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A SYLVESTER–KAC MATRIX TYPE AND THE LAPLACIAN CONTROLLABILITY OF HALF GRAPHS*

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Abstract. In this paper, we provide a new family of tridiagonal matrices whose eigenvalues are perfect squares. This result motivates the computation of the spectrum of a particular antibidiagonal matrix. As an application, we consider the Laplacian controllability of a particular subclass of chain graphs known as half graphs.

Key words. Tridiagonal matrix, Sylvester–Kac matrix, Chain graph, Controllable graph.

AMS subject classifications. 15A15, 15A18, 05C50, 93B05.

1. Introduction. The first occurrence of the so-called Sylvester–Kac matrix type dates back to 1854 when J.J. Sylvester, in the brief note [35], conjectured that the eigenvalues of

$$\begin{bmatrix} 0 & 1 & & & & \\ n & 0 & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & 0 & n \\ & & & & 1 & 0 \end{bmatrix},$$

are:

$$\pm\, n, \pm\,(n-2), \pm\,(n-4), \ldots .$$

A complete proof to this claim is given by M. Kac in his celebrated work [23], almost a century after the original formulation. For distinct proofs, approaches, and some historical remarks, the reader is referred to [4, 6–10, 12, 14–19, 22, 24, 25, 27, 30, 32, 34, 36]. Nowadays, a *Sylvester–Kac type matrix* is a tridiagonal integral matrix with integer spectrum satisfying a certain regularity property. In some instances, the integral condition is dropped.

A straightforward computation shows that the matrix $A_{n+1} = [a_{ij}]$ of order $n+1$ given by

$$\begin{aligned} a_{kk} &= (a+(m+k-1)c)(a+(m+n-k+1)c) + (k-1)\,(n-k+2)\,c^2 \\ a_{k,k+1} &= -c(n-k+1)(a+(m+k-1)c) \\ a_{k+1,k} &= -kc(a+(m+n-k+1)c) \end{aligned} \tag{1.1}$$

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Electronic Journal of Linear Algebra, ISSN 1081-3810
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is the square of the matrix

$$B_{n+1} = \begin{bmatrix} & & & & (a+mc) \\ & & & (a+(m+1)\,c) & -c \\ & & (a+(m+2)\,c) & -2c & \\ & \iddots & \iddots & & \\ (a+(m+n)c) & -nc & & & \end{bmatrix}_{(n+1)\times(n+1)},$$

that is, $A_{n+1} = B_{n+1}^2$. We consider the initial matrices of order $n+1$.

Example 1.1. If we take $n = 4, m = 0$ and $c = 1$, then the matrix A_5 is of the form

$$A_5 = \begin{bmatrix} a\,(a+4) & -4a & 0 & 0 & 0 \\ -a-4 & (a+1)\,(a+3)+4 & -3a-3 & 0 & 0 \\ 0 & -2a-6 & (a+2)^2+6 & -2a-4 & 0 \\ 0 & 0 & -3a-6 & (a+1)\,(a+3)+6 & -a-3 \\ 0 & 0 & 0 & -4a-4 & a\,(a+4)+4 \end{bmatrix}.$$

Our first aim is to show that, for $a, m, c \geq 0$, the spectrum of A_{n+1}, $\sigma(A_{n+1})$, is

$$\left\{(a+mc)^2,\ (a+(m+1)\,c)^2,\ (a+(m+2)\,c)^2, \ldots, (a+(m+n)\,c)^2\right\}.$$

Afterwards, we prove that

$$\sigma\left(B_{n+1}\right) = \left\{a+mc, -(a+(m+1)c),\ a+(m+2)\,c, \ldots, (-1)^n\,(a+(m+n)\,c)\right\}.$$

As a consequence, in Section 3, we prove that the eigenvalues of the antibidiagonal matrix

$$\tilde{B}_n = \begin{bmatrix} & & & & & 1 \\ & & & & 2 & -1 \\ & & & 3 & -2 & \\ & & 4 & -3 & & \\ & \iddots & \iddots & & & \\ n & -(n-1) & & & & \end{bmatrix},$$

are

$$1, -2, 3, -4, \ldots, (-1)^{n-1}n,$$

taking into account that the spectrum of

$$\tilde{B}_n^2 = \begin{bmatrix} n & -(n-1) & & & & \\ -n & 3n-3 & -2(n-2) & & & \\ & -2(n-1) & \ddots & \ddots & & \\ & & \ddots & \ddots & -2(n-2) & \\ & & & -3(n-2) & 4n-6 & -(n-1) \\ & & & & -2(n-1) & 2n-1 \end{bmatrix},$$

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contains the first n nonzero integer squares (each one with multiplicity 1). According to its spectral structure, we may say that $\tilde{B}_n^2$ is of a Sylvester–Kac type. In Section 3, we show that a certain type of antibidiagonal matrices has eigenvectors with nonzero entries in specific positions, the essential ingredient in the study of controllability of dynamical leader–follower systems. As an application, a relation with particular chain graphs and their Laplacian controllability is established in Section 4, representing the second goal of this work.

