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LINEAR MAPS PRESERVING THE LORENTZ SPECTRUM: THE 2×2 CASE*

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Abstract. In this paper, a complete description of the linear maps $\phi : W_n \to W_n$ that preserve the Lorentz spectrum is given when $n = 2$, and W_n is the space M_n of $n \times n$ real matrices or the subspace S_n of M_n formed by the symmetric matrices. In both cases, it has been shown that $\phi(A) = PAP^{-1}$ for all $A \in W_2$, where P is a matrix with a certain structure. It was also shown that such preservers do not change the nature of the Lorentz eigenvalues (that is, the fact that they are associated with Lorentz eigenvectors in the interior or on the boundary of the Lorentz cone). These results extend to $n = 2$ those for $n \geq 3$ obtained by Bueno, Furtado, and Sivakumar (2021). The case $n = 2$ has some specificities, when compared to the case $n \geq 3$, due to the fact that the Lorentz cone in $\mathbb{R}^2$ is polyedral, contrary to what happens when it is contained in $\mathbb{R}^n$ with $n \geq 3$. Thus, the study of the Lorentz spectrum preservers on $W_n = M_n$ also follows from the known description of the Pareto spectrum preservers on M_n.

Key words. Lorentz cone, Lorentz eigenvalues, Linear map preserver, 2×2 matrices.

AMS subject classifications. 15A18, 58C40.

1. Introduction. Given a matrix A in M_n, the algebra of $n \times n$ matrices with real entries, and a closed convex cone $K \subseteq \mathbb{R}^n$, the *eigenvalue complementarity problem* consists of finding a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in K, \quad Ax - \lambda x \in K^*, \quad x^T(A - \lambda I_n)x = 0,$$

where

$$K^* := \{y \in \mathbb{R}^n : x^T y \geq 0, \ \forall x \in K\}$$

denotes the (positive) dual cone of K. If $K = \mathbb{R}^n$, then the eigenvalue complementarity problem reduces to the usual eigenvalue problem for the matrix A.

The eigenvalue complementarity problem originally arose in the solution of a contact problem in mechanics and has since been used in other applications in physics, economics, and engineering, including, for example, the stability of dynamical systems [4].

In this work, we consider the complementarity eigenvalue problem associated with the Lorentz cone, defined, for $n \geq 2$, by

$$\mathcal{K}^n := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : ||x|| \leq x_n\},$$

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also known as the ice-cream cone. By $||x||$ we denote the 2-norm of x. If n is clear from the context, we may simply write $\mathcal{K}$ instead of $\mathcal{K}^n$. The Lorentz cone is widely used in optimization theory as an instance of a second-order cone, which has special importance in linear and quadratic programming [1].

It is well known that the Lorentz cone is self-dual, that is, $(\mathcal{K}^n)^* = \mathcal{K}^n$. Therefore, for $A \in M_n$, the eigenvalue complementarity problem relative to $\mathcal{K}^n$ consists of finding a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}^n, \quad (A - \lambda I)x \in \mathcal{K}^n, \quad x^T(A - \lambda I)x = 0, \tag{1.1}$$

where, here and throughout, I denotes the identity matrix of the appropriate order. By Corollary 2.1 in [5], it is guaranteed that (1.1) always admits a solution.

If a scalar λ and a nonzero vector x satisfy (1.1), we call λ a *Lorentz eigenvalue* of A and x an associated *Lorentz eigenvector* of A. We call the set of all Lorentz eigenvalues of A the *Lorentz spectrum of* A and denote it by $\sigma_{\mathcal{K}}(A)$. For brevity, we write L-eigenvalue, L-eigenvector, and L-spectrum instead of Lorentz eigenvalue, Lorentz eigenvector, and Lorentz spectrum, respectively. We classify the L-eigenvalues of a matrix $A \in M_n$ by whether they correspond to L-eigenvectors in the interior or on the boundary of the Lorentz cone. In the first case, we call them *interior L-eigenvalues*, and in the second case, we call them *boundary L-eigenvalues*. We denote the set of interior L-eigenvalues by $\sigma_{\mathcal{K}}^{int}(A)$ and the set of boundary L-eigenvalues by $\sigma_{\mathcal{K}}^{bd}(A)$.

The roots of the characteristic polynomial of a matrix $A \in M_n$ will be called the *standard eigenvalues* of A, to distinguish them from the L-eigenvalues.

In [3] the authors focused on the problem of studying the linear maps $\phi : W_n \to W_n$ that preserve the L-spectrum, that is, such that $\sigma_{\mathcal{K}}(\phi(A)) = \sigma_{\mathcal{K}}(A)$, for all $A \in W_n$, where W_n is a subspace of M_n and $n \geq 3$. The authors started by characterizing such maps ϕ for the following subspaces W_n of M_n: the subspace of diagonal matrices; the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is symmetric; and the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is a generic matrix. In each of these cases, it was shown that the maps should be what were called *standard maps*, that is, maps of the form $\phi(A) = PAQ$ for all $A \in W_n$ or $\phi(A) = PA^TQ$ for all $A \in W_n$, for some matrices $P, Q \in M_n$. However, when W_n is either M_n or the subspace S_n of symmetric matrices in M_n, just the standard linear maps $\phi : W_n \to W_n$ that preserve the L-spectrum were described, and it was conjectured that linear maps that are not standard do not preserve the L-spectrum. (See also the recent paper [7] in which the linear preservers $\phi : M_n \to M_n$ are investigated.)

The goal of this paper is to consider the case $n = 2$. The main differentiating feature between the cases $n \geq 3$ and $n = 2$ is that the Lorentz cone in $\mathbb{R}^2$ is *polyhedral*, i.e., it can be expressed as the intersection of a finite number of half-spaces. This implies that the L-spectrum of a matrix in M_2 is always finite, contrary to what happens for matrices of order $n \geq 3$, which can have infinite L-spectrum.

To our knowledge, the only polyhedral cone whose spectral linear preservers on M_n have been studied in depth in the literature is the Pareto cone [2]. It can be easily verified that, for $n = 2$, the Pareto cone is a clockwise rotation of the Lorentz cone by an angle of $\frac{\pi}{4}$. So, a description of the L-spectrum preservers on M_2 follows from the one of the Pareto spectrum preservers on M_n given in [2], taking $n = 2$, and reciprocally.

