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MAJORIZATION INEQUALITIES VIA CONVEX FUNCTIONS*

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Abstract. Convex functions have been well studied in the literature for scalars and matrices. However, other types of convex functions have not received the same attention given to the usual convex functions. The main goal of this article is to present matrix inequalities for many types of convex functions, including log-convex, harmonically convex, geometrically convex, and others. The results extend many known results in the literature in this direction.
For example, it is shown that if A, B are positive definite matrices and f is a continuous $\sigma\tau$-convex function on an interval containing the spectra of A, B, then

$$\lambda^{\downarrow}(f(A\sigma B)) \prec_w \lambda^{\downarrow}(f(A)\tau f(B)),$$

for the matrix means $\sigma, \tau \in \{\nabla_\alpha, !_\alpha\}$ and $\alpha \in [0,1]$. Further, if $\sigma = \sharp_\alpha$, then

$$\lambda^{\downarrow}\left(f\left(e^{A\nabla_\alpha B}\right)\right) \prec_w \lambda^{\downarrow}\left(f(e^A)\tau f(e^B))\right).$$

Similar inequalities will be presented for two-variable functions too.

Key words. Majorization, Convexity and joint convexity, Extensions of matrix Jensen inequality, Matrix means.

AMS subject classifications. 26B25, 47A30, 47B15, 15A60, 15A45.

1. Introduction.

Convex functions have played a central role in the study of Mathematical inequalities. Recall that a continuous function $f : J \to \mathbb{R}$ is said to be convex on the interval J if it satisfies the inequality:

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y),$$

for all $0 \le \alpha \le 1$ and $x, y \in J$. Simplifying the notations, we denote the quantity $\alpha x + (1-\alpha)y$ by $x\nabla_\alpha y$. Consequently, f is convex if it satisfies the inequality:

$$f(x\nabla_\alpha y) \le f(x)\nabla_\alpha f(y), \qquad (x, y \in J, 0 \le \alpha \le 1). \tag{1.1}$$

The quantity $x\nabla_\alpha y$ is usually referred to as the arithmetic mean of x and y. Therefore, the inequality (1.1) can be looked at as a comparison of the image of the arithmetic mean with the arithmetic mean of the images.

Speaking of means and convex functions, there are many other means, which in turns define certain types of convexity. More precisely, given two positive numbers x, y, the geometric and harmonic means of x, y are defined, respectively, by:

$$x\sharp_\alpha y = x^\alpha y^{1-\alpha} \quad \text{and} \quad x!_\alpha y = \left(\alpha x^{-1} + (1-\alpha)y^{-1}\right)^{-1}, \quad 0 \le \alpha \le 1.$$

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Comparing between the three aforementioned means, the following is well known:

$$x!_\alpha y \le x\sharp_\alpha y \le x\nabla_\alpha y; \qquad (x, y > 0, 0 \le \alpha \le 1). \tag{1.2}$$

It is customary to use the notations $\nabla, \sharp, !$ instead of $\nabla_{\frac{1}{2}}, \sharp_{\frac{1}{2}}, !_{\frac{1}{2}}$, respectively.

While convex functions deal with the arithmetic mean ∇, we will be interested in looking at other functions that deal with the other means, as well. More precisely, we have the following definition [1].

Definition 1.1. *Let $f : J \subseteq (0, \infty) \to (0, \infty)$ be continuous and let $\sigma, \tau \in \{\nabla, \sharp, !\}$. We say that f is $\sigma\tau$-convex if it satisfies the inequality:*

$$f(x\sigma y) \le f(x)\tau f(y), \ \forall x, y \in J.$$

Therefore, f is convex if it is $\nabla\nabla$- convex. Further, when f is $\nabla\sharp$-convex, then f satisfies

$$f(x\nabla y) \le f(x)\sharp f(y);$$

which means that f is a log-convex function. On the other hand, f is $\sharp\sharp$-convex if it satisfies the inequality:

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)},$$

which means that f is geometrically convex.

It should be noted that since, by definition, f is continuous, $\sigma\tau$-convexity is equivalent to $\sigma_\alpha\tau_\alpha$-convexity, $0 \le \alpha \le 1$.

The following lemma presents basic relations among the different types of $\sigma\tau$-convex functions. The reader is referred to [1, 7, 10] for proofs and more information.

Lemma 1.2. *Let $f : (0, \infty) \to (0, \infty)$ be continuous.*

(i) *f is $\nabla\sharp$-convex if and only if $\log f$ is convex;*
(ii) *f is $\nabla!$-convex if and only if $1/f$ is concave;*
(iii) *f is $\sharp\nabla$-convex (concave) if and only if $f \circ \exp$ is $\nabla\nabla$-convex (concave);*
(iv) *If h is $\nabla\nabla$-convex (concave), then $f(t) = h(\log t)$ is $\sharp\nabla$-convex (concave);*
(v) *f is $\sharp\sharp$-convex if and only if the function $h = \log \circ f \circ \exp$ is convex;*
(vi) *f is $\sharp\sharp$-convex if and only if $h = \log \circ f$ is $\sharp\nabla$-convex;*
(vii) *f is $\sharp!$-convex (concave) if and only if $f \circ \exp$ is $\nabla!$-convex (concave);*
(iix) *f is $!\sharp$-convex if and only if $h(t) = t \log f(t)$ is $\nabla\nabla$-convex;*
(ix) *f is $!\sharp$-convex if and only if $\log f$ is $!\nabla$-convex;*
(x) *f is $!!$-convex (concave) if and only if $h(t) = t/f(t)$ is $\nabla\nabla$-concave (convex).*

Remark 1.3. The notations $\nabla, \sharp$ and $!$ are well known and are commonly used to denote the arithmetic, geometric, and harmonic means, respectively. In the literature, the notation AA is used to denote $\nabla\nabla$, for example. In this article, we tend to use $\sigma\tau$ in general for two reasons. First, it is an attempt to remind the reader of the general form, and second, it is more convenient in this article due to the many results and other notations that may cause confusion.

The main purpose of this article is to study matrix inequalities for $\sigma\tau$-convex functions, where $\sigma, \tau \in \{\nabla_\alpha, \sharp_\alpha, !_\alpha\}$. To do so, we need to recall some background about matrices and matrix means inequalities.

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Let $\mathbb{M}_n$ denote the algebra of all $n \times n$ complex matrices. Given a matrix $A \in \mathbb{M}_n$, we use the notation $sp(A)$ to denote the spectrum of A, that is, the set of eigenvalues of A. If D is a complex region containing the spectrum of A, and if $f : D \to \mathbb{C}$ is a given analytic function in D, the matrix $f(A)$ is defined via the Dunford integral:

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz,$$

where Γ is a simple closed curve in $\mathbb{C}$ that surrounds the spectrum of A, and lies entirely in D.

