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# GRAPH DEGENERACY AND ORTHOGONAL VECTOR REPRESENTATIONS*

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**Abstract.** We apply a technique of Sinkovic and van der Holst for constructing orthogonal vector representations of a graph whose complement has given treewidth to graphs whose complement has given degeneracy.

**Key words.** Minimum semidefinite rank, Orthogonal vector representations, Graph degeneracy.

**AMS subject classifications.** 05C50, 05C07.

We will only consider finite simple graphs $G = (V, E)$ and assume knowledge of some standard graph theory definitions [5]. For notation, $|G|$ is the number of vertices of $G$, $N_G(v)$ is the set of neighbors of the vertex $v$ in $G$, and $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degrees, respectively, over the vertices of $G$.

A graph $G$ is $d$-degenerate if none of its subgraphs has minimum degree larger than $d$. The *degeneracy* of $G$ is the smallest integer $k$ such that $G$ is $k$-degenerate [10].

The *coloring number* of $G$, $\mathrm{col}(G)$, is the smallest positive integer $k$ for which there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $G$ such that every $v_i$ has fewer than $k$ neighbors $v_j \in N_G(v_i)$ with $j < i$ [6]. The coloring number and degeneracy are connected as $\mathrm{col}(G)$ is equal to one more than the degeneracy of $G$ [13].

The graph of a real symmetric $n$-by-$n$ matrix $M = [m_{ij}]$ has vertices $1, 2, \ldots, n$ and edges $ij$ when $i \neq j$ and $m_{ij} \neq 0$. Given a graph $G$, let $P(G)$ be the set of positive semidefinite matrices whose graph is $G$.

A positive semidefinite matrix $A \in P(G)$ satisfies the Strong Arnold Hypothesis if the only real symmetric matrix $X$ satisfying $AX = A \circ X = I \circ X = 0$ is the zero matrix, where $\circ$ is the entrywise product and $I$ is the identity matrix. The Colin de Verdière parameter $\nu(G)$ is defined as the maximum nullity among matrices in $P(G)$ that satisfy the Strong Arnold Hypothesis [4].

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$, their Gram matrix is $A = [a_{ij}]$ where $a_{ij} = \vec{v}_i \cdot \vec{v}_j$ is the standard dot product of the vectors [8]. The vectors comprise a *faithful orthogonal vector representation* (just *vector representation* hereafter) of a graph $G$ if their Gram matrix is in $P(G)$ [11, 14].

Sinkovic and van der Holst [12] proved that if a graph $G$ is the complement of a partial $k$-tree, then $\nu(G) \geq |G| - k - 2$. The proof involved the construction of a vector representation whose Gram matrix satisfies the Strong Arnold Hypothesis. They were able to apply their result to prove, for certain graphs, the Graph Complement Conjecture [7] for $\nu$, that $\nu(G) + \nu(\overline{G}) \geq |G| - 2$.

In this note, we apply Sinkovic and van der Holst's idea to graphs whose complement $\overline{G}$ has degeneracy $l$. Our main result is that $\nu(G) \geq |G| - 2l - 1$ for all such graphs $G$.

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## 1. Main Results.

THEOREM 1. *For any graph $G$ whose complement $\overline{G}$ has degeneracy $l$, $\nu(G) \geq |G| - 2l - 1$.*

*Proof.* We will prove a stronger statement: Let $G$ be a graph whose complement $\overline{G}$ is $l$-degenerate. Then there exists a vector representation of $G$ in $\mathbb{R}^{2l+1}$ such that the vectors representing any $l+1$ vertices are linearly independent. Further, if $A$ is the Gram matrix of the resulting vector representation, then $A$ satisfies the Strong Arnold Hypothesis.

Let $n = |G|$. Since the graph $\overline{G}$ is $l$-degenerate, we may choose an ordering $v_1, v_2, \ldots, v_n$ of the vertices of $\overline{G}$ such that every $v_i$ has at most $l$ neighbors $v_j$ with $j < i$. For each $i$ with $1 \leq i \leq n$, let $G_i$ be the subgraph of $G$ induced by the vertices $v_1, v_2, \ldots, v_i$.

Let $m = \min\{n, 2l+1\}$. Begin by choosing a positive definite $m$-by-$m$ matrix $A$ for $G_m$ (for example, choose $A = L(G)+I$, where $L(G)$ is the Laplacian matrix of $G$ and $I$ is the identity matrix) and let $B = \sqrt{A}$. Then the columns of $B$ are linearly independent and give the desired vector representation of $G_m$. Since $A$ is positive definite, it also satisfies the Strong Arnold Hypothesis. If $n \leq 2l+1$, we are done.