In this paper, we give an independent proof of the characterization of the linear maps $\phi : W_2 \to W_2$ that preserve the L-spectrum when $W_2 = M_2$ and, in addition, consider the new case $W_2 = S_2$, the subspace of M_2 of symmetric matrices. Also, for both cases of W_2, we show that the nature of the L-eigenvalues (being

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associated with L-eigenvectors in the interior or on the boundary of the Lorentz cone) is not changed by the L-spectrum preservers. To prove our results, we introduce techniques that explore the knowledge of the Lorentz spectrum of matrices in M_2 and hope that the ideas behind our proofs can be extended to complete the study of the L-spectrum preservers studied in [3] for matrices in M_n, with $n \geq 3$, in which case no connection exists with the Pareto cone. It follows from our characterization that such preservers on M_2 are standard and that, in the case $W_2 = M_2$, their form is less restrictive than the one for $n \geq 3$. (See Theorem 2.4 where the result for $n \geq 3$ is recalled.)

We next give the main results of this paper. Recall that M_2 denotes the space of 2×2 real matrices and S_2 denotes the subspace of M_2 of symmetric matrices.

THEOREM 1.1. *Let $\phi : W_2 \to W_2$ be a linear map, with $W_2 \in \{M_2, S_2\}$. Then, ϕ preserves the L-spectrum if and only if $\phi(A) = PAP^{-1}$ for all $A \in W_2$, or $\phi(A) = QAQ^{-1}$ for all $A \in W_2$, where*

$$P = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -\alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \tag{1.2}$$

for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 - \beta^2 = 1$, and $\beta = 0$ if $W_2 = S_2$.

COROLLARY 1.2. *Let $\phi : W_2 \to W_2$ be a linear map. If ϕ preserves the L-spectrum, then, for all $A \in W_2$,*

$$\sigma_{\mathcal{K}}^{int}(A) = \sigma_{\mathcal{K}}^{int}(\phi(A)) \quad \text{and} \quad \sigma_{\mathcal{K}}^{bd}(A) = \sigma_{\mathcal{K}}^{bd}(\phi(A)).$$

The paper is organized as follows. In Section 2, we introduce some known results in the literature regarding the L-spectrum of a matrix $A \in M_n$ and its linear preservers. In Section 3, we obtain a description of the L-eigenvalues of a generic matrix in M_2 and give some related results that will be helpful in the proof of Theorem 1.1. In Section 4, we deduce some conditions that should be satisfied by the images of matrices in certain bases for S_2 and M_2, respectively, under an L-spectrum linear preserver. Finally, in Section 5, we prove Theorem 1.1 and Corollary 1.2. We conclude the paper with some final remarks in Section 6.

2. Background.

In this section, we present some results known in the literature concerning the characterization of the L-spectrum of a matrix in M_n and properties of linear preservers of the L-spectrum. We also introduce some related useful concepts and notation.

2.1. L-spectrum of a matrix.

We first observe that

$$\sigma_{\mathcal{K}}(A) = \sigma_{\mathcal{K}}^{int}(A) \cup \sigma_{\mathcal{K}}^{bd}(A),$$

where this union is not necessarily disjoint. (Recall the definitions of interior and boundary eigenvalues in the introduction.)

We also note that any L-eigenvector $[x \ x_n]^T$ of $A \in M_n$, with $x_n \in \mathbb{R}$, can be normalized to have $x_n = 1$ while remaining in the Lorentz cone. Such a normalized L-eigenvector corresponds to an interior L-eigenvalue if $||x|| < 1$ and to a boundary L-eigenvalue if $||x|| = 1$.

The next characterization of interior and boundary L-eigenvalues of a matrix $A \in M_n$ is known [6].

PROPOSITION 2.1. *Let $A \in M_n$. Then,*

1. *λ is an interior L-eigenvalue of A if and only if λ is a standard eigenvalue of A associated with an eigenvector in the interior of $\mathcal{K}^n$.*

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2. *λ is a boundary L-eigenvalue of A if and only if there is some $s \geq 0$ and a vector $x \in \mathbb{R}^{n-1}$, with $||x|| = 1$, such that*
$$(A - \lambda I) \begin{bmatrix} x \\ 1 \end{bmatrix} = s \begin{bmatrix} -x \\ 1 \end{bmatrix}.$$

From Proposition 2.1, we have the following useful observation.

COROLLARY 2.2. *Let $A \in M_n$. Then, $\lambda \in \sigma_{\mathcal{K}}^{int}(A)$ if and only if $-\lambda \in \sigma_{\mathcal{K}}^{int}(-A)$.*

In contrast with interior L-eigenvalues, a boundary L-eigenvalue may or may not be a standard eigenvalue. A surprising fact, compared with the classical eigenvalue problem, is that a matrix may have infinitely many boundary L-eigenvalues, though this does not occur in the 2×2 case since the Lorentz cone for $n = 2$ is a polyhedral cone. (See [6] for a proof that there are only finitely many complementarity eigenvalues relative to a polyhedral cone.)

2.2. Linear preservers of the L-spectrum. In [3] the following important result was shown for matrices of size $n \geq 3$, although the presented proof is also valid for 2×2 matrices. By W_n we denote any of the spaces M_n or S_n, the subspace of symmetric matrices.

PROPOSITION 2.3 ([3]). *Let $n \geq 2$. If $\phi : W_n \to W_n$ is a linear map preserving the L-spectrum, then ϕ is bijective and $\phi(I) = I$.*

An immediate consequence of Proposition 2.3 is that if $\phi : W_n \to W_n$ is a linear map preserving the L-spectrum, then ϕ^{-1} also preserves the L-spectrum.

For completeness and for purpose of comparison with our main result, Theorem 1.1, we next state the characterization obtained in [3] of the standard linear maps $\phi : W_n \to W_n$ that preserve the L-spectum, when $n \geq 3$.

THEOREM 2.4 ([3]). *Let $n \geq 3$ and let $\phi : W_n \to W_n$ be a standard map. Then, ϕ preserves the L-spectrum if and only if there exists an orthogonal matrix $Q \in M_{n-1}$ such that*
$$\phi(A) = (Q \oplus [1])A(Q^T \oplus [1]),$$
for all $A \in W_n$.

3. L-spectrum of 2×2 matrices. In the next theorem, we present a characterization of the L-eigenvalues of 2×2 matrices and then we give some related properties.

THEOREM 3.1. *Let*
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2. \tag{3.3}$$
Then,

1. *a is an interior L-eigenvalue of A if and only if $b = 0$ and either $a = d$ or $|a - d| < |c|$;*
2. *$\lambda \in \mathbb{R} \setminus \{a\}$ is an interior L-eigenvalue of A if and only if $\lambda \in \left\{ \frac{a+d\pm\sqrt{(a-d)^2+4bc}}{2} \right\} \subseteq \mathbb{R}$ and $|b| < |a - \lambda|$;*
3. *λ is a boundary L-eigenvalue of A if and only if one of the following holds:*

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(a) $\lambda = \frac{(a+d)+(b+c)}{2}$ *and* $a - d \leq c - b$,
(b) $\lambda = \frac{(a+d)-(b+c)}{2}$ *and* $a - d \leq b - c$.