When A is Hermitian, the above Dunford integral is equivalent to

$$f(A) = U\text{diag}[f(\lambda_i)]U^*,$$

where $A = U\text{diag}[\lambda_i]U^*$ is the spectral decomposition of A, in which U is unitary.

Further, if $A = \sum_{i=1}^{n} \lambda_i P_i$ is the spectral decomposition of the Hermitian matrix $A \in \mathbb{M}_n$, when λ_i's are eigenvalues of A and P_i's are projections with $\sum_{i=1}^{n} P_i = I$, then $f(A) = \sum_{i=1}^{n} f(\lambda_i)P_i$, see [6].

For convenience, we will use the notation J_A for an interval that contains the spectrum of the Hermitian matrix A, and $J_{A,B}$ for the interval that contains the spectra of both A and B. According to the last observation, when A is Hermitian and $f : J_A \to \mathbb{R}$ is a continuous function, $f(A)$ is Hermitian too.

Speaking of Hermitian matrices, we define the order $A \le B$ between two Hermitian matrices $A, B \in \mathbb{M}_n$ as follows:

$$A \le B \Leftrightarrow B - A \ge 0;$$

where the notation $C \ge 0$ means that C is a positive semi-definite matrix, that is, $\langle Cx, x\rangle \ge 0$ for all $x \in \mathbb{C}^n$. If $\langle Cx, x\rangle > 0$ for all nonzero vectors $x \in \mathbb{C}^n$, C is said to be positive definite and is denoted by $C > 0$. We denote the set of all positive definite matrices in $\mathbb{M}_n$ by $\mathbb{P}_n$.

Given a $\sigma\tau$-convex function $f : J_{A,B} \to \mathbb{R}$, it is of interest to study convex inequalities similar to the real case. That is to find relations between $f(A\sigma B)$ and $f(A)\tau f(B)$, where for two positive definite matrices A, B, these notations will be used in this article as follows

$$A\nabla_\alpha B = \alpha A + (1-\alpha)B,\ A\sharp_\alpha B = A^\alpha B^{1-\alpha},\ A!_\alpha B = \left(\alpha A^{-1} + (1-\alpha)B^{-1}\right)^{-1},$$

for $0 \le \alpha \le 1$.

We should remark that the notation $\sharp_\alpha$ is not used in the literature as above, but it is used to denote the weighted geometric mean (see, e.g., [6]).

It is well established in the literature that a $\sigma\tau$-convex function $f : J_{A,B} \to (0,\infty)$ does not necessarily satisfy the matrix inequality:

$$f(A\sigma B) \le f(A)\tau f(B), \tag{1.3}$$

see [4, Example V.1.4, p. 114] for example, where the case $\nabla\nabla$ is treated.

Therefore, it has been an interesting topic in the literature to study when such extensions are valid.

Weaker forms of matrix inequalities have been studied and verified, when the concept of "majorization" is involved: given a real vector $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $\mathbf{x}^\uparrow$ be the vector obtained from $\mathbf{x}$ by rearranging

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the coordinates of $\mathbf{x}$ in a non-decreasing order. Similarly, we use the notation $\mathbf{x}^{\downarrow}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x}$ is weakly majorized (or submajorized) by $\mathbf{y}$ and we write $\mathbf{x}^{\downarrow} \prec_w \mathbf{y}^{\downarrow}$ (or $\mathbf{x} \prec_w \mathbf{y}$) if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}, \qquad \forall 1 \leq k < n.$$

If in addition $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, then we say that $\mathbf{x}$ is majorized by $\mathbf{y}$, and we write $\mathbf{x}^{\downarrow} \prec \mathbf{y}^{\downarrow}$. On the other hand, if

$$\sum_{j=1}^{k} x_j^{\uparrow} \geq \sum_{j=1}^{k} y_j^{\uparrow}, \forall 1 \leq k \leq n,$$

we say that $\mathbf{x}$ is weakly supermajorized by $\mathbf{y}$, and we write $\mathbf{x} \prec^w y$. Further, for $\mathbf{x}, \mathbf{y}$ with positive components, we write

$$\mathbf{x} \prec_{w \log} \mathbf{y} \Leftrightarrow \prod_{j=1}^{k} x_j^{\downarrow} \leq \prod_{j=1}^{k} y_j^{\downarrow}, \forall 1 \leq k \leq n.$$

If $\mathbf{x} \prec_{w \log} \mathbf{y}$ and $\prod_{j=1}^{n} x_j^{\downarrow} = \prod_{j=1}^{n} y_j^{\downarrow}$, we write $\mathbf{x} \prec_{\log} \mathbf{y}$.

For two Hermitian matrices $A, B \in \mathbb{M}_n$, we use the above majorization inequalities if the corresponding inequalities are valid for their eigenvalues vectors. For example, we write $A \prec_w B$ if $\lambda^{\downarrow}(A) \prec_w \lambda^{\downarrow}(B)$, where $\lambda(A)$ denotes the vector of eigenvalues of A.

The following two lemmas about majorization will be needed in the sequel.

Lemma 1.4 ([4, Theorem II.1.10, Theorem II.2.8]). *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(i) $\mathbf{x} \prec \mathbf{y}$ *if and only if* $\mathbf{x}$ *is in the convex hull of all vectors obtained by permutating the coordinates of* $\mathbf{y}$.
(ii) $\mathbf{x} \prec_w \mathbf{y}$ *if and only if there exists a vector* $\mathbf{u}$ *such that* $\mathbf{x} \leq \mathbf{u}$ *and* $\mathbf{u} \prec \mathbf{y}$.

Lemma 1.5 ([4, Theorem II.3.3., Corollary II.3.4]). *Let f be a real convex function. If* $\mathbf{x} \prec \mathbf{y}$*, then* $f(\mathbf{x}) \prec_w f(\mathbf{y})$*. If in addition f is monotone, then* $\mathbf{x} \prec_w \mathbf{y}$ *implies that* $f(\mathbf{x}) \prec_w f(\mathbf{y})$.

Regarding extension of convex inequalities from the scalar to the matrix setting like (1.3), it was shown in [3] that

$$\lambda^{\downarrow}\left(f\left(A \nabla_\alpha B\right)\right) \prec_w \lambda^{\downarrow}\left(f(A) \nabla_\alpha f(B)\right), 0 \leq \alpha \leq 1, \tag{1.4}$$

for all Hermitian matrices $A, B \in \mathbb{M}_n$ and convex functions $f: J_{A,B} \to \mathbb{R}$.