If $n > 2l+1$, assume for some $k$ with $k < n$ we have already constructed a vector representation $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ of $G_k$ in $\mathbb{R}^{2l+1}$ that meets the given conditions and such that the Gram matrix of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ satisfies the Strong Arnold Hypothesis.

Let $Q$ be the set of vectors representing the vertices of $N_{\overline{G_{k+1}}}(v_{k+1})$, and let $L = \operatorname{span}(Q)^{\perp}$.

We claim all of the following subspaces are proper subspaces of $L$:

- $L \cap \operatorname{span}(\vec{w})^{\perp}$ for each $w \in V(G_k) \setminus N_{\overline{G_{k+1}}}(v_{k+1})$,
- $L \cap \operatorname{span}(X)$ for each set $X$ of vectors representing $l$ vertices.

To prove the claim, consider first a vertex $w$ such that $w \in V(G_k) \setminus N_{\overline{G_{k+1}}}(v_{k+1})$. Since any $l+1$ vectors from $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent by the induction hypothesis and since $Q$ has at most $l$ vectors because $\overline{G}$ has degeneracy $l$, the vector $\vec{w}$ representing vertex $w$ is not in $\operatorname{span}(Q)$. Thus, $L \cap \operatorname{span}(\vec{w})^{\perp} \neq L$. Second, again by the induction hypothesis, if $X$ is a set of vectors representing $l$ vertices, then $\operatorname{span}(X)$ is $l$-dimensional. From above, $|Q| \leq l$ and the vectors of $Q$ are linearly independent, so that $L$ has dimension $2l+1-|Q| \geq l+1$ as a subspace of $\mathbb{R}^{2l+1}$. Thus, $L \cap \operatorname{span}(X) \neq L$.

Having established the claim, we can choose $\vec{v} \in L$ so that $\vec{v}$ does not belong to any of those (finitely many) proper subspaces.

By construction, $\vec{v}$ is orthogonal to $\vec{q}$ for each $q \in Q$. For any $w \in V(G_k) \setminus N_{\overline{G_k}}(v_k)$, since $\vec{v} \notin L \cap \operatorname{span}(\vec{w})^{\perp}$, $\vec{w} \cdot \vec{v} \neq 0$. Thus, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}$ is a vector representation of $G_{k+1}$.

By the induction hypothesis, any $l+1$ of these vectors not including $\vec{v}$ are linearly independent. If $X \cup \vec{v}$ is a set of $l+1$ vectors, the vectors of $X$ are linearly independent by the induction hypothesis and $\vec{v}$ is not in their span since $\vec{v} \in L$ and $\vec{v} \notin L \cap \operatorname{span}(X)$. Thus, the vectors of $X \cup \vec{v}$ are linearly independent for every such $X$.

Finally, let $B$ be the matrix with column vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}$ so that $A = B^{\mathsf{T}}B$ is the Gram matrix of those vectors. By the induction hypothesis, the Gram matrix of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ satisfies the Strong Arnold Hypothesis. Thus, if $A$ does not satisfy the Strong Arnold Hypothesis, the nonzero symmetric matrix $X$ with $AX = A \circ X = I \circ X = 0$ must have a nonzero row and column corresponding to $v_k$. Since $AX = 0$

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implies $XB^{\mathsf{T}}BX = 0$ implies $BX = 0$, that the column of $X$ corresponding to $v_k$ is nonzero then implies there is a non-trivial zero linear combination of $\vec{v}$ and the vectors of $Q$. But that is at most $l+1$ vectors, contradicting that they were constructed to be linearly independent. ∎

For a graph $G$ whose complement $\overline{G}$ has treewidth $k$ and degeneracy $l$, Theorem 1 improves on the bound of Sinkovic and van der Holst when $k > 2l-1$:

EXAMPLE 2. The square grid graph $L_{n,n} = P_n \,\square\, P_n$ (where $P_n$ is the path on $n$ vertices and $\square$ is the cartesian product of graphs) has treewidth $n$ [3] and degeneracy 2 (order the vertices lexicographically). In particular, $\nu(\overline{L_{n,n}}) \geq n^2 - 5$.

A graph $H$ has Hadwiger number $h(H)$ at most three if and only if its treewidth is at most two [1], and the Hadwiger number of a planar graph is at most four, so $h(L_{n,n}) = 4$ for $n \geq 3$. Since, $\nu(G)$ is at least $h(G)-1$ by minor monotonicity, $\nu(L_{n,n}) \geq 3$ for $n \geq 3$. Thus, all of the $L_{n,n}$ satisfy the Graph Complement Conjecture.