2. The spectrum of A_{n+1}. In this section, we will prove the claim in the introduction about eigenvalues of the matrix A_{n+1} as defined in (1.1).

Let $\mathbf{v} \in \mathbb{R}^{n+1}$ be defined by

$$\mathbf{v} \;=\; \left(\binom{n}{0},\quad -\binom{n}{1},\quad \binom{n}{2},\quad -\binom{n}{3},\quad \ldots,\quad (-1)^n\binom{n}{n}\right)^{\intercal}.$$

Then, for $\lambda = (a + (m+n)\,c)^2$, we have

$$\mathbf{v}A_{n+1} = \lambda\mathbf{v},$$

which means that $\mathbf{v}$ is a left-eigenvector of A corresponding to the eigenvalue λ.

Define the matrix T as

$$T = \begin{bmatrix} \binom{n}{0} & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \cdots & (-1)^n\binom{n}{n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & 0 & & 0 \\ & & 0 & 1 & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix}.$$

Its inverse follows easily:

$$T^{-1} = \begin{bmatrix} \binom{n}{0} & \binom{n}{1} & -\binom{n}{2} & \binom{n}{3} & \cdots & (-1)^{n+1}\binom{n}{n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & 0 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix}.$$

Then,

$$TA_{n+1}T^{-1} = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline -c\sqrt{\lambda} & & & \\ 0 & & & \\ \vdots & & W & \\ 0 & & & \end{array}\right],$$

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where the $n \times n$ matrix $W = [w_{ij}]$ is given by

$$\begin{aligned}
w_{11} &= (a + c\,(m - n + 1))\,(a + c\,(m + n - 1))\,,\\
w_{kk} &= (a\,(m + k)\,c)\,(a + (m + n)\,c) + k\,(n - k + 1)\,c^2, \quad \text{for} \quad 2 \le k \le n,\\
w_{12} &= \frac{1}{2}c\,(n-1)\left(-2\,(a + c) + an + c\left(n^2 - 2m + mn\right)\right),\\
w_{k,k+1} &= -c\,(n - k)\,(a + (m + k)\,c)\,, \quad \text{for} \quad 2 \le k \le n - 1,\\
w_{k+1,k} &= -c\,(k + 1)\,(a + (m + n - k)\,c)\,, \quad \text{for} \quad 1 \le k \le n - 1.
\end{aligned}$$

For the bidiagonal matrix U of order n,

$$U = \frac{1}{n}\begin{bmatrix}
1 & n-1 & & & & & \\
 & 2 & n-2 & & & & \\
 & & 3 & \ddots & & & \\
 & & & \ddots & 3 & & \\
 & & & & n-2 & 2 & \\
 & & & & & n-1 & 1 \\
 & & & & & & n
\end{bmatrix},$$

the inverse is given by

$$U^{-1} = \begin{bmatrix}
\binom{n}{1}/\binom{n-1}{0} & -\binom{n}{2}/\binom{n-1}{0} & \binom{n}{3}/\binom{n-1}{0} & -\binom{n}{4}/\binom{n-1}{0} & \cdots & -\binom{n}{n-1}/\binom{n-1}{0} & \binom{n}{n}/\binom{n-1}{0} \\
 & \binom{n}{2}/\binom{n-1}{1} & -\binom{n}{3}/\binom{n-1}{1} & \binom{n}{4}/\binom{n-1}{1} & \cdots & \binom{n}{n-1}/\binom{n-1}{1} & -\binom{n}{n}/\binom{n-1}{1} \\
 & & \binom{n}{3}/\binom{n-1}{2} & -\binom{n}{4}/\binom{n-1}{2} & \cdots & -\binom{n}{n-1}/\binom{n-1}{2} & \binom{n}{n}/\binom{n-1}{2} \\
 & & & \ddots & \ddots & \vdots & \vdots \\
 & & & & & \binom{n}{n-1}/\binom{n-1}{n-2} & -\binom{n}{n}/\binom{n-1}{n-2} \\
 & & & & & & \binom{n}{n}/\binom{n-1}{n-1}
\end{bmatrix}.$$

Since

$$U^{-1}WU = A_n,$$

for $\lambda = (a + (m + n)c)^2$, it follows

$$TA_{n+1}T^{-1} = \left[\begin{array}{c|ccc}
\lambda & 0 & \cdots & 0 \\
\hline
-c\sqrt{\lambda} & & & \\
0 & & & \\
\vdots & & UA_nU^{-1} & \\
0 & & &
\end{array}\right],$$

which proves our main claim. Therefore, we may conclude the next result.