Proof. Conditions 1 and 2 follow immediately from the fact that, by Proposition 2.1, λ is an interior L-eigenvalue of A if and only if there is some $x \in \mathbb{R}$, with $|x| < 1$, such that

$$0 = (A - \lambda I) \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} (a-\lambda)x + b \\ cx + (d-\lambda) \end{bmatrix}. \tag{3.4}$$

Now we show Condition 3. By Proposition 2.1, we have that λ is a boundary L-eigenvalue of A if and only if there is some $s \geq 0$ and $x \in \{-1, 1\}$ such that

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = s \begin{bmatrix} -x \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} (a-\lambda+s)x + b \\ cx + (d-\lambda-s) \end{bmatrix} = 0.$$

When $x = 1$, this is equivalent to

$$\begin{cases} \lambda = a + b + s \\ \lambda = c + d - s \end{cases} \text{for some } s \geq 0,$$

that is,

$$\lambda = \frac{a+b+c+d}{2} \quad \text{and} \quad a - d \leq c - b.$$

When $x = -1$, we get

$$\begin{cases} \lambda = a - b + s \\ \lambda = d - c - s \end{cases} \text{for some } s \geq 0,$$

that is,

$$\lambda = \frac{a+d-b-c}{2} \quad \text{and} \quad a - d \leq b - c.$$

∎

Based on the characterization of the boundary L-eigenvalues of a matrix in M_2 given in Theorem 3.1, we introduce the following definitions.

Definition 3.2. *Let $A \in M_2$. We say that λ is a* type + boundary L-eigenvalue *of A (resp. a* type – boundary L-eigenvalue *of A) if Condition* 3*a (resp. Condition* 3*b) in Theorem* 3.1 *holds.*

Moreover, we say that a boundary L-eigenvalue λ of A is strict *if λ is of type + and $a - d < c - b$, or if λ is of type $-$ and $a - d < b - c$. If λ is a boundary L-eigenvalue of both type + and type $-$, then λ is strict if at least one of the previous strict inequalities holds.*

We next present some immediate consequences of Theorem 3.1. We first introduce two useful concepts.

Definition 3.3. *Let $A \in M_2$ be as in* (3.3)*. The* trace *of A, denoted by* $\mathrm{tr}(A)$, *is the sum of the diagonal entries of A, that is,* $\mathrm{tr}(A) = a + d$. *The* anti-trace *of A, denoted by* $\mathrm{antitr}(A)$, *is the sum of the antidiagonal entries of A, that is,* $\mathrm{antitr}(A) = b + c$.

Corollary 3.4. *Let $A \in M_2$. If A has a type + boundary L-eigenvalue λ_1 and a type $-$ boundary L-eigenvalue λ_2, then*

1. $\lambda_1 + \lambda_2 = \mathrm{tr}(A)$.
2. $|\lambda_1 - \lambda_2| = |\,\mathrm{antitr}(A)|$.

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Corollary 3.5. *Let $A \in M_2$ be as in (3.3) and let λ be a boundary L-eigenvalue of A. Then, λ is a standard eigenvalue of A if and only if A has a non-strict boundary L-eigenvalue.*

Proof. By Theorem 3.1, if λ is a type + boundary L-eigenvalue of A, then

$$\lambda = \frac{a+d+b+c}{2} \quad \text{and} \quad a-d \leq c-b,$$

and if λ is a type $-$ boundary L-eigenvalue of A, then

$$\lambda = \frac{a+d-b-c}{2} \quad \text{and} \quad a-d \leq b-c.$$

An elementary calculation shows that, in any case,

$$\det(A-\lambda I) = \frac{1}{4}\left((b-c)^2-(a-d)^2\right),$$

which is zero if and only if $|a-d| = |b-c|$. Thus, the claim follows. $\square$

The next result says that if we change the signs of both b and c in a matrix A as in (3.3), then the interior and the boundary L-eigenvalues of A get preserved.

Corollary 3.6. *Let $A \in M_2$ and $B = TAT$, where*

$$T = [-1] \oplus [1]. \tag{3.5}$$

Then A and B have the same L-spectrum. Moreover, we have $\sigma_{\mathcal{K}}^{int}(A) = \sigma_{\mathcal{K}}^{int}(B)$ and $\sigma_{\mathcal{K}}^{bd}(A) = \sigma_{\mathcal{K}}^{bd}(B)$. Additionally, λ is a type + boundary L-eigenvalue of A if and only if λ is a type $-$ boundary L-eigenvalue of B.

By using Theorem 3.1, we next give the explicit L-spectrum of the matrices in a basis of M_2 and S_2, which will be used in the characterization of the linear maps preserving the L-spectrum. In each case, the L-spectrum is presented as the union of two sets, namely, $\sigma_{\mathcal{K}}^{int}(A) \cup \sigma_{\mathcal{K}}^{bd}(A)$. Here and throughout, for $i, j \in \{1, 2\}$, E_{ij} denotes the 2×2 matrix with all entries 0 except the one in position (i, j) which is 1.

Corollary 3.7. *We have*

- $\sigma_{\mathcal{K}}(E_{11}) = \{0\} \cup \emptyset$
- $\sigma_{\mathcal{K}}(E_{21}) = \{0\} \cup \{1/2\}$
- $\sigma_{\mathcal{K}}(E_{22}) = \{1\} \cup \{1/2\}$
- $\sigma_{\mathcal{K}}(E_{12}+E_{21}) = \emptyset \cup \{-1, 1\}$

4. Images of matrices in a basis of W_2 under an L-spectrum preserver. Let us consider a linear map $\phi : W_2 \to W_2$ preserving the L-spectrum, with $W_2 \in \{M_2, S_2\}$. In this section, we obtain a generic form that $\phi(A)$ should have when A is a matrix in a specific basis of W_2, namely, the basis $\{E_{11}, E_{22}, E_{12}+E_{21}\}$ if $W_2 = S_2$, and the basis $\{E_{11}, E_{22}, E_{21}, E_{12}+E_{21}\}$ if $W_2 = M_2$. For $E_{12}+E_{21}$, the possible images under ϕ are exactly determined.

We begin with a result which shows that under certain conditions, a linear preserver of the L-spectrum preserves the interior and boundary L-eigenvalues. This will be key in proving the remaining results.