In the same reference, it was shown, for the same parameters, that

$$\lambda^{\downarrow}\left(f\left(X^* A X\right)\right) \prec_w \lambda^{\downarrow}\left(X^* f(A) X\right), \tag{1.5}$$

for all contractions $X \in \mathbb{M}_n$. This latter inequality is an extension of the simple Jensen inequality that states [6, Theorem 1.2]

$$f\left(x^* A x\right) \leq x^* f(A) x \text{ (equivalently } f\left(\langle Ax, x\rangle\right) \leq \langle f(A)x, x\rangle), \tag{1.6}$$

for any unit vector $x \in \mathbb{R}^n$, the Hermitian matrix $A \in \mathbb{M}_n$ and the continuous convex function $f: J_A \to \mathbb{R}$.

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Related to the quantities appearing in (1.5) and (1.6), and in alignment of the theme of this paper, we adopt the following notations. For a positive definite matrix $A \in \mathbb{P}_n$ and a unit vector $x \in \mathbb{C}^n$, we set

$$\nabla(x;A) = x^*Ax, \qquad \sharp(x;A) = \exp\left(x^*(\log A)x\right), \qquad !(x;A) = \left(x^*A^{-1}x\right)^{-1}. \tag{1.7}$$

For $C \in \mathbb{M}_n$, we define similarly

$$\nabla(C;A) = C^*AC, \qquad \sharp(C;A) = \exp\left(C^*(\log A)C\right), \qquad !(C;A) = \left(C^*A^{-1}C\right)^{-1}. \tag{1.8}$$

So, (1.5) and (1.6) can be read using our new notations as follows:

$$\lambda^{\downarrow}\{f\left(\nabla(X;A)\right)\} \prec_w \lambda^{\downarrow}\left\{\nabla(X;f(A))\right\} \quad \text{and} \quad f\left(\nabla(x;A)\right) \le \nabla(x;f(A)). \tag{1.9}$$

It is the sole goal of this paper to extend the inequalities (1.4) and (1.9) to the context of other quantities. More precisely, we will extend (1.4) to the context of geometric and harmonic means, and we will extend (1.9) to the context of other forms (1.7) and (1.8).

In the sequel, we will deal with positive functions defined on positive intervals, to avoid any miscalculation due to the well definiteness of the underlying quantities.

Before we present our main results, we would like to note that variants of Lemma 1.5 for $\sigma\tau$-convex functions hold as follows.

Proposition 1.6. *Let* $f:(0,\infty)\to(0,\infty)$ *and let* $\mathbf{x},\mathbf{y}\in(0,\infty)^n$.

(i) *If f is a (monotone) $\nabla!$-convex function, then ($\mathbf{x}\prec_w\mathbf{y}$) $\mathbf{x}\prec\mathbf{y}$ implies that* $f(\mathbf{x})^{-1}\prec^w f(\mathbf{y})^{-1}$.
(ii) *If f is a (monotone) $\nabla\sharp$-convex function, then ($\mathbf{x}\prec_w\mathbf{y}$) $\mathbf{x}\prec\mathbf{y}$ implies that* $f(\mathbf{x})\prec_{w\log} f(\mathbf{y})$.
(iii) *If f is a (monotone deceasing) $!\nabla$-convex function, then ($\mathbf{x}\prec_w\mathbf{y}$) $\mathbf{x}\prec\mathbf{y}$ implies that* $f(\mathbf{x}^{-1})\prec_w f(\mathbf{y}^{-1})$.
(iv) *If f is a (monotone) $\sharp\nabla$-convex function, then ($\mathbf{x}\prec_{w\log}\mathbf{y}$) $\mathbf{x}\prec_{\log}\mathbf{y}$ implies that* $f(\mathbf{x})\prec_w f(\mathbf{y})$.
(v) *If f is a (monotone) $\sharp!$-convex function, then ($\mathbf{x}\prec_{w\log}\mathbf{y}$) $\mathbf{x}\prec_{\log}\mathbf{y}$ implies that* $f(\mathbf{x})^{-1}\prec^w f(\mathbf{y})^{-1}$.
(vi) *If f is a (monotone decreasing) $!\sharp$-convex function, then ($\mathbf{x}\prec_w\mathbf{y}$) $\mathbf{x}\prec\mathbf{y}$ implies that* $f(\mathbf{x}^{-1})\prec_{w\log} f(\mathbf{y}^{-1})$.
(vii) *If f is a (monotone decreasing) $!!$-convex function, then ($\mathbf{x}\prec_w\mathbf{y}$) $\mathbf{x}\prec\mathbf{y}$ implies that* $f(\mathbf{x}^{-1})^{-1}\prec^w f(\mathbf{y}^{-1})^{-1}$.
(viii) *If f is a (monotone) $\sharp\sharp$-convex function, then ($\mathbf{x}\prec_{w\log}\mathbf{y}$) $\mathbf{x}\prec_{\log}\mathbf{y}$ implies that* $f(\mathbf{x})\prec_{w\log} f(\mathbf{y})$.

Proof. The proof follows from Lemma 1.2 and Lemma 1.5. We only give a short explanation for part (i). Let f be a $\nabla!$-convex function so that the function $g(t)=-1/f(t)$ is convex by Lemma 1.2. If $\mathbf{x}\prec\mathbf{y}$, then Lemma 1.5 implies that $g(\mathbf{x})\prec_w g(\mathbf{y})$, that is,

$$\sum_{j=1}^k g(x_j)^{\downarrow} \le \sum_{j=1}^k g(y_j)^{\downarrow}.$$

in which $g(x_1)^{\downarrow}\ge g(x_2)^{\downarrow}\ge\cdots\ge g(x_n)^{\downarrow}$ and so $-f(x_1)^{-1}\ge -f(x_2)^{-1}\ge\cdots\ge -f(x_n)^{-1}$. Eventually, we have

$$\sum_{j=1}^k \left[f(y_j)^{-1}\right]^{\uparrow} \le \sum_{j=1}^k \left[f(x_j)^{-1}\right]^{\uparrow}, \qquad (k=1,\ldots,n),$$

which means that $f(\mathbf{x})^{-1}\prec^w f(\mathbf{y})^{-1}$. Suppose in addition f is monotone increasing, so that $g(t)=-1/f(t)$ is monotone increasing, too. If $\mathbf{x}\prec_w\mathbf{y}$, then we find a vector $\mathbf{u}$ by Lemma 1.4 such that $\mathbf{x}\le\mathbf{u}$ and $\mathbf{u}\prec\mathbf{y}$.

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Therefore, $g(\mathbf{x}) \le g(\mathbf{u})$ and $g(\mathbf{u}) \prec_w g(\mathbf{y})$, whence $g(\mathbf{x}) \prec_w g(\mathbf{y})$ again by Lemma 1.4. This completes the proof of (i). To see part (iv), we recall that when $\mathbf{x} \prec_{\log} \mathbf{y}$ we have $\log(\mathbf{x}) \prec \log(\mathbf{y})$. If f is a (monotone) $\sharp\nabla$-convex function, then $g(t) = f(\exp t)$ is a (monotone) convex function and Lemma 1.5 gives (iv). ∎

The two next lemmas are well-known facts in matrix analysis.