THEOREM 2.1. *The eigenvalues of the matrix A_{n+1} defined in (1.1) are*

$$(a + (m + k)c)^2\,, \quad \textit{for} \quad 0 \le k \le n\,.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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As a consequence, we have

$$\det(A_{n+1}) = \lambda \det(A_n) = (a + (m+n)\,c)^2 \det(A_n),$$

and therefore

$$\det(A_{n+1}) = \prod_{k=0}^{n} (a + (m+k)\,c)^2 .$$

3. Two antibidiagonal matrices. In this section, we consider the matrix B_{n+1}, for $a, m, c \geq 0$ and obtain its spectrum. We start by noticing that if λ^2 is an eigenvalue of A_n, then either λ or $-\lambda$ is an eigenvalue of B_n. Henceforth, our task is to determine the sign of λ.

Let P_σ be the permutation matrix for the permutation

$$\sigma = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & \frac{n+1}{2} & \frac{n+1}{2}+1 & \cdots & n & n+1 \\ 1 & 3 & \cdots & n & n+1 & \cdots & 4 & 2 \end{pmatrix} & \text{if } n \text{ is odd;} \\ \begin{pmatrix} 1 & 2 & \cdots & \lceil\frac{n+1}{2}\rceil & \lceil\frac{n+1}{2}\rceil+1 & \cdots & n & n+1 \\ 1 & 3 & \cdots & n+1 & n & \cdots & 4 & 2 \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Set $\ell = \lfloor \frac{n+2}{2} \rfloor$. Then for $b_k = a + (m+k)c,\ 0 \leq k \leq n$,

$$P_\sigma^t B_{n+1} P_\sigma = \begin{bmatrix} 0 & b_0 & & & & & & & \\ b_n & 0 & -nc & & & & & & \\ & -c & 0 & b_1 & & & & & \\ & & b_{n-1} & 0 & -(n-1)c & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & & 0 & b_{\ell-1} & \\ & & & & & b_\ell & 0 & -\ell c \\ & & & & & & -(\ell-1)c & \ell c \end{bmatrix},$$

for n even and

$$P_\sigma^t B_{n+1} P_\sigma = \begin{bmatrix} 0 & b_0 & & & & & & & \\ b_n & 0 & -nc & & & & & & \\ & -c & 0 & b_1 & & & & & \\ & & b_{n-1} & 0 & -(n-1)c & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & & 0 & -(\ell+1)c & \\ & & & & & -(\ell-1)c & 0 & b_{\ell-1} \\ & & & & & & b_\ell & -\ell c \end{bmatrix},$$

if n is odd. For a more general setting, the reader is referred to [13]. It is well-known that the previous matrices can be symmetrized by taking off diagonal entries $(i, i+1)$ and $(i+1, i)$ both equal to $\sqrt{(P_\sigma^t B_{n+1} P_\sigma)_{i,i+1} (P_\sigma^t B_{n+1} P_\sigma)_{i+1,1}}$, $1 \leq i \leq n+1$ (see [33]).

- According to [20, Corollary 2], if n is even, the spectrum of any tridiagonal matrix of the first form is a real $(n+1)$-tuple $(\mu_1, \mu_2, \dots, \mu_{n+1})$ such that $\mu_1 > -\mu_2 > \cdots > (-1)^n \mu_{n+1} \geq 0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 559-571, September 2022.

- For n odd, $(P_\sigma^t B_{n+1} P_\sigma)_{n+1,n+1} = -\ell c$. In this case, we apply the result of [20] to $-P_\sigma^t B_{n+1} P_\sigma$. Therefore its eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n+1}$ satisfy $-\mu_1 > \mu_2 > \cdots > (-1)^{n+1}\mu_{n+1} \geq 0$.

As we observed previously, there are two possibilities for the eigenvalue λ_k: either $a + (m+k)\,c$ or $-\left(a + (m+k)\,c\right)$. The eigenvalue with the least module is $a + mc$. Taking into account that as the modules of the remaining eigenvalues increase their signs alternate, we arrive to desired conclusion:

$$\sigma\left(B_{n+1}\right) = \left\{(-1)^k \left(a + (m+k)\,c\right) \;:\; 0 \leq k \leq n\right\}.$$

If we now set $a = m = 0$ and $c = 1$ in B_{n+1}, we immediately obtain the spectrum of $\tilde{B}_{n+1}$, namely,

$$\sigma(\tilde{B}_{n+1}) = \{1, -2, 3, -4, \ldots, (-1)^n(n+1)\}.$$

In the sequel, we show that for a certain type of antibidiagonal matrices some eigenvectors' entries are always nonzero.