Lemma 4.1. *Let $\phi : W_2 \to W_2$ be a linear map that preserves the L-spectrum. If $A \in W_2$ has two distinct strict boundary L-eigenvalues, then*

$$\sigma_{\mathcal{K}}^{int}(A) = \sigma_{\mathcal{K}}^{int}(\phi(A)) \neq \emptyset \quad \text{and} \quad \sigma_{\mathcal{K}}^{bd}(A) = \sigma_{\mathcal{K}}^{bd}(\phi(A)). \tag{4.6}$$

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Proof. Let A be as in (3.3). Since A has two distinct strict boundary L-eigenvalues, say λ_1 and λ_2, by Theorem 3.1 we have $a-d<c-b$ and $a-d<b-c$. This implies that $-A$ does not have any boundary L-eigenvalues and, consequently, has at least one interior L-eigenvalue since every matrix has a nonempty L-spectrum. Hence, we have

$$\sigma_{\mathcal{K}}^{bd}(A)=\{\lambda_1,\lambda_2\}, \quad \sigma_{\mathcal{K}}^{int}(-A)\neq\emptyset, \quad \text{and} \quad \sigma_{\mathcal{K}}^{bd}(-A)=\emptyset.$$

Taking into account Corollary 2.2 and the fact that, by Corollary 3.5, λ_1 and λ_2 are not standard eigenvalues of A, we have

$$\sigma_{\mathcal{K}}^{int}(A)=-\sigma_{\mathcal{K}}^{int}(-A), \quad \sigma_{\mathcal{K}}^{int}(A)\neq\emptyset, \quad \text{and} \quad \sigma_{\mathcal{K}}^{int}(A)\cap\{\lambda_1,\lambda_2\}=\emptyset.$$

Since ϕ preserves the L-spectrum, for $i\in\{1,2\}$ we should have $\lambda_i\in\sigma_{\mathcal{K}}^{bd}(\phi(A))$, as otherwise $\lambda_i\in\sigma_{\mathcal{K}}^{int}(\phi(A))$, which implies, by Corollary 2.2, that $-\lambda_i\in\sigma_{\mathcal{K}}^{int}(\phi(-A))$, a contradiction since $-\lambda_i$ is not an L-eigenvalue of $-A$. Then, since $\phi(A)$ has two boundary L-eigenvalues, which are the boundary L-eigenvalues of A, it follows that the interior L-eigenvalues of A are also interior L-eigenvalues of $\phi(A)$. ∎

Before we fulfill the main purpose of this section, we state a simple consequence of Lemma 4.1 that will be used in the proof of Theorem 1.1 in the next section.

Lemma 4.2. *Let $\phi: W_2\to W_2$ be a linear map that preserves the L-spectrum. Then, $\phi(E_{11}+E_{21})$ is singular.*

Proof. Let $\varepsilon>0$ and $A_\varepsilon:=(-1-\varepsilon)E_{11}-E_{21}$. The matrix A_ε has two distinct strict boundary L-eigenvalues, implying, by Lemma 4.1, that $\phi(A_\varepsilon)$ has the same interior L-eigenvalues as A_ε. Since 0 is an interior L-eigenvalue of A_ε, $\phi(A_\varepsilon)$ is singular. By continuity, $\phi(-E_{11}-E_{21})$ is singular, and hence, so is $\phi(E_{11}+E_{21})$. ∎

4.1. Necessary forms for the images of a basis.

Lemma 4.3. *Let $\phi: W_2\to W_2$ be a linear map that preserves the L-spectrum. Then,*

$$\phi(E_{11})=\begin{bmatrix}1-a & \mp\sqrt{a^2-a}\\ \pm\sqrt{a^2-a} & a\end{bmatrix}, \quad \phi(E_{22})=\begin{bmatrix}a & \pm\sqrt{a^2-a}\\ \mp\sqrt{a^2-a} & 1-a\end{bmatrix},$$

for some $a\leq 0$, and

$$\phi(E_{12}+E_{21})=\begin{bmatrix}m & r\\ -r\pm 2 & -m\end{bmatrix},$$

for some $m,r\in\mathbb{R}$. In particular, if $W_2=S_2$, then

$$\phi(E_{11})=E_{11}, \quad \phi(E_{22})=E_{22},$$

and

$$\phi(E_{12}+E_{21})=\begin{bmatrix}m & r\\ r & -m\end{bmatrix},$$

for some $m\in\mathbb{R}$ and $r\in\{-1,1\}$.

Proof. For $\varepsilon\in\mathbb{R}\setminus\{0\}$, let $G_\varepsilon:=E_{22}+\varepsilon(E_{12}+E_{21})$, whose standard eigenvalues are $(1\pm\sqrt{1+4\varepsilon^2})/2$. By Theorem 3.1,

$$\sigma_{\mathcal{K}}^{int}(G_\varepsilon)=\left\{\frac{1+\sqrt{1+4\varepsilon^2}}{2}\right\} \text{ and } \sigma_{\mathcal{K}}^{bd}(G_\varepsilon)=\left\{\frac{1}{2}\pm\varepsilon\right\},$$

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and both boundary L-eigenvalues are strict. Thus, by Lemma 4.1, (4.6) holds with A replaced by G_ε. Let

$$\phi(E_{22}) := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \phi(E_{12}+E_{21}) := \begin{bmatrix} m & r \\ p & q \end{bmatrix}.$$

Then, by Corollary 3.4 applied to $\phi(G_\varepsilon)$,

$$a+d+\varepsilon(m+q)=1, \quad b+c+\varepsilon(r+p)=\pm 2\varepsilon.$$

Since $\varepsilon \neq 0$ is arbitrary, we have

$$a+d=1, \quad m+q=0, \quad b+c=0, \quad r+p=\pm 2.$$

Hence,

$$\phi(E_{22}) = \begin{bmatrix} a & b \\ -b & 1-a \end{bmatrix} \text{ and } \quad \phi(E_{12}+E_{21}) = \begin{bmatrix} m & r \\ -r\pm 2 & -m \end{bmatrix}.$$

From the obtained form of $\phi(E_{22})$, we conclude, by Theorem 3.1, that 1 is not a boundary L-eigenvalue of $\phi(E_{22})$. Since $\sigma_{\mathcal{K}}(\phi(E_{22})) = \sigma_{\mathcal{K}}(E_{22}) = \{1, 1/2\}$, it follows that 1 is an interior L-eigenvalue of $\phi(E_{22})$. This implies that

$$\det(\phi(E_{22})-I) = b^2 - a^2 + a = 0.$$

By Theorem 3.1, $b \neq 0$. Moreover, $|b| < |a-1|$, i.e., $b^2 < (a-1)^2$. Since $b^2 = a(a-1) \geq 0$, we get $a \leq 0$,

$$\phi(E_{22}) = \begin{bmatrix} a & \pm\sqrt{a^2-a} \\ \mp\sqrt{a^2-a} & 1-a \end{bmatrix}, \text{ and}$$

$$\phi(E_{11}) = \phi(I-E_{22}) = I-\phi(E_{22}) = \begin{bmatrix} 1-a & \mp\sqrt{a^2-a} \\ \pm\sqrt{a^2-a} & a \end{bmatrix}, \tag{4.7}$$

where the second equality in (4.7) follows from Proposition 2.3.