LEMMA 1.7 ([4]). *If $X \in \mathbb{M}_n$ is a Hermitian matrix, then*

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(X) = \max \sum_{j=1}^{k} \nabla(u_j; X) \qquad (k = 1, \dots, n),$$

in which the maximum is taken over all orthonormal sets $\{u_1, \dots, u_k\}$ of vectors in $\mathbb{C}^n$.

LEMMA 1.8 (The Minimax principle [4, Theorem III.1.2]). *Let $X \in \mathbb{M}_n$ be a Hermitian matrix. Then,*

$$\lambda_k^{\downarrow}(X) = \max_{\mathcal{M} \subseteq \mathbb{C}^n, \dim \mathcal{M} = k} \ \min_{u \in \mathcal{M}, \|u\|=1} \nabla(u; X),$$

for every $k = 1, \dots, n$.

The next lemma gives variants of the matrix Jensen inequality (1.6) for $\sigma\tau$-convex functions. The proof follows easily from Lemma 1.2 and the inequality (1.6).

LEMMA 1.9. *Let $f : (0, \infty) \to (0, \infty)$ be $\sigma\tau$-convex for $\sigma, \tau \in \{\nabla, \sharp, !\}$. Then,*

$$f(\sigma(u; X)) \le \tau(u; f(X)), \tag{1.10}$$

for every positive definite matrix X and every unit vector $u \in \mathbb{C}^n$.

2. Majorization inequalities for $\sigma\tau-$convex functions.

We begin with the following main result that extends (1.4) to the context of $\sigma\tau$-convex functions. Although the proof of this result uses a standard argument as in the proof of [3, Theorem 2.9], it cannot be derived easily from [3, Theorem 2.9] .

THEOREM 2.1. *Let A and B be positive definite matrices and $f : J_{A,B} \subseteq (0, \infty) \to (0, \infty)$ be a continuous $\sigma\tau$-convex function. If $\sigma, \tau \in \{\nabla_\alpha, !_\alpha\}$, then*

$$\lambda^{\downarrow}(f(A\sigma B)) \prec_w \lambda^{\downarrow}\left(f(A)\tau f(B)\right), \tag{2.11}$$

for $\alpha \in [0, 1]$. Further, if $\sigma = \sharp_\alpha$, then

$$\lambda^{\downarrow}\left(f\left(e^{A\nabla_\alpha B}\right)\right) \prec_w \lambda^{\downarrow}\left(f(e^A)\tau f(e^B)\right). \tag{2.12}$$

If $\tau = \sharp_\alpha$, then the weak majorization $\prec_w$ is replaced by the weak log-majorization $\prec_{w\log}$ in (2.11) and (2.12). Furthermore, if the function f is monotone and $\sigma, \tau \neq \sharp_\alpha$, then the weak majorization $\prec_w$ can be replaced with an inequality.

Proof. **Case I:** If $\sigma = \nabla_\alpha$ and $\tau \in \{\nabla_\alpha, \sharp_\alpha\}$, then the result was shown in [3]. That is

$$\lambda^{\downarrow}(f(\alpha A + (1-\alpha)B)) \prec_w \lambda^{\downarrow}\left(\alpha f(A) + (1-\alpha) f(B)\right), \tag{2.13}$$

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when f is $\nabla\nabla$-convex; and

$$\lambda^{\downarrow}(f(\alpha A+(1-\alpha)B)) \prec_{w\log} \lambda^{\downarrow}\left(f(A)^{\alpha}f(B)^{1-\alpha}\right), \tag{2.14}$$

when f is $\nabla\sharp$-convex.

Case II: Let f be $\nabla!$-convex. Assume that $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $\alpha A+(1-\alpha)B$ arranged in such a way that $f(\lambda_1)\geq\cdots\geq f(\lambda_n)$. Then,

$$\sum_{j=1}^{k}\lambda_j^{\downarrow}(f(\alpha A+(1-\alpha)B))=\sum_{j=1}^{k}f\left(\alpha u_j^*Au_j+(1-\alpha)u_j^*Bu_j\right),$$

in which $\{u_1,\ldots,u_n\}$ is an orthonormal system of eigenvectors corresponding to $\lambda_1,\ldots,\lambda_n$. So, if f is a $\nabla!$-convex function, then $-1/f$ is convex by Lemma 1.2. Then Lemma 1.9 implies that for every $j=1,\ldots,n$,

$$\begin{aligned} f\left(\alpha u_j^*Au_j+(1-\alpha)u_j^*Bu_j\right) &\leq \left(\alpha u_j^*f(A)^{-1}u_j+(1-\alpha)u_j^*f(B)^{-1}u_j\right)^{-1} \\ &= \left(u_j^*\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)u_j\right)^{-1} \\ &\leq u_j^*\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)^{-1}u_j, \end{aligned}$$

where the last inequality follows from (1.6) using the convex function $t\mapsto t^{-1}$. Therefore,

$$\begin{aligned} \sum_{j=1}^{k}\lambda_j^{\downarrow}(f(\alpha A+(1-\alpha)B)) &\leq \sum_{j=1}^{k}u_j^*\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)^{-1}u_j \\ &\leq \sum_{j=1}^{k}\lambda_j^{\downarrow}\left(\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)^{-1}\right), \end{aligned} \tag{2.15}$$

where we have used Lemma 1.7 to obtain the last inequality. The latter inequality is equivalent to saying

$$\lambda^{\downarrow}\left(f(A\nabla_{\alpha}B)\right) \prec_w \lambda^{\downarrow}\left(f(A)!_{\alpha}f(B)\right).$$

This proves the theorem when f is $\nabla!$-convex.

Case III: Let f be $!\nabla$-convex. Then $t\mapsto f(1/t)$ is convex by Lemma 1.2. Applying (2.13) for this convex function gives

$$\lambda^{\downarrow}\left(f\left((\alpha A+(1-\alpha)B)^{-1}\right)\right) \prec_w \lambda^{\downarrow}\left(\alpha f(A^{-1})+(1-\alpha)f(B^{-1})\right).$$

Substituting A and B by A^{-1} and B^{-1}, respectively, we obtain

$$\lambda^{\downarrow}\left(f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right) \prec_w \lambda^{\downarrow}\left(\alpha f(A)+(1-\alpha)f(B)\right), \tag{2.16}$$

which implies the desired result when f is $!\nabla$-convex.