Theorem 3.1. *Let*

$$B = \begin{bmatrix} & & & & b_n \\ & & & b_{n-1} & -b_n \\ & & & & \\ & & \iddots & \iddots & \\ & & & & \\ & b_2 & & & \\ b_1 & -b_2 & & & \end{bmatrix},$$

where $b_i \neq 0$ for any $i \in \{1, 2, \ldots, n\}$. Then the $\lceil \frac{n+1}{2} \rceil$th entry of any eigenvector of B is nonzero.

Proof. Suppose on the contrary, that there exists an eigenvector $\mathbf{x}$ of B for the eigenvalue μ, such that $x_{\lceil \frac{n+1}{2} \rceil} = 0$.

If $n = 2k$, then we first consider the $(k+1)$th equation in $B\mathbf{x} = \mu\mathbf{x}$

$$b_k x_k - b_{k+1} x_{k+1} = \mu x_{k+1}.$$

It is equivalent to $x_k = 0$. Next from the kth equation, using the similar argument, we obtain $x_{k+2} = 0$. We proceed in the same way, by considering first the $(k+1-i)$th and afterwards $(k+1+i)$th equation, for $2 \leq i \leq k-1$. The first one provides $x_{k+1-i} = 0$, while the second one gives $x_{k+1+i} = 0$ for $2 \leq i \leq k-1$. In the end we obtain $\mathbf{x} = \mathbf{0}$, which is a contradiction.

If $n = 2k+1$, we start from the $(k+1)$th equation and we obtain $x_{k+2} = 0$. Then we first observe the $(k+i+1)$th equation and after it the $(k+1-i)$th equation, for $1 \leq i \leq k-1$. In each step, we obtain $x_{k-i} = x_{k+i+1} = 0$, which altogether lead to $\mathbf{x} = \mathbf{0}$. Thus, we have arrived at a contradiction, and the proof is completed. ∎

4. Controllability of chain graphs. In this section, we present our main motivation for studying the matrices $\tilde{B}_n$: the Laplacian controllability of some particular chain graphs.

Let $G = (V(G), E(G))$ be a simple graph (without loops or multiple edges) of order $n = |V(G)|$. We use $A(G)$ to denote its standard $(0,1)$-adjacency matrix, $D(G)$ for the diagonal matrix of vertex degrees and

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 559-571, September 2022.

$L(G) = D(G) - A(G)$ for the Laplacian matrix. Since $L(G)$ is symmetric and semidefinite, we may assume that its eigenvalues, indexed in nonincreasing order, are

$$\mu_1(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G)\,.$$

In addition, the least eigenvalue $\mu_n(G)$ is zero (associated with a constant eigenvector).

Following [31], we proceed with more details on controllability of specified dynamical control systems. We consider a multiagent system with n agents modeled by a graph G. Let $x_i(t)$ be the state of the agent i at time t, whose dynamics is described by the single integrator $\dot{x}_i(t) = u_i(t)$, where $u_i(t)$ is agent's i control input. To compute the control of the agent i, the information that it receives from its neighbours is also taken into consideration. If we let $u_i(t) = -\sum_{i\sim j}(x_i(t) - x_j(t))$, then the single integrator dynamics can be represented as the Laplacian dynamics of the form

$$\dot{x}(t) = -L(G)x(t).$$

The set $V(G)$ is now disjoint union of the set of followers and the set of leaders, $V(G) = \ell \cup f$. This designation induce the following partition of $L(G)$:

$$L(G) = \begin{bmatrix} L_f(G) & l_{f\ell} \\ l_{f\ell}^{\mathsf{T}} & L_l(G) \end{bmatrix}.$$

Since

$$\begin{bmatrix} \dot{x}_f(t) \\ \dot{u}(t) \end{bmatrix} = -\begin{bmatrix} L_f(G) & l_{f\ell} \\ l_{f\ell}^{\mathsf{T}} & L_l(G) \end{bmatrix} \begin{bmatrix} x_f(t) \\ u(t) \end{bmatrix},$$

we consider the leader–follower control system, where followers develop through the Laplacian based dynamics

$$\dot{x}_f(t) = -L_f(G)x_f(t) - l_{f\ell}\,u(t), \tag{4.2}$$

and u stands for the outer control signal imposed by the leaders' states.

The system described by (4.2) is said to be controllable if it can be driven from any initial state to any desired final state in finite time. In the study of the controllability of multiagent systems, the main problem is to determine the locations of leaders under which the controllability can be realized. A multiagent system (4.2) is said to be k leaders controllable if there exist minimum number of $|\ell| = k$ leaders to make (4.2) controllable. Especially, if $k = 1$, the system (4.2) is called single leader controllable.