The particular claim in the statement for $W_2 = S_2$ follows since $\phi(E_{11})$ and $\phi(E_{12}+E_{21})$ are symmetric and $a \leq 0$. ∎

Notice that, if $\phi : W_2 \to W_2$ is a linear map preserving the L-spectrum, by Lemma 4.3, ϕ preserves the trace of E_{11}, E_{22}, and $E_{12}+E_{21}$, and therefore it preserves the trace of all matrices in S_2. Also, observe that ϕ preserves the modulus of the anti-trace of E_{11}, E_{22}, and $E_{12}+E_{21}$. Moreover, if ϕ preserves the anti-trace of $E_{12}+E_{21}$, then ϕ preserves the anti-trace of all matrices in S_2; otherwise, the anti-traces of A and $\phi(A)$ have opposite signs for all $A \in S_2$. These results are contained in the following corollary and extended to the case $\phi : M_2 \to M_2$.

Corollary 4.4. *Let $\phi : W_2 \to W_2$ be a linear map that preserves the L-spectrum. Then,*

$$\mathrm{tr}(A) = \mathrm{tr}(\phi(A)) \quad \textit{for all } A \in W_2,$$

and either

$$\mathrm{antitr}(A) = \mathrm{antitr}(\phi(A)) \quad \textit{for all } A \in W_2,$$

or

$$\mathrm{antitr}(A) = -\,\mathrm{antitr}(\phi(A)) \quad \textit{for all } A \in W_2.$$

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Proof. Let A be as in (3.3) and let

$$\phi(A) := \begin{bmatrix} r & s \\ p & q \end{bmatrix}.$$

Let δ be an arbitrary real number such that

$$a - d < \delta + c - b, \quad a - d < \delta + b - c, \quad \text{and} \quad b + c \neq 2\delta.$$

Let $A_\delta = A + \delta E_{22} - \delta(E_{12} + E_{21})$. Notice that A_δ has two strict boundary L-eigenvalues, namely

$$\lambda_1 = \frac{a + d + b + c - \delta}{2} \quad \text{and} \quad \lambda_2 = \frac{a + d - b - c + 3\delta}{2}, \tag{4.8}$$

which are distinct since $b + c \neq 2\delta$. Thus, by Lemma 4.1, λ_1 and λ_2 are also boundary L-eigenvalues of $\phi(A_\delta)$. Taking into account the form of $\phi(\delta E_{22} - \delta(E_{12} + E_{21}))$ that follows from Lemma 4.3, the boundary L-eigenvalues of $\phi(A_\delta)$ are

$$\beta_1 = \frac{r + q + s + p - \delta}{2}, \quad \beta_2 = \frac{r + q - s - p + 3\delta}{2}, \tag{4.9}$$

if $\operatorname{antitr}(\phi(E_{12} + E_{21})) = 2$, and

$$\beta_1 = \frac{r + q + s + p + 3\delta}{2}, \quad \beta_2 = \frac{r + q - s - p - \delta}{2}, \tag{4.10}$$

if $\operatorname{antitr}(\phi(E_{12} + E_{21})) = -2$. As $\{\lambda_1, \lambda_2\} = \{\beta_1, \beta_2\}$, we have

$$\lambda_1 + \lambda_2 = \beta_1 + \beta_2,$$

and

$$\lambda_1 - \lambda_2 = \beta_1 - \beta_2 \quad \text{or} \quad \lambda_1 - \lambda_2 = -(\beta_1 - \beta_2).$$

Since $\lambda_1 + \lambda_2 = a + d + \delta$ and $\beta_1 + \beta_2 = r + q + \delta$, we get $a + d = r + q$. We also have $\lambda_1 - \lambda_2 = b + c - 2\delta$. Moreover, $\beta_1 - \beta_2 = s + p - 2\delta$ if (4.9) holds, and $\beta_1 - \beta_2 = s + p + 2\delta$ if (4.10) holds. In the first case, $\lambda_1 - \lambda_2 = -(\beta_1 - \beta_2)$ only for $\delta = \frac{b+c+s+p}{4}$. Thus, for $\delta \neq \frac{b+c+s+p}{4}$, we have $\lambda_1 - \lambda_2 = \beta_1 - \beta_2$, implying $b + c = s + p$. In the second case, $\lambda_1 - \lambda_2 = \beta_1 - \beta_2$ only for $\delta = \frac{b+c-s-p}{4}$. Thus, for $\delta \neq \frac{b+c-s-p}{4}$, we have $\lambda_1 - \lambda_2 = -(\beta_1 - \beta_2)$, implying $b + c = -(s + p)$. Since δ is an arbitrary number satisfying (4.8), it ranges over an infinite set, and hence the claim follows. ∎

We next describe the generic structure of the image of E_{21} under a linear map preserving the L-spectrum.

LEMMA 4.5. *Let $\phi : M_2 \to M_2$ be a linear map that preserves the L-spectrum. Then,*

$$\phi(E_{21}) = \begin{bmatrix} \pm\sqrt{b^2 + b} & \mp b \\ \pm(b + 1) & \mp\sqrt{b^2 + b} \end{bmatrix}, \quad b \geq 0.$$

Proof. By Corollary 4.4,

$$\phi(E_{21}) = \begin{bmatrix} a & b \\ -b \pm 1 & -a \end{bmatrix},$$

for some $a, b \in \mathbb{R}$. By Theorem 3.1, this implies $\sigma_{\mathcal{K}}^{bd}(\phi(E_{21})) \subseteq \{-1/2, 1/2\}$. On the other hand, by Corollary 3.7, $\sigma_{\mathcal{K}}(E_{21}) = \{0, 1/2\}$. Thus, since ϕ preserves the L-spectrum, 0 is an interior L-eigenvalue of $\phi(E_{21})$. Hence, by Theorem 3.1, either $a = b = 0$, or $|b| < |a|$ (i.e., $b^2 < a^2$). Since $\phi(E_{21})$ is singular, we also have $a^2 = b^2 \mp b$. Thus, $a^2 = b^2 + b$ if $b > 0$ and $a^2 = b^2 - b$ if $b < 0$, implying the claim. ∎

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4.2. Explicit image of $E_{12}+E_{21}$. The following two lemmas will be used in determining $\phi(E_{12}+E_{21})$ under a linear L-spectrum preserver ϕ. By $||\cdot||_F$ we denote the Frobenius norm of a matrix.