Case IV: Let f be $!\sharp$-convex. Then, Lemma 1.2 ensures that the function $t\mapsto\log f$ is $!\nabla$-convex. Applying (2.16) for $\log f$ then implies that

$$\begin{aligned} \lambda^{\downarrow}\left(\log f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right) &\prec_w \lambda^{\downarrow}\left(\alpha\log f(A)+(1-\alpha)\log f(B)\right) \\ &= \lambda^{\downarrow}\left(\log f(A)^{\alpha}+\log f(B)^{1-\alpha}\right). \end{aligned} \tag{2.17}$$

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It is known [2] that for all positive definite matrices $A, B \in \mathbb{P}_n$ the matrix $\log A + \log B$ is majorized by the matrix $\log(A^{1/2}BA^{1/2})$. Hence, (2.17) gives

$$\lambda^{\downarrow}\left(\log f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right) \prec_w \lambda^{\downarrow}\left(\log\left(f(A)^{\alpha/2}f(B)^{1-\alpha}f(A)^{\alpha/2}\right)\right). \tag{2.18}$$

Since the logarithm function is increasing, we have $\lambda_j^{\downarrow}(\log X) = \log(\lambda_j^{\downarrow}(X))$ for every positive definite matrix X so that (2.18) yields

$$\begin{aligned}\sum_{j=1}^{k}\log\left(\lambda_j^{\downarrow}\left\{f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right\}\right) &\le \sum_{j=1}^{k}\log\left(\lambda_j^{\downarrow}\left(f(A)^{\alpha/2}f(B)^{1-\alpha}f(A)^{\alpha/2}\right)\right)\\ &= \sum_{j=1}^{k}\log\left(\lambda_j^{\downarrow}\left(f(A)^{\alpha}f(B)^{1-\alpha}\right)\right),\end{aligned}$$

whence

$$\prod_{j=1}^{k}\lambda_j^{\downarrow}\left\{f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right\} \le \prod_{j=1}^{k}\lambda_j^{\downarrow}\left(f(A)^{\alpha}f(B)^{1-\alpha}\right).$$

for every $k = 1, \ldots, n$. Equivalently,

$$\lambda^{\downarrow}\left(f\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right) \prec_{w\log} \lambda^{\downarrow}\left(f(A)^{\alpha}f(B)^{1-\alpha}\right), \tag{2.19}$$

and we obtain the result, when f is $!\sharp$-convex.

Case V: Let f be $\sharp\sharp$-convex. Then, $f \circ \exp$ is $\nabla\sharp$-convex by Lemma 1.2. Utilizing (2.14) for the $\nabla\sharp$-convex function $f \circ \exp$ gives

$$\lambda^{\downarrow}\left\{f\left(\exp\left(\alpha A+(1-\alpha)B\right)\right)\right\} \prec_{w\log} \lambda^{\downarrow}\left(f(\exp A)^{\alpha}f(\exp B)^{1-\alpha}\right), \tag{2.20}$$

and we obtain the desired result in the case where f is $\sharp\sharp$-convex.

Case VI: Next, suppose that f is a $!!$-convex function. If $w \in \mathbb{C}^n$ is a unit vector, then

$$\begin{aligned}f\left(\left(w^*\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)w\right)^{-1}\right) &= f\left(\left(\alpha w^*A^{-1}w+(1-\alpha)w^*B^{-1}w\right)^{-1}\right)\\ &\le \left(\alpha f\left(\left(w^*A^{-1}w\right)^{-1}\right)^{-1}+(1-\alpha)f\left(\left(w^*B^{-1}w\right)^{-1}\right)^{-1}\right)^{-1}.\end{aligned} \tag{2.21}$$

On the other hand, Lemma 1.9 implies that

$$w^*f(A)^{-1}w \le f\left(\left(w^*A^{-1}w\right)^{-1}\right)^{-1} \quad \text{and} \quad w^*f(B)^{-1}w \le f\left(\left(w^*B^{-1}w\right)^{-1}\right)^{-1}. \tag{2.22}$$

From (2.21) and (2.22), we have

$$f\left(\left(w^*\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)w\right)^{-1}\right) \le \left(w^*\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)w\right)^{-1}. \tag{2.23}$$

Now suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}$ arranged in such a way that $f(\lambda_1) \ge \cdots \ge f(\lambda_n)$. Since $\lambda_j^{\downarrow}(X^{-1}) = \lambda_j^{\uparrow -1}(X)$, we can write

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j^{\downarrow}\left(f\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right) &= \sum_{j=1}^{k}f\left(\lambda_j^{\downarrow}\left(\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\right)\\ &= \sum_{j=1}^{k}f\left(\lambda_j^{\uparrow -1}\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)\right).\end{aligned} \tag{2.24}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 179-194, February 2022.

If $\lambda_j^{\uparrow}\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)=w_j^*\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)w_j$, where $\{w_1,\ldots,w_n\}$ is an orthonormal system of eigenvectors, then

$$\begin{aligned}\sum_{j=1}^k f\left(\lambda_j^{\uparrow-1}\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)\right)&=\sum_{j=1}^k f\left(\left(w_j^*\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)w_j\right)^{-1}\right)\\ &\leq\sum_{j=1}^k\left(w_j^*\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)w_j\right)^{-1},\end{aligned}\tag{2.25}$$

in which the inequality follows form (2.23). We will use a variant of Lemma 1.7 as follows:

$$\sum_{j=1}^k\lambda_j^{\downarrow}(X^{-1})\geq\max\sum_{j=1}^k(w_j^*Xw_j)^{-1}\qquad(k=1,\ldots,n),\tag{2.26}$$

in which the maximum is taken over all orthonormal set $\{w_1,\ldots,w_k\}$ of vectors in $\mathbb{C}^n$. Indeed, this is a consequence of Lemma 1.7, when we use $(w^*Xw)^{-1}\leq w^*X^{-1}w$. Combining (2.24), (2.25), and (2.26), we reach the desired result when f is !!-convex. That is,

$$\lambda^{\downarrow}\left(f\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)^{-1}\right)\prec_w\lambda^{\downarrow}\left(\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)^{-1}\right).\tag{2.27}$$

Case VII: If $\sigma=\sharp_\alpha$ and $\tau=!_\alpha$, then by Lemma 1.2 we can consider the function $f\circ\exp$ with $\sigma=\nabla_\alpha$ and $\tau=!_\alpha$ so that (2.15) yields

$$\lambda^{\downarrow}\left\{f\left(\exp\left(\alpha A+(1-\alpha)B\right)\right)\right\}\prec_w\left(\left(\alpha f(\exp A)^{-1}+(1-\alpha)f(\exp B)^{-1}\right)^{-1}\right),\tag{2.28}$$

and the result follows for $\sharp$!-convex functions.

Case VIII: If $\sigma=\sharp_\alpha$ and $\tau=\nabla_\alpha$, then Lemma 1.2 ensures that the function $f\circ\exp$ is $\nabla\nabla$-convex and so (2.12) follows from (2.11).