The following useful lemma was proven in [31].

Lemma 4.1. ([31]) *The system (4.2) is controllable if and only if there is no nonzero vector* $\mathbf{v}$ *such that the following equations are met simultaneously*

$$\begin{aligned} L_f(G)\mathbf{v} &= \lambda\mathbf{v} \\ l_{f\ell}(G)^{\mathsf{T}}\mathbf{v} &= O. \end{aligned} \tag{4.3}$$

Moreover, if $L_f(G)$ *does not have distinct eigenvalues, then (4.2) is not controllable.*

We recall another important argument for further analysis of controllability.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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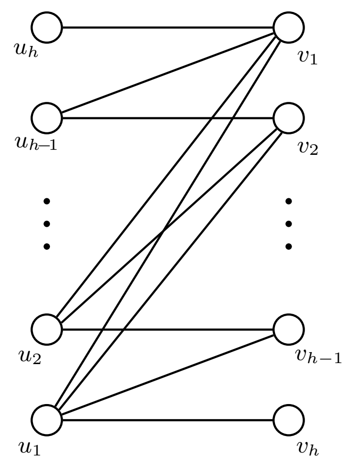

FIGURE 1. *The half graph* $H_{2h} = \mathrm{DNG}(1, \ldots, 1; 1, \ldots, 1)$.

LEMMA 4.2. ([31]) *The system* (4.2) *is controllable if and only if there is no eigenvector of* $L(G)$ *taking* 0 *on the elements corresponding to the leaders, i.e., if and only if* $L(G)$ *and* $L_f(G)$ *do not share any common eigenvalues.*

Next, we define the structure of *half graphs*. These are bipartite graphs and the corresponding colour classes are partitioned into the h singletons $\bigcup_{i=1}^{h}\{u_i\}$ and $\bigcup_{i=1}^{h}\{v_i\}$, respectively in such a way that a vertex u_s is joined (by cross edges) to all vertices in $\bigcup_{k=1}^{h+1-s}\{v_k\}$, for $1 \leq s \leq h$ (see Figure 1).

These graphs are denoted by

$$\mathrm{DNG}(\underbrace{1, 1, \ldots, 1}_{h}; \underbrace{1, 1, \ldots, 1}_{h}).$$

The abbreviation DNG stands for *double nested graph* – an alternative name reflecting the double nesting property: from left to right and from right to left, i.e., for any $s, t \in \{1, \ldots, h-1\}$, $N_G(u_{s+1}) \subset N_G(u_s)$ and $N_G(v_{t+1}) \subset N_G(v_t)$ (by $N_G(u)$ we denote the set of neighbours of the vertex u). Since, a half graph is of the order $2h$ we shortly denote it by H_{2h}.

We start with a useful lemma, whose proof can be found in [28]. We use $\dot\cup$ to denote the sum of two multisets, i.e., the multiset in which the multiplicity of an element is the sum of its multiplicities in the summands.

LEMMA 4.3. [28, Lemma 4.1] *Let* A *and* B *be symmetric matrices of the same order* n, *and let*

$$C = \begin{bmatrix} B & A \\ A & B \end{bmatrix}.$$

Then $\sigma(C) = \sigma(A+B)\,\dot\cup\,\sigma(B-A)$.

Remark 4.4. Since all the matrices $C, A+B$ and $B-A$ are symmetric, the multisets $\sigma(C), \sigma(A+B), \sigma(B-A)$ consist of respectively $2n, n$ and n eigenvalues. This fact shows that the corresponding sum of Lemma 4.3 is obtained by "pasting" $\sigma(B-A)$ to $\sigma(B+A)$.

In the next step, we prove that a half graph has no multiple Laplacian eigenvalues. The Laplacian matrix of such a graph is of the form:

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 559-571, September 2022.

$$L(G) = \begin{bmatrix} D & -A \\ -A & D \end{bmatrix},$$

where

$$D = \begin{bmatrix} h & & & & \\ & h-1 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & \\ \vdots & \vdots & \iddots & & \\ 1 & 1 & & & \\ 1 & & & & \end{bmatrix}, \tag{4.4}$$

with respect to the vertex ordering $u_1, u_2, \dots, u_h, v_1, v_2, \dots, v_h$.

Remark 4.5. For the matrices A and D given in (4.4), $A^{-1}D = \tilde{B}_h$ is the antibidiagonal matrix with eigenvalues $(-1)^{i-1}i$, for $1 \le i \le h$.

Next, we determine $\sigma(D-A)$ and $\sigma(D+A)$.