LEMMA 4.6. *Let $A \in M_2$ be as in (3.3). Suppose A has two distinct standard real eigenvalues and at least one of them, say λ_A, is an interior L-eigenvalue. Moreover, suppose that $\lambda_A \neq a$. Then, for any $\varepsilon > 0$, there is some $\delta > 0$ such that any $B \in M_2$ with $||B - A||_F < \delta$ has an interior L-eigenvalue λ_B satisfying $|\lambda_A - \lambda_B| < \varepsilon$. That is, sufficiently small perturbations of A have an interior L-eigenvalue arbitrarily close to λ_A.*

Proof. Suppose that λ_A is an interior L-eigenvalue of A. By Theorem 3.1, since $\lambda_A \neq a$, we have $|b| < |a-\lambda_A|$, that is, $b^2 - (a-\lambda_A)^2 < 0$. Since λ_A depends continuously on the entries of A, any sufficiently small perturbation of A, say

$$A_\varepsilon := \begin{bmatrix} a_\varepsilon & b_\varepsilon \\ c_\varepsilon & d_\varepsilon \end{bmatrix},$$

has a real eigenvalue λ_A^ε arbitrarily close to λ_A and such that $\lambda_A^\varepsilon \neq a_\varepsilon$ and $|b_\varepsilon| < |a_\varepsilon - \lambda_A^\varepsilon|$. Note that, since A has distinct real eigenvalues, for ε sufficiently small, both eigenvalues of A_ε are also distinct and real. By Theorem 3.1, λ_A^ε is an interior L-eigenvalue of A_ε. ∎

LEMMA 4.7. *Let $\lambda \in \{-1, 1\}$. Then, there is some $\varepsilon > 0$ such that, in any neighborhood of $E_{12} + E_{21}$, there is a matrix with no L-eigenvalue at distance from λ smaller than ε.*

Proof. Let $H := E_{12} + E_{21}$. For any $\delta \in \mathbb{R}$, the matrices

$$H_\delta := H + \delta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.11}$$

and $-H_\delta$ have standard eigenvalues $\beta_1 = -\sqrt{\delta^2+1}$ and $\beta_2 = \sqrt{\delta^2+1}$. Notice that, for $i \in \{1, 2\}$,

$$1 \geq (\delta - \beta_i)^2 \;\Leftrightarrow\; 1 - \delta^2 - \beta_i^2 \geq -2\delta\beta_i \;\Leftrightarrow\; \delta^2 \leq \delta\beta_i, \tag{4.12}$$

where the last inequality follows from the second one by noting that $\beta_i^2 = \delta^2 + 1$.

Suppose that $\lambda = 1$ and let $\delta > 0$. From (4.12), $|1| \geq |\delta - \beta_2|$, implying by Theorem 3.1 that β_2 is not an interior L-eigenvalue of H_δ. On the other hand, H_δ has no boundary L-eigenvalues. Hence, the only L-eigenvalue of H_δ is β_1 whose distance from 1 is at least 2, regardless of the value of $\delta > 0$.

With a similar argument, we can see that, for $\delta < 0$, the only L-eigenvalue of $-H_\delta$ is β_2 whose distance from -1 is at least 2, regardless of the value of $\delta < 0$.

Thus, for each $\lambda \in \{1, -1\}$, there is some $\delta \in \mathbb{R}$ such that one of the matrices H_δ or $-H_\delta$ has no L-eigenvalues arbitrarily close to λ. ∎

LEMMA 4.8. *Suppose that $\phi : W_2 \to W_2$ is a linear map that preserves the L-spectrum. Then*

$$\phi(E_{12} + E_{21}) = E_{12} + E_{21} \quad \textit{or} \quad \phi(E_{12} + E_{21}) = -(E_{12} + E_{21}).$$

Proof. Let $H := E_{12} + E_{21}$. By Corollary 3.7 and Corollary 3.6, we have $\sigma_{\mathcal{K}}(H) = \sigma_{\mathcal{K}}(-H) = \{-1, 1\}$.

We start by proving that 1 and -1 are not interior L-eigenvalues of $\phi(H)$. To show this fact, suppose first that $\lambda \in \{-1, 1\}$ is an interior L-eigenvalue of $\phi(H)$. Then, since by Corollary 4.4, $\mathrm{tr}(\phi(H)) = \mathrm{tr}(H) = 0$, and interior L-eigenvalues are standard eigenvalues, $\phi(H)$ has distinct standard eigenvalues 1 and -1.

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We first show that the entry in position (1,1) of $\phi(H)$ is different from λ. This is clear by Theorem 3.1, if the entry in position $(1,2)$ of $\phi(H)$ is nonzero. If the entry in position $(1,2)$ of $\phi(H)$ is zero, then $\phi(H)$ is a lower triangular matrix with main diagonal entries 1 and -1, and the $(2,1)$ entry of $\phi(H)$ has modulus 2 (since by Corollary 4.4, the modulus of the anti-trace is preserved). Then, the entry in position $(1,1)$ of $\phi(H)$ is different from λ, as otherwise, by Theorem 3.1, λ would not be an interior L-eigenvalue of $\phi(H)$.

By Lemma 4.6, any matrix B in a sufficiently small neighborhood of $\phi(H)$ has an interior L-eigenvalue arbitrarily close to λ. By the continuity of ϕ^{-1}, and since ϕ^{-1} preserves the L-spectrum, any matrix in a sufficiently small neighborhood of H has an L-eigenvalue arbitrarily close to λ, which is impossible by Lemma 4.7.

Thus, 1 and -1 are not interior L-eigenvalues of $\phi(H)$. By Corollary 2.2, neither 1 nor -1 is an interior L-eigenvalue of $-\phi(H)$. Since $\sigma_{\mathcal{K}}(H) = \sigma_{\mathcal{K}}(-H) = \{1, -1\}$, we conclude that 1 and -1 are boundary L-eigenvalues of both $\phi(H)$ and $-\phi(H)$. By Corollary 3.4, there are $x, y \in \mathbb{R}$ such that

$$1)\ \phi(H) = \begin{bmatrix} x & y \\ 2-y & -x \end{bmatrix} \quad \text{or} \quad 2)\ \phi(H) = \begin{bmatrix} x & y \\ -2-y & -x \end{bmatrix}.$$

Suppose that Case 1 holds. Then, by Condition 3 of Theorem 3.1, applied to both $\phi(H)$ and $-\phi(H)$, we have

$$x + y = -x + 2 - y \ \text{ and}$$
$$x - y = -x - (2 - y),$$

implying that

$$x = 0 \text{ and } y = 1.$$

A similar argument applied to Case 2 yields $x = 0$ and $y = -1$. Thus, the claim follows. □

5. Proof of the main results.

Theorem 1.1.