Case IX: Finally, assume that f is a monotone function. It was proved in [3] that utilizing the Minimax principle, Lemma 1.8, the majorization can be replaced with inequality in (2.13), that is

$$\lambda^{\downarrow}(f(\alpha A+(1-\alpha)B))\leq\lambda^{\downarrow}\left(\alpha f(A)+(1-\alpha)f(B)\right).\tag{2.29}$$

Let f be an increasing ∇!-convex function so that $t\mapsto -1/f(t)$ is an increasing convex function. Indeed (2.29) gives

$$\lambda_j^{\downarrow}(-f(\alpha A+(1-\alpha)B)^{-1})\leq\lambda_j^{\downarrow}\left(-\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)\right)\quad(j=1,\ldots,n).$$

Noting that $\lambda_j^{\downarrow}(-X)=-\lambda_j^{\uparrow}(X)$ and $\lambda_j^{\downarrow-1}(X)=\lambda_j^{\uparrow}(X^{-1})$ for every positive definite matrix X, we reach

$$\lambda^{\downarrow}(f(\alpha A+(1-\alpha)B))\leq\lambda^{\downarrow}\left(\left(\alpha f(A)^{-1}+(1-\alpha)f(B)^{-1}\right)^{-1}\right).\tag{2.30}$$

Employing the monotone increasing convex function $t\mapsto -f(t^{-1})$ and the monotone decreasing convex function $t\mapsto -1/f(t^{-1})$ and a similar argument as above show that the majorization can be replaced with inequalities in both (2.16) and (2.27). ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 179-194, February 2022.

It has been remarked in [2, 3] that if f is a monotone $\nabla\sharp$-convex (log-convex), then the log-majorization $\prec_{w\log}$ cannot be replaced with an inequality in (2.14). We remark that this is a more general fact, that is, when $\sigma = \sharp$ or $\tau = \sharp$, then the majorizations in (2.11) and (2.12) cannot be replaced with an inequality. We give some examples.

Example 2.2.

1. Let $\sigma = \sharp_\alpha, \tau = \nabla_\alpha$ and $f(t) = 1/\log t$. Then f is $\sharp\nabla$-convex. Put
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad \alpha = 1/3.$$
Then, $\lambda_2^\downarrow\left(f\left(e^{A\nabla_\alpha B}\right)\right) = 0.3094 \nleq 0.2514 = \lambda_2^\downarrow\left(f(e^A)\nabla_\alpha f(e^B)\right)\big)$.
2. Noting that $f(e^A)\nabla_\alpha f(e^B) \geq f(e^A)!_\alpha f(e^B)$ by the Arithmetic-Geometric means inequality, the last example also works for $\sigma = \sharp_\alpha$ and $\tau = !_\alpha$.
3. Let $\sigma = \sharp_\alpha, \tau = \sharp_\alpha$, and $f(t) = e^t$. Then f is an increasing $\sharp\sharp$-convex function. If
$$A = \begin{bmatrix} 1.7854 & 1.7881 \\ 1.7881 & 1.8018 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \alpha = 1/2,$$
then $\lambda_2^\downarrow\left(f\left(e^{A\nabla_\alpha B}\right)\right) = 3.5260 \nleq 3.4761 = \lambda_2^\downarrow\left(f(e^A)\sharp_\alpha f(e^B)\right)$.

The next consequence of Theorem 2.1 gives a better estimate than [3, Corollary 3.8].

Corollary 2.3. *If f is a multiplicatively convex function, that is, $\sharp\sharp$-convex, on $(0,\infty)$, then*

$$\lambda^\downarrow\left\{f\left(\exp\left(A\nabla_\alpha B\right)\right)\right\} \prec_{w\log} \lambda^\downarrow\left(f(\exp A)^\alpha f(\exp B)^{1-\alpha}\right), \tag{2.31}$$

for all $A, B \in \mathbb{P}_n$ and every $\alpha \in [0,1]$.

The next result follows by applying Theorem 2.1 for the $!\sharp$-convex function $f(t) = t^q$ on $(0,\infty)$ for every $q \geq 0$. It was presented in [3, Corollary 3.9] with the weak majorization instead of the weak log-majorization.

Corollary 2.4. *If $r \geq 0$, then*

$$\lambda^\downarrow\left\{\left(\alpha A^{-1} + (1-\alpha)B^{-1}\right)^{-r}\right\} \prec_{w\log} \lambda^\downarrow\left\{A^{\alpha r}B^{(1-\alpha)r}\right\}. \tag{2.32}$$

for all $A, B \in \mathbb{P}_n$ and every $\alpha \in [0,1]$.

The functions $f(t) = t^p$ and $g(t) = -t^q$ are both $\nabla!$-convex functions on $(0,\infty)$, when $p \in [-1,0]$ and $q \in (-\infty,-1) \cup (0,\infty)$. Although we considered all functions to be positive, this particular case will not cause any problem and we get the next result, see [3] and [8, Remark 2.3].

Corollary 2.5. *If $q \geq 0$ or $q \leq -1$, then*

$$\lambda^\downarrow\left(2\left(A^{-q} + B^{-q}\right)^{-1}\right) \leq \lambda^\downarrow\left(2^{-q}(A+B)^q\right). \tag{2.33}$$

Using the well-known Fan Dominance Theorem [4, Theorem IV.2.2], the following norm inequalities hold.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 179-194, February 2022.

Corollary 2.6. *Let $A, B \in \mathbb{P}_n$, $\sigma \in \{\nabla_\alpha, !_\alpha\}, \tau \in \{\nabla_\alpha, \sharp_\alpha, !_\alpha\}$ with $0 \le \alpha \le 1$, $f : J_{A,B} \subseteq (0,\infty) \to (0,\infty)$ and let $|||\cdot|||$ be a unitarily invariant norm on $\mathbb{M}_n$. If f is $\sigma\tau$-convex, then*

$$|||f(A\sigma B)||| \le |||f(A)\tau f(B)|||\,, \tag{2.34}$$

for every $\alpha \in [0,1]$.

On the other hand, if $\sigma = \sharp_\alpha$, then

$$|||f \circ \exp(A\nabla_\alpha B)||| \le \left|\left|\left|f(e^A)\tau f(e^B)\right|\right|\right|. \tag{2.35}$$

The next corollary is a generalization of [5, Theorem 2.2] and the proof is similar to that in [5] by using Lemma 1.9.

Corollary 2.7. *Let $\sigma, \tau \in \{\nabla, \sharp, !\}$, $X \in \mathbb{P}_n$, and $f : J_X \subseteq (0,\infty) \to (0,\infty)$ be a monotone $\sigma\tau$-convex function. Then,*

$$\lambda_k^{\downarrow}\left[f\left(\sigma(C;X)\right)\right] \le \lambda_k^{\downarrow}\left[\tau\left(C; f(X)\right)\right] \qquad (k = 1,\dots,n), \tag{2.36}$$

for every isometry $C \in \mathbb{M}_n$.