We next show that the bound of Theorem 1 is best possible in terms of degeneracy.

EXAMPLE 3. For a given positive integer $l$, let $H_l$ be the graph constructed as follows (see Figure 1 for examples): Start with a clique on vertices $c_1, c_2, \ldots, c_l$; add vertices $u_1, u_2, \ldots, u_{l+1}$ each adjacent to all of the previous vertices $c_1, c_2, \ldots, c_l$; end by adding vertices $w_1, w_2, \ldots, w_{l+1}$ with each $w_i$ adjacent to all of the $u_1, u_2, \ldots, u_{l+1}$ except $u_i$.

The ordering $c_1, c_2, \ldots, c_l, u_1, u_2, \ldots, u_{l+1}, w_1, w_2, \ldots, w_{l+1}$ shows that $H_l$ has degeneracy at most $l$. Since $\delta(H_l) = l$, the degeneracy of $H_l$ is exactly $l$.

We claim that $\nu(\overline{H_l}) = |H_l| - 2l - 1$. From Theorem 1, $\nu(\overline{H_l}) \geq |H_l| - 2l - 1$. For the other inequality, consider a vector representation of $\overline{H_l}$. The vectors representing $c_1, c_2, \ldots, c_l$ must be mutually orthogonal, and the vectors representing $u_1, u_2, \ldots, u_{l+1}$ must each be orthogonal to the span of the vectors representing $c_1, c_2, \ldots, c_l$. Finally, for each $i$ with $1 \leq i \leq l+1$, the vector representing $u_i$ cannot be in the span of the vectors representing the other $u$ vertices because those vectors are each orthogonal to the vector representing vertex $w_i$ while the vector representing vertex $u_i$ cannot be. Since any set of vectors such that any vector is not in the span of the others is linearly independent (any non-trivial zero linear combination will place one vector in the span of the others), the vectors representing the vertices $u_1, u_2, \ldots, u_{l+1}$ are linearly independent. Thus, any vector representation of $\overline{H_l}$ must include $2l+1$ linearly independent vectors representing the vertices $c_1, c_2, \ldots, c_l, u_1, u_2, \ldots, u_{l+1}$.

As a result, for any given degeneracy $l$, there is a graph $\overline{H_l}$ whose complement has degeneracy $l$ and that achieves equality in the inequality of Theorem 1.

For $l > 1$, we also note that contracting the edges $u_i w_{i+1}$ for $1 \leq i \leq l$ and $u_{l+1} w_1$ in $H_l$ results in a complete graph on $2l+1$ vertices. Thus, $\nu(H_l) \geq 2l$ for $l > 1$ and all of the $H_l$ satisfy the Graph Complement Conjecture.

Finally, since adding all possible edges among the $u_i$ results in a chordal graph with maximum clique size $2l+1$, each $H_l$ with $l > 1$ has treewidth $2l$. Thus, the $H_l$ (for $l > 1$) also provide a family where Theorem 1 improves on the bound of Sinkovic and van der Holst.

We can also relate $\nu(G)$ and $\delta(G)$ using Theorem 1:

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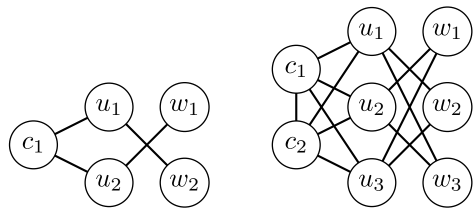


FIG. 1. *The graphs $H_1$ and $H_2$ of Example 3.*

THEOREM 4. *For any graph $G$, $\nu(G) \geq 2\delta(G) - |G| + 1$.*

*Proof.* In general, $\mathrm{col}(\overline{G}) \leq \Delta(\overline{G})+1$ [9, p. 78], $\mathrm{col}(\overline{G}) = l+1$ in Theorem 1, and $\delta(G)+\Delta(\overline{G}) = |G|-1$. □

Unfortunately, Theorem 4 is only nontrival when $\delta(G) > |G|/2$, perhaps only interesting if $\delta(G)$ is close to $|G|$ and seems far from the conjectured $\nu(G) \geq \delta(G)$ (the "delta conjecture" [2]).

Finally, note that, although we established Theorem 1 is best possible (for degeneracy) in Example 3, the graphs $\overline{H_l}$ from Example 3 do satisfy $\nu(\overline{H_l}) = l + 1 = \delta(\overline{H_l})$.

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