Proof. Let $\phi : W_2 \to W_2$ be a linear map that preserves the L-spectrum. By Corollary 4.4, either A and $\phi(A)$ have the same anti-trace for all $A \in W_2$, or A and $\phi(A)$ have opposite anti-traces for all $A \in W_2$. When proving Theorem 1.1, we only consider the case in which ϕ preserves the anti-trace. The case when the anti-trace of A and $\phi(A)$ are opposite for all $A \in W_2$ can be obtained by considering the orthogonal similarity via the matrix $T = [-1] \oplus [1]$. More precisely, assume that A and $\phi(A)$ have opposite anti-traces. Then, $\pi(A) = T\phi(A)T$, for $A \in W_2$, is a linear map that preserves the anti-trace and symmetry, and, taking into account Corollary 3.6, π preserves the L-spectrum if and only if ϕ does. Hence, by the result that we next show, π preserves the L-spectrum if and only if there is some $P \in M_2$, as in (1.2), such that $\pi(A) = PAP^{-1}$ for any $A \in W_2$, that is, $\phi(A) = (TP)A(TP)^{-1}$ for any $A \in W_2$. Thus, the claim follows with $Q = TP$.

Necessity: Suppose that ϕ preserves the anti-trace. For $u, v \in \mathbb{R}$, let

$$P(u, v) := \begin{bmatrix} u & v \\ v & u \end{bmatrix}.$$

Case 1: Assume that $W_2 = S_2$. By Lemmas 4.3 and 4.8, we have $\phi(E_{11}) = E_{11}$, $\phi(E_{22}) = E_{22}$, and $\phi(E_{12} + E_{21}) = E_{12} + E_{21}$. Thus, $\phi(A) = PAP^{-1}$ for all $A \in S_2$, where $P = P(1, 0) = I$.

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Case 2: Assume now that $W_2 = M_2$. By Lemma 4.3, for some $a \le 0$, we have

$$\begin{aligned}\phi(E_{11}) &= \begin{bmatrix} 1-a & \mp\sqrt{a^2-a} \\ \pm\sqrt{a^2-a} & a \end{bmatrix} =: \begin{bmatrix} \alpha^2 & -\alpha\beta \\ \alpha\beta & -\beta^2 \end{bmatrix} \\ &= P(\alpha,\beta)E_{11}P^{-1}(\alpha,\beta).\end{aligned}$$

Without loss of generality, we assume $\alpha \ge 0$, implying $\alpha \ge 1$ since $\alpha^2 = 1-a$ and $a \le 0$.

By Lemma 4.5 and taking into account that ϕ preserves the anti-trace, for some $b \ge 0$, we have

$$\begin{aligned}\phi(E_{21}) &= \begin{bmatrix} \pm\sqrt{b^2+b} & -b \\ b+1 & \mp\sqrt{b^2+b} \end{bmatrix} =: \begin{bmatrix} \gamma\delta & -\delta^2 \\ \gamma^2 & -\gamma\delta \end{bmatrix} \\ &= P(\gamma,\delta)E_{21}P^{-1}(\gamma,\delta).\end{aligned}$$

As above, we assume $\gamma \ge 0$, implying $\gamma \ge 1$.

Then

$$\begin{aligned}\phi(E_{11}+E_{21}) &= \begin{bmatrix} \alpha^2 & -\alpha\beta \\ \alpha\beta & -\beta^2 \end{bmatrix} + \begin{bmatrix} \gamma\delta & -\delta^2 \\ \gamma^2 & -\gamma\delta \end{bmatrix} \\ &= \begin{bmatrix} \alpha^2+\gamma\delta & -\alpha\beta-\delta^2 \\ \alpha\beta+\gamma^2 & -\beta^2-\gamma\delta \end{bmatrix}.\end{aligned}$$

Since, by Lemma 4.2, $\phi(E_{11}+E_{21})$ is singular, we have

$$\det(\phi(E_{11}+E_{21})) = (\alpha\gamma-\beta\delta)\,(\beta\gamma-\alpha\delta) = 0.$$

Note that $\alpha\gamma - \beta\gamma \ne 0$, as otherwise $(\alpha\gamma)^2 = (\beta\delta)^2$, or equivalently, $a = 1+b$, a contradiction since $a \le 0$ and $1+b > 0$. Thus,

$$\beta\gamma = \alpha\delta, \tag{5.13}$$

implying

$$0 = (\alpha\delta)^2 - (\beta\gamma)^2 = (1-a)b + a(1+b) = a+b.$$

Hence, $a = -b$ which yields $\alpha = \gamma$. Since α and γ are nonzero, from (5.13) we get $\beta = \delta$. Now let $P := P(\alpha,\beta)$. Then,

$$\phi(E_{11}) = PE_{11}P^{-1} \quad \text{and} \quad \phi(E_{21}) = PE_{21}P^{-1},$$

implying

$$\begin{aligned}\phi(E_{22}) &= I - \phi(E_{11}) = I - PE_{11}P^{-1} \\ &= P(I-E_{11})P^{-1} = PE_{22}P^{-1}.\end{aligned}$$

Moreover, taking into account Lemma 4.8 and the fact that ϕ preserves the anti-trace, we have

$$\phi(E_{12}+E_{21}) = E_{12}+E_{21} = P(E_{12}+E_{21})P^{-1}.$$

Thus, since $\phi(A) = PAP^{-1}$ for all the matrices A in a basis for M_2, we have $\phi(A) = PAP^{-1}$ for all $A \in M_2$.

Sufficiency: Let $A \in W_2$ and let P be as in (1.2) with $\alpha^2 - \beta^2 = 1$. We assume that $\alpha > 0$ as, otherwise, since $PAP^{-1} = (-P)A(-P)^{-1}$, we may consider $-P$ instead of P. It is enough to prove $\sigma_{\mathcal{K}}(A) \subseteq \sigma_{\mathcal{K}}(\phi(A))$,

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since by applying this result to ϕ^{-1}, we get $\sigma_{\mathcal{K}}(\phi(A)) \subseteq \sigma_{\mathcal{K}}(A)$. (Note that $\phi^{-1}(A) = P^{-1}AP$, where P^{-1} still has the form of P in (1.2), with β replaced by $-\beta$.)