Related to the majorization results, Bourin [5] showed that for every positive definite matrix $X \in \mathbb{P}_n$ and every isometry C, if $f : J_X \to \mathbb{R}$ is a convex monotone function, then there exists a unitary U such that the matrix inequality $f\left(\nabla(C;X)\right) \le U^*\nabla\left(C; f(X)\right)U$ holds. As a consequence of our results in this section, we can extend this result to any $\sigma\tau$-convex function as follows.

Corollary 2.8. *Let $\sigma, \tau \in \{\nabla, \sharp, !\}$, $X \in \mathbb{P}_n$, and $f : J_X \subseteq (0,\infty) \to (0,\infty)$ be a monotone $\sigma\tau$-convex function. If $C \in \mathbb{M}_n$ is an isometry, then there exists a unitary U such that the matrix inequality:*

$$f\left(\sigma(C;X)\right) \le U^*\tau\left(C; f(X)\right)U, \tag{2.37}$$

holds.

3. Majorization inequalities for two-variables functions.

Let $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_m$ be positive definite matrices with spectral decompositions $A = \sum_{i=1}^n \lambda_i P_i$ and $B = \sum_{i=1}^m \mu_i Q_i$. When f is a two-variable real function defined on $J_1 \times J_2 \subseteq (0,\infty)\times(0,\infty)$, we can define a new matrix $f(A,B)$ by:

$$f(A,B) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\lambda_i, \mu_j) P_i \otimes Q_j,$$

and so f becomes a matrix function of two variables from $\mathbb{M}_n \times \mathbb{M}_m$ to $\mathbb{M}_{nm}$. We have the next lemma.

Lemma 3.1. *Let $\sigma \in \{\nabla, !, \sharp\}$. If $h : J_A \times J_B \subseteq (0,\infty)\times(0,\infty) \to (0,\infty)$ is separately $\sigma\sigma$-convex, then*

$$h\left(\sigma(u;A), \sigma(v;B)\right) \le \sigma(u \otimes v; h(A,B)), \tag{3.38}$$

for all unit vectors $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$ and all positive definite matrices $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_m$.

Proof. Let $A = \sum_{i=1}^n \lambda_i P_i$ and $B = \sum_{i=1}^m \mu_i Q_i$ be spectral decompositions of the matrices A and B. Assume that $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$ are unit vectors so that $\sum_{i=1}^n u^* P_i u = 1 = \sum_{j=1}^m v^* Q_j v$.

Case I: First, we note that when $\sigma = \nabla$, the inequality (3.38) has been proved in [9].

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Case II: If h is separately !!-convex, then,

$$h\left(!(u;A),!(v;B)\right) = h\left(\left(\sum_{i=1}^{n}{\lambda_i}^{-1}u^*P_iu\right)^{-1}, b\right) \le \left(\sum_{i=1}^{n}h(\lambda_i,b)^{-1}u^*P_iu\right)^{-1}, \tag{3.39}$$

where $b =!(v;B)$ and the inequality follows from the !!-convexity of h in the first variable. Furthermore, for every $i = 1,\dots,n$, the !!-convexity of h in the second variable gives

$$h(\lambda_i,b) = h\left(\lambda_i, \left(\sum_{j=1}^{m}{\mu_j}^{-1}v^*Q_jv\right)^{-1}\right) \le \left(\sum_{j=1}^{m}h(\lambda_i,\mu_j)^{-1}v^*Q_jv\right)^{-1}. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\begin{aligned} h\left(!(u;A),!(v;B)\right) &\le \left(\sum_{i=1}^{n}u^*P_iu\sum_{j=1}^{m}h(\lambda_i,\mu_j)^{-1}v^*Q_jv\right)^{-1} \\ &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m}h(\lambda_i,\mu_j)^{-1}(u\otimes v)^*(P_i\otimes Q_j)(u\otimes v)\right)^{-1} \\ &=!(u\otimes v; h(A,B)), \end{aligned}$$

and we obtain (3.38) in the case where h is separately !!-convex.

Case III: Next suppose that h is separately ♯♯-convex. Utilizing the ♯♯-convexity of h in its first variable, we can write

$$h\left(\sharp(u;A),\sharp(v;B)\right) = h\left(\prod_{i=1}^{n}{\lambda_i}^{u^*P_iu}, b\right) \le \prod_{i=1}^{n}h(\lambda_i,b)^{u^*P_iu}, \tag{3.41}$$

in which $b = \sharp(v;B) = \prod_{j=1}^{m}{\mu_j}^{v^*Q_jv}$. Moreover, using ♯♯-convexity of h in the second variable gives

$$h(\lambda_i,b) = h\left(\lambda_i, \prod_{j=1}^{m}{\mu_j}^{v^*Q_jv}\right) \le \prod_{j=1}^{m}h(\lambda_i,\mu_j)^{v^*Q_jv}, \tag{3.42}$$

for every $i = 1,\dots,n$. From (3.41) and (3.42), we obtain

$$\begin{aligned} h\left(\sharp(u;A),\sharp(v;B)\right) &\le \prod_{i=1}^{n}\prod_{j=1}^{m}h(\lambda_i,\mu_j)^{u^*P_iuv^*Q_jv} \\ &= \prod_{i=1}^{n}\prod_{j=1}^{m}h(\lambda_i,\mu_j)^{(u\otimes v)^*(P_i\otimes Q_j)(u\otimes v)} = \sharp(u\otimes v; h(A,B)), \end{aligned}$$

which completes the proof. □

Remark 3.2. Let us give some examples explaining Lemma 3.1 for perspective functions. We showed in [7] that if f is !!-convex, then the associated perspective function g is !!-convex in its both variables and so (3.38) holds by Lemma 3.1. For example, the function $f(t) = t^r$ is !!-convex for every $r \in [0,1]$ and so is

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 179-194, February 2022.

$g(t,s) = sf(t/s) = s^{1-r}t^r$ is !!-convex in its both variables. Note that in this particular example, we have $g(A,B) = A^r \otimes B^{1-r}$. Now (3.38) implies that

$$\langle A^{-1}\eta, \eta\rangle^{-r}\langle B^{-1}\zeta, \zeta\rangle^{r-1} \leq \left\langle (A^r \otimes B^{1-r})\eta \otimes \zeta, \eta \otimes \zeta \right\rangle.$$

Note that because the function $g(t,s)$ in this example can be decomposed as $g(t,s) = g_1(t)g_2(s)$, the above inequality follows directly from Corollary 2.8.