THEOREM 4.6. *Let D, A be the matrices of order $h \ge 1$ given in (4.4). Then $\sigma(D-A) = \{0, 1 \dots, h\} \setminus \{\lceil \frac{h}{2} \rceil\}$ with the eigenvectors $\mathbf{v}_i$ of the following form (up to nonzero scalar multiple)*

- $\mathbf{v}_0 = (1, 1, \dots, 1)^\intercal$ *for* $\mu = 0$*;*
- $\mathbf{v}_i = (\underbrace{0, 0, \dots 0}_{i}, \underbrace{1, 1, \dots, 1}_{h-2i}, -(h-2i), \underbrace{0, 0, \dots 0}_{i-1})^\intercal$ *for* $\mu = i, \quad 1 \le i < \lceil \frac{h}{2} \rceil$*;*
- $\mathbf{v}_i = (\underbrace{0, 0, \dots 0}_{h-i}, h+1-2i, \underbrace{1, 1, \dots, 1}_{2i-h-1}, \underbrace{0, 0, \dots 0}_{h-i})^\intercal$ *for* $\mu = i, \quad \lceil \frac{h}{2} \rceil < i \le h$.

Proof. By direct computation. ∎

To find $\sigma(D+A)$, we consider the formula for the characteristic polynomial $\phi_{D+A}(t)$.

THEOREM 4.7. *Let D and A be the matrices of order $h \ge 1$ given in (4.4). Then*

$$\phi_{D+A}(t) = t\Big(t - \Big\lceil \frac{h}{2} \Big\rceil\Big) \prod_{i=1}^{h} (t-i) \left(\frac{1}{(t-1)(t-h)} - \sum_{j=2}^{h} \frac{1}{(t-(j-1))(t-j)(t-(h+1-j))} \right). \tag{4.5}$$

Proof. Let $H_{2h} = \mathrm{DNG}(1, 1, \dots, 1; 1, 1, \dots, 1)$ be a half graph. According to [3, Theorem 3.5], the Laplacian characteristic polynomial of H_{2h} up to sign is

$$t \prod_{i=1}^{h} (t - d_i^*)(t - d_{h+1-i}) \left(\frac{1}{p_1} + t \sum_{j=2}^{h} \frac{1}{(t - d_{h+2-j})p_j} + \frac{1}{t - d_1} \right),$$

where $d_i^* = d_i = h + 1 - i$, $1 \le i \le h$ and $p_j = (d_j^* - t)(t - d_{h+1-j}) = (h+1-j-t)(t-j)$. Taking into account that $\phi_{L(H_{2h})}(t) = \phi_{D+A}(t) \cdot \phi_{D-A}(t)$ and $\phi_{D-A}(t) = \prod_{\substack{0 \le i \le h \\ i \ne \lceil \frac{h}{2} \rceil}} (t-i)$ (by Theorem 4.6) we obtain

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 559-571, September 2022.

$$
\begin{aligned}
\phi_{D+A}(t) &= \frac{-t\prod_{i=1}^{h}(t-(h+1-i))(t-i)\left(\frac{1}{(h-t)(t-1)}+t\sum_{j=2}^{h}\frac{1}{(t-(j-1))(h+1-j-t)(t-j)}+\frac{1}{t-h}\right)}{\prod_{\substack{0\le i\le h\\ i\ne\lceil\frac{h}{2}\rceil}}(t-i)}\\
&= \frac{-t\prod_{i=1}^{h}(t-i)^2\left(-\frac{t}{(t-1)(t-h)}-t\sum_{j=2}^{h}\frac{1}{(t-(j-1))(t-j)(t-(h+1-j))}\right)}{\prod_{\substack{0\le i\le h\\ i\ne\lceil\frac{h}{2}\rceil}}(t-i)}\\
&= t\Big(t-\Big\lceil\frac{h}{2}\Big\rceil\Big)\prod_{i=1}^{h}(t-i)\left(\frac{1}{(t-1)(t-h)}-\sum_{j=2}^{h}\frac{1}{(t-(j-1))(t-j)(t-(h+1-j))}\right),
\end{aligned}
$$

as claimed. $\square$

THEOREM 4.8. *Let D, A be the matrices of order $h \geq 1$ given in (4.4). Then:*

- *For any $i \in \{1, 2, \ldots, \lceil\frac{h}{2}\rceil - 1, \lceil\frac{h}{2}\rceil + 2, \ldots, h\}$, the matrix $D + A$ has an eigenvalue in the interval $(i-1, i)$;*
- *If h is an odd number, then $D + A$ has an eigenvalue in the interval $(\lceil\frac{h}{2}\rceil, \lceil\frac{h}{2}\rceil + 1)$;*
- *If h is an even number, then $D + A$ has an eigenvalue in the interval $(\lceil\frac{h}{2}\rceil - 1, \lceil\frac{h}{2}\rceil)$;*
- *The largest eigenvalue of $D + A$ is greater than h.*