We show that if (λ, x) is an L-eigenpair of A, then (λ, Px) is an L-eigenpair of $\phi(A) = PAP^{-1}$. For this purpose, we start by proving two facts. First, P preserves the Lorentz cone, that is, if $x \in \mathcal{K}$, then $Px \in \mathcal{K}$. Second, P preserves orthogonality, that is, if $x^T y = 0$, then $(Px)^T(Py) = 0$, for $x, y \in \mathcal{K}$.

Let $x = [x_1\ x_2]^T \in \mathcal{K}$ and

$$[z_1\ z_2]^T := Px = [x_1\alpha + x_2\beta, x_1\beta + x_2\alpha]^T. \tag{5.14}$$

Then, $Px \in \mathcal{K}$ if and only if

$$|z_1| = |x_1\alpha + x_2\beta| \leq x_1\beta + x_2\alpha = z_2.$$

Since $|\beta| < \alpha$ and $|x_1| \leq x_2$, it follows that $z_2 = x_1\beta + x_2\alpha \geq 0$. Also, because of

$$z_1^2 - z_2^2 = x_1^2 - x_2^2 \leq 0, \tag{5.15}$$

we get that $Px \in \mathcal{K}$.

Now note that, if x and y are nonzero orthogonal vectors in $\mathcal{K}$, then they lie on the boundary of $\mathcal{K}$. More specifically, one is a positive multiple of $[1\ 1]^T$ and the other one is a positive multiple of $[-1\ 1]^T$. Since

$$P[1\ 1]^T = [\alpha + \beta, \alpha + \beta]^T \quad \text{and} \quad P[-1\ 1]^T = [-\alpha + \beta, \alpha - \beta]^T,$$

are orthogonal, it follows that P also preserves orthogonality.

Suppose that (λ, x) is an L-eigenpair of A, that is,

$$x \neq 0, \quad x \in \mathcal{K}, \quad (A - \lambda I)x \in \mathcal{K}, \quad \text{and} \quad x^T(A - \lambda I)x = 0.$$

Since P is invertible, we have $Px \neq 0$. Moreover, as P preserves the Lorentz cone, we have $y := Px \in \mathcal{K}$ and

$$(\phi(A) - \lambda I)y = P(A - \lambda I)P^{-1}Px = P[(A - \lambda I)x] \in \mathcal{K}.$$

From the orthogonality of x and $(A - \lambda I)x$ and the fact that P preserves orthogonality, it follows that $y^T(\phi(A) - \lambda I)y = 0$. Thus, (λ, Px) is an L-eigenpair of $\phi(A)$. ∎

Corollary 1.2.

Proof. By Theorem 1.1, and arguing as in its proof, we may assume that ϕ preserves the anti-trace, that is, $\phi(A) = PAP^{-1}$ for P as in (1.2) with $\alpha^2 - \beta^2 = 1$. Moreover, we may assume that $\alpha > 0$, as otherwise we consider $-P$ instead of P.

Assume that (λ, x) is an L-eigenpair of A, with $x = [x_1\ x_2]^T$. Let $z = [z_1\ z_2]^T$ be as in (5.14). It was shown in the sufficiency part of the proof of Theorem 1.1 that (λ, z) is an L-eigenpair of $\phi(A)$. Since by (5.15), $|x_1| < x_2$ if and only if $|z_1| < z_2$, it follows that z is an L-eigenvector of $\phi(A)$ in the interior of $\mathcal{K}$ if and only if x is an L-eigenvector of A in the interior of $\mathcal{K}$. Since A and $\phi(A)$ have the same L-spectrum, the claim follows. ∎

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6. Conclusions. Let M_n denote the space of $n \times n$ real matrices and S_n denote the subspace of M_n formed by the symmetric matrices. In this paper, for $W_2 \in \{M_2, S_2\}$, we described the linear maps $\phi : W_2 \to W_2$ that preserve the Lorentz spectrum (L-spectrum for short), that is, those maps ϕ for which A and $\phi(A)$ have the same L-spectrum for all $A \in W_2$. We have shown that $\phi(A) = PAP^{-1}$, where P is a matrix with a certain structure. In the case $W_2 = S_2$, P is a diagonal orthogonal matrix. In addition, we proved that such preservers on W_2 do not change the nature (interior or boundary) of the L-eigenvalues.

In the case $n \geq 3$, the characterization of the linear maps $\phi : W_n \to W_n$ that preserve the L-spectrum and are standard was given in [3]. (See [7] in which the case $W_n = M_n$ was also studied.) Recall that a linear map $\phi : W_n \to W_n$ is said to be standard if there exist matrices $P, Q \in M_n$ such that $\phi(A) = PAQ$ for all $A \in W_n$ or $\phi(A) = PA^TQ$ for all $A \in W_n$. In [3], a conjecture was made that all maps $\phi : W_n \to W_n$ that preserve the L-spectrum are, in fact, standard, as has been shown here to happen for $n = 2$. We also have seen here that these preservers on $W_2 = S_2$ have the same form as the standard ones on S_n, for $n \geq 3$. However, if $W_2 = M_2$, they have a more general form than those on M_n for $n \geq 3$.

Contrary to what happens when $n \geq 3$, the Lorentz cone in $\mathbb{R}^n$ with $n = 2$ is a polyhedral cone, which is a rotation of the Pareto cone. Thus, our characterization of the linear maps $\phi : M_2 \to M_2$ also follows from the characterization of the linear maps that preserve the Pareto spectrum [2]. However, we gave here an independent proof hoping that it gives tools that may be helpful in proving the still open conjecture stated in [3] that any linear preservers of the L-spectrum on S_n or M_n, for $n \geq 3$, are standard maps, which, together with the results in that reference, would complete the description of such linear preservers.

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REFERENCES

[1] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Math. Prog.*, 95:3–51, 2003.
[2] R. Alizadeh and F. Shakeri. Linear maps preserving Pareto eigenvalues. *Linear Multilinear Algebra*, 65:1053–1061, 2017.
[3] M.I. Bueno, S. Furtado, and K.C. Sivakumar. Linear maps preserving the Lorentz-cone spectrum in certain subspaces of M_n. *Banach J. Math. Anal.*, 15:article 58, 2021.
[4] J.A.C. Martins, A. Pinto da Costa, I.N. Figueiredo, and J.J. Judice. The directional instability problem in systems with frictional contacts. *Comput. Methods Appl. Mech. Eng.*, 193:357–384, 2004.
[5] A. Seeger. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. *Linear Algebra Appl.* 292:1–14, 1999.
[6] A. Seeger and M. Torki. On eigenvalues induced by a cone constraint. *Linear Algebra Appl.*, 372:181–206, 2003.
[7] A. Seeger and M. Torki. On spectral maps induced by convex cones. *Linear Algebra Appl.*, 592:651–692, 2020.