9/10 < 19/20 = r_1(B)/|b_{11}|$; Theorem 2.3 is not valid since $A_{21} \neq O$; Theorem 2.5 is not valid because $|a_{32} + b_{21}| = 11 > 1 = |b_{21}|$.

We close this section with a simple sufficient condition ensuring that the k-subdirect sum of two Nekrasov matrices is still a Nekrasov matrix.

Theorem 2.11. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be Nekrasov matrices satisfying (2.4). If*

$$\frac{h_{n_1-k+i}(A)}{|a_{n_1-k+i,n_1-k+i}|} = \frac{h_i(B)}{|b_{ii}|} \quad \forall i \in \{1, 2, \ldots, k-1\}, \tag{2.9}$$

and

$$\frac{h_{n_1}(A)}{|a_{n_1,n_1}|} \leq \frac{h_k(B)}{|b_{kk}|}, \tag{2.10}$$

then $C = A \bigoplus_k B$ is a Nekrasov matrix.

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Proof. Since A and B are two Nekrasov matrices, it is sufficient to prove that

$$\frac{h_i(C)}{|c_{ii}|}\begin{cases} = \frac{h_i(A)}{|a_{ii}|}, & \text{if } i \in S_1; \\ \leq \frac{h_{i-n_1+k}(B)}{|b_{i-n_1+k,i-n_1+k}|}, & \text{if } i \in S_2 \cup S_3. \end{cases} \tag{2.11}$$

It is clear that $h_i(C)/|c_{ii}| = h_i(A)/|a_{ii}| < 1$ for $i \in S_1$. For $i = n_1 - k + 1$, by (2.9) we have

$$\frac{h_i(C)}{|c_{ii}|} = \frac{h_i(A) + h_1(B)}{|a_{ii}| + |b_{11}|} = \frac{h_1(B)}{|b_{11}|}.$$

Assume that (2.11) holds for all $i < n_1 - k + t \leq n_1$. Consider $i = n_1 - k + t$.

$$\begin{aligned}
\frac{h_i(C)}{|c_{ii}|} &= \frac{l_{i,n_1-k}(C) + \sum\limits_{j=n_1-k+1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum\limits_{j=i+1}^{n} |c_{ij}|}{|a_{ii} + b_{tt}|} \\
&\leq \frac{l_{i,n_1-k}(A) + \sum\limits_{j=n_1-k+1}^{i-1} (|a_{ij}| + |b_{t,j-n_1+k}|) \frac{h_j(A)}{|a_{jj}|} + \sum\limits_{j=i+1}^{n_1} |a_{ij}| + \sum\limits_{j=t+1}^{n_2} |b_{tj}|}{|a_{ii}| + |b_{tt}|} \\
&= \frac{l_{i,n_1-k}(A) + \sum\limits_{j=n_1-k+1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum\limits_{j=i+1}^{n_1} |a_{ij}| + \sum\limits_{j=1}^{t-1} |b_{tj}| \frac{h_j(B)}{|b_{jj}|} + \sum\limits_{j=t+1}^{n_2} |b_{tj}|}{|a_{ii}| + |b_{tt}|} \\
&= \frac{h_i(A) + h_t(B)}{|a_{ii}| + |b_{tt}|} \leq \frac{h_t(B)}{|b_{tt}|}.
\end{aligned}$$

Remark that (2.10) ensures the last inequality holds for $i = n_1$. Hence, (2.11) holds for $i \in S_2$.

For $i = n_1 + 1$, we have

$$\begin{aligned}
\frac{h_i(C)}{|c_{ii}|} &\leq \frac{\sum\limits_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum\limits_{j=i+1}^{n} |c_{ij}|}{|c_{ii}|} \\
&\leq \frac{\sum\limits_{j=1}^{k} |b_{k+1,j}| \frac{h_{j+n_1-k}(C)}{|c_{j+n_1-k,j+n_1-k}|} + \sum\limits_{j=k+2}^{n_2} |b_{k+1,j}|}{|b_{k+1,k+1}|} \\
&\leq \frac{\sum\limits_{j=1}^{k} |b_{k+1,j}| \frac{h_j(B)}{|b_{jj}|} + \sum\limits_{j=k+2}^{n_2} |b_{k+1,j}|}{|b_{k+1,k+1}|} \\
&= \frac{h_{k+1}(B)}{|b_{k+1,k+1}|} < 1.
\end{aligned}$$

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We assume that (2.11) holds for $i < n_1 + t \leq n$. Now consider the case $i = n_1 + t$.

$$\begin{aligned}
\frac{h_i(C)}{|c_{ii}|} &\leq \frac{\sum_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^{n} |c_{ij}|}{|c_{ii}|} \\
&\leq \frac{\sum_{j=1}^{k+t-1} |b_{k+t,j}| \frac{h_{j+n_1-k}(C)}{|c_{j+n_1-k,j+n_1-k}|} + \sum_{j=k+t+1}^{n_2} |b_{k+t,j}|}{|b_{k+t,k+t}|} \\
&\leq \frac{\sum_{j=1}^{k+t-1} |b_{k+t,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+t+1}^{n_2} |b_{k+t,j}|}{|b_{k+t,k+t}|} \\
&= \frac{h_{k+t}(B)}{|b_{k+t,k+t}|} < 1.
\end{aligned}$$

Hence, (2.11) holds for $i \in S_3$. This completes the proof. ◻

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