Proof. Any $h \geq 1$ is of the form $h = 2k$ or $h = 2k - 1$, for some $k \geq 1$. Hence, $\lceil\frac{h}{2}\rceil = k$. We compute the values of $\phi_{D+A}(t)$ at $0, 1, \ldots, h$ using (4.5):

- $\phi_{D+A}(0) = (-1)^h(2k)(h-1)!$;
- $\phi_{D+A}(1) = 2(-1)^{h-1}(k-1)(h-3)!$;
- For any $2 \leq \ell \leq k-1$,
$$\phi_{D+A}(\ell) = 2\ell(-1)^{h-\ell}(k-\ell)\prod_{i=1}^{\ell-2}(\ell-i)\prod_{i=\ell+2}^{h-1-\ell}(i-\ell)\prod_{i=h+2-\ell}^{h}(i-\ell);$$
- $\phi_{D+A}(k) = (-1)^k k!(h-k)!$;
- For any $k+1 \leq \ell \leq h-1$,
$$\phi_{D+A}(\ell) = 2\ell(-1)^{h-\ell+1}(\ell-k)\prod_{i=1}^{h-1-\ell}(\ell-i)\prod_{i=h+2-\ell}^{\ell-2}(\ell-i)\prod_{i=\ell+2}^{h}(i-\ell);$$
- $\phi_{D+A}(h) = (-2)(h-k)(h-2)!$.

Based on the obtained values, we conclude that $\phi_{D+A}(0), \phi_{D+A}(1), \ldots, \phi_{D+A}(k-1)$ alternate in sign. Therefore, for any $i \in \{1, 2, \ldots, k-1\}$, we have $\phi_{D+A}(t) = 0$, for some $t \in (i-1, i)$. A similar argument holds for $\phi_{D+A}(k+1), \phi_{D+A}(k+2), \ldots, \phi_{D+A}(h)$ and consequently for any $i \in \{k+2, k+3, \ldots, h\}$, we have $\phi_{D+A}(t) = 0$, for some $t \in (i-1, i)$. Also, if h is odd, then $\phi_{D+A}(k)$ and $\phi_{D+A}(k+1)$ differ in sign. If h is even, the same holds for $\phi_{D+A}(k-1)$ and $\phi_{D+A}(k)$. Therefore, there is an eigenvalue in the interval $(k, k+1)$ (resp. $(k-1, k)$) if h is odd (resp. if h is even).

So far, we proved that, for any h, $D + A$ has $h - 1$ eigenvalues less than h. Since $\phi_{D+A}(t)$ is a monic polynomial and $\phi_{D+A}(h) < 0$, we easily conclude that the largest eigenvalue of $D + A$ is greater than h. $\square$

Taking into account Theorems 4.6 and 4.8, we conclude that H_{2h} has no multiple Laplacian eigenvalues. The structure of the corresponding eigenvectors is given in the following theorem.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 559-571, September 2022.

THEOREM 4.9. *Let $G = H_{2h}$, $h \geq 1$ be a half graph, D and A matrices given in (4.4). Then the eigenvectors of $L(G)$ are either of the form $\alpha(\mathbf{v}_j^\intercal, \mathbf{v}_j^\intercal)^\intercal$, where $\mathbf{v}_j$ is an eigenvector of $D - A$, or $\alpha(\mathbf{v}_c^\intercal, -\mathbf{v}_c^\intercal)^\intercal$ where $\mathbf{v}_c$ is an eigenvector of $D + A$, for $\alpha \in \mathbb{R} \setminus \{0\}$.*

Remark 4.10. In what follows we point out that the integral part of the spectrum of half graphs of order $2h$ coincides with the spectrum of antiregular graphs of order h. The set of vertex degrees of an antiregular graph of order h consists of $h-1$ distinct integers. Let T_n be an antiregular graph of order n. Then T_n is a threshold graph that is generated either by the binary sequence $\underbrace{(01)(01)\cdots(01)}_{k}$ if $n = 2k$ or by the binary sequence $\underbrace{(0^21)(01)\cdots(01)}_{k}$ if $n = 2k+1$. (For more details on the generating procedure, spectral and structural properties of threshold graphs, the reader is referred to [2,5]). According to the vertex ordering, where the vertices are ordered according to their vertex degrees in non-increasing order, the Laplacian matrix of T_n is

$$L(T_n) = D'_{2k} - \begin{bmatrix} J_k - I_k & A_k \\ A_k & O_k \end{bmatrix}, \tag{4.6}$$

where $D'_{2k} = \mathrm{diag}(2k-1, \ldots, k, k, \ldots 2, 1)$, if $n = 2k$. Otherwise,

$$L(T_n) = D'_{2k+1} - \begin{bmatrix} J_k - I_k & A'_{k\times(k+1)} \\ A'^{\intercal}_{k\times(k+1)} & O_{k+1} \end{bmatrix}, \tag{4.7}$$

where $A'_{k\times(k+1)} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & \\ \vdots & \vdots & \iddots & & \\ 1 & 1 & & & \end{bmatrix}$ and $D'_{2k+1} = \mathrm{diag}(2k, \ldots, k, k, \ldots 2, 1)$, if $n = 2k+1$. It is easy to see that if $G = H_{2h}$, then $D - A = L(T_h)$. Therefore, all the Laplacian eigenvalues of T_h are also the eigenvalues of H_{2h} and the corresponding eigenvectors are of the form $(\mathbf{v}^\intercal, \mathbf{v}^\intercal)^\intercal$, where $\mathbf{v}$ is the eigenvector of T_h for the same eigenvalue. In the light of this connection, we see that the results of Theorem 4.6 coincide with those obtained in [1], taking into account the different vertex ordering.

In [29] one can find how the Laplacian spectrum and eigenspaces of a threshold graph are modified by $(0,1)$-operations in its binary generating procedure. Also [21] explains how to identify the Laplacian eigenvalues of a threshold graph from its Laplacian matrix.

LEMMA 4.11. *Let $G = H_{2h}$ and $\mathbf{v} = (v_1, v_2, \ldots, v_{2h})^\intercal$ be an eigenvector of G associated with $\mu \in \sigma(D + A)$. Then $v_{\lceil \frac{h+1}{2} \rceil} \neq 0$.*

Proof. According to Theorem 4.9, we may assume that $\mathbf{v} = (\mathbf{x}^\intercal, -\mathbf{x}^\intercal)^\intercal$, where $\mathbf{x} = (x_1, x_2, \ldots, x_h) \neq \mathbf{0}$ and $(D + A)\mathbf{x} = \mu\mathbf{x}$. Suppose on the contrary that $x_{\lceil \frac{h+1}{2} \rceil} = 0$. From $(D + A)\mathbf{x} = \mu\mathbf{x}$ we obtain $\mathbf{x} = A^{-1}(\mu I - D)\mathbf{x}$, i.e., $\mathbf{x}$ is an eigenvector of the antibidiagonal matrix $A^{-1}(\mu I - D)$ for the eigenvalue $\lambda = 1$. Taking into account that

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 559-571, September 2022.

$$A^{-1}(\mu I - D) = \begin{bmatrix} & & & & \mu-1 \\ & & & \mu-2 & -(\mu-1) \\ & & \iddots & \iddots & \\ & \mu-(h-1) & & & \\ \mu-h & -(\mu-(h-1)) & & & \end{bmatrix},$$

and that $\mu \notin \mathbb{Z}$, it follows that $\mu - i \neq 0$. Now the result follows by Theorem 3.1. ∎

Finally, we are in the position to prove the main result of this section.

THEOREM 4.12. *Let $G = H_{2h}$ be a half graph. Then the system (4.2) modeled by G is a single leader controllable, with vertices of degree $\lceil \frac{h}{2} \rceil$ representing the leaders.*

Proof. If μ is an integer eigenvalue of G and $(\mathbf{x}^\intercal, \mathbf{x}^\intercal)^\intercal$ is the corresponding eigenvector, then according to Theorem 4.6 we have $\mathbf{x}_{\lceil \frac{h+1}{2} \rceil} \neq 0$. If μ is a non-integer eigenvalue, then by Lemma 4.11, the $\lceil \frac{h+1}{2} \rceil$th entry in the corresponding eigenvector is nonzero. By Lemma 4.2 we conclude that the system (4.2) modeled by G is a single leader controllable with vertices of degrees$\lceil \frac{h}{2} \rceil$ in the role of leaders. ∎

Example 4.13. For $G = H_{10}$, the system (4.2) is a single leader controllable. The leader ℓ can be any of the vertices u_3, v_3. In Figure 2 the leader is $\ell = u_3$ and it is connected to the followers v_1, v_2, v_3.

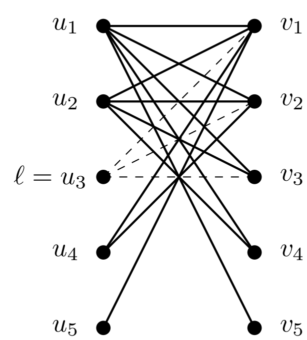

FIGURE 2. *A single leader controllable system modelled by H_{10}.*

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