In the sequel, $h : J \times I \subseteq (0,\infty) \times (0,\infty) \to (0,\infty)$ is said to be jointly $\sigma\sigma$-convex if it satisfies

$$h(a\sigma b, x\sigma y) \leq h(a,x)\sigma h(b,y),\ \forall a,b \in I, x,y \in J.$$

Theorem 3.3. *Let $A, B \in \mathbb{P}_n, X, Y \in \mathbb{P}_m$ and $h : J_{A,B} \times I_{X,Y} \subseteq (0,\infty) \times (0,\infty) \to (0,\infty)$. If h is jointly $\sigma\sigma$-convex, where $\sigma \in \{\nabla_\alpha, !_\alpha\}$, $0 \leq \alpha \leq 1$, then*

$$\lambda^{\downarrow}\left[h\left(A\sigma B, X\sigma Y\right)\right] \prec_w \lambda^{\downarrow}\left[h(A,X)\sigma h(B,Y)\right]. \tag{3.43}$$

If $\sigma = \sharp_\alpha$, then

$$\lambda^{\downarrow}\left[h\left(e^{A\nabla_\alpha B}, e^{X\nabla_\alpha Y}\right)\right] \prec_{w\log} \lambda^{\downarrow}\left[h(e^A, e^X)^\alpha h(e^B, e^Y)^{1-\alpha}\right]. \tag{3.44}$$

Proof. **Case I:** Let $\sigma = \nabla_\alpha$ and let $C = A\nabla_\alpha B$ and $Z = X\nabla_\alpha Y$. Let $\{\lambda_i\}_{i=1}^n$ and $\{\mu_j\}_{j=1}^m$ be the eigenvalues of C and Z, respectively.

Let $\{\gamma_\ell\}_{\ell=1}^{nm}$ be the eigenvalues of the matrix $h(C,Z)$, arranged in a non-increasing order. We notice that for each $1 \leq \ell \leq nm$, there exist two indices i_ℓ and j_ℓ such that

$$\gamma_\ell = h(\lambda_{i_\ell}, \mu_{j_\ell}); 1 \leq i_\ell \leq n, 1 \leq j_\ell \leq m.$$

Since $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{nm}$, we have

$$h(\lambda_{i_1}, \mu_{j_1}) \geq \ldots \geq h(\lambda_{i_{nm}}, \mu_{j_{nm}}),$$

in which i_ℓ, j_ℓ are integers satisfying $i_\ell \in \{1,\ldots,n\}$ and $j_\ell \in \{1,\ldots,m\}$. Note that there are m equal indices i_ℓ and n equal indices j_ℓ.

Now if $1 \leq k \leq nm$, then

$$\sum_{\ell=1}^{k} \lambda_\ell^{\downarrow}(h(C,Z)) = \sum_{\ell=1}^{k} \gamma_\ell = h(\lambda_{i_1}, \mu_{j_1}) + \ldots + h(\lambda_{i_k}, \mu_{j_k}).$$

Now assume that $\{u_1,\ldots,u_n\}$ and $\{v_1,\ldots,v_m\}$ are orthonormal systems of eigenvectors corresponding to $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_m$, respectively. Then,

$$h(\lambda_{i_r}, \mu_{j_s}) = h(u_{i_r}^* C u_{i_r}, v_{j_s}^* Z v_{j_s}), \qquad (i_r \in \{1,\ldots,n\},\ j_s \in \{1,\ldots,m\}).$$

Moreover, from the joint convexity of h, we have

$$\begin{aligned} h(u_{i_r}^* C u_{i_r}, v_{j_s}^* Z v_{j_s}) &\leq \alpha h(u_{i_r}^* A u_{i_r}, v_{j_s}^* X v_{j_s}) + (1-\alpha) h(u_{i_r}^* B u_{i_r}, v_{j_s}^* Y v_{j_s}) \\ &\leq \alpha (u_{i_r} \otimes v_{j_s})^* h(A,X)(u_{i_r} \otimes v_{j_s}) + (1-\alpha)(u_{i_r} \otimes v_{j_s})^* h(B,Y)(u_{i_r} \otimes v_{j_s}), \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 179-194, February 2022.

where the second inequality follows from Lemma 3.1. Therefore,

$$\sum_{\ell=1}^{k}\lambda_\ell^{\downarrow}(h(C,Z)) = h(\lambda_{i_1},\mu_{j_1})+\ldots+h(\lambda_{i_k},\mu_{j_k})$$

$$\begin{aligned}&\le \alpha(u_{i_1}\otimes v_{j_1})^*h(A,X)(u_{i_1}\otimes v_{j_1})+(1-\alpha)(u_{i_1}\otimes v_{j_1})^*h(B,Y)(u_{i_1}\otimes v_{j_1})\\&+\cdots\\&+\alpha(u_{i_p}\otimes v_{j_q})^*h(A,X)(u_{i_p}\otimes v_{j_q})+(1-\alpha)(u_{i_k}\otimes v_{j_k})^*h(B,Y)(u_{i_k}\otimes v_{j_k}).\end{aligned}\tag{3.45}$$

Finally, noting that the set $\{(u_i\otimes v_j);\ i=1,\ldots,n,\ j=1,\ldots,m\}$ is an orthonormal set of vectors in $\mathbb{C}^{nm}$, Lemma 1.7 implies that the right side of (3.45) is dominated by:

$$\sum_{\ell=1}^{k}\lambda_\ell^{\downarrow}\left(\alpha h(A,X)+(1-\alpha)h(B,Y)\right),$$

as required.

Case II: Next suppose that $\sigma=!_\alpha$. If $u\in\mathbb{C}^n$ and $v\in\mathbb{C}^m$ are unit vectors, then

$$\begin{aligned}&h\left(\left(u^*\left(\alpha A^{-1}+(1-\alpha)B^{-1}\right)u\right)^{-1},\left(v^*\left(\alpha X^{-1}+(1-\alpha)Y^{-1}\right)v\right)^{-1}\right)\\&\quad=h\left(\left(\alpha u^*A^{-1}u+(1-\alpha)u^*B^{-1}u\right)^{-1},\left(\alpha v^*X^{-1}v+(1-\alpha)v^*Y^{-1}v\right)^{-1}\right)\\&\quad\le\left(\alpha h\left(\left(u^*A^{-1}u\right)^{-1},\left(v^*X^{-1}v\right)^{-1}\right)^{-1}+(1-\alpha)h\left(\left(u^*B^{-1}u\right)^{-1},\left(v^*Y^{-1}v\right)^{-1}\right)^{-1}\right)^{-1},\end{aligned}\tag{3.46}$$

by the joint !!-convexity of h. Moreover, utilizing Lemma 3.1 with $\sigma=!_\alpha$, we obtain

$$h\left(\left(u^*A^{-1}u\right)^{-1},\left(v^*X^{-1}v\right)^{-1}\right)^{-1}\ge(u\otimes v)^*h(A,X)^{-1}(u\otimes v),\tag{3.47}$$

and

$$h\left(\left(u^*B^{-1}u\right)^{-1},\left(v^*Y^{-1}v\right)^{-1}\right)^{-1}\ge(u\otimes v)^*h(B,Y)^{-1}(u\otimes v).\tag{3.48}$$

Combining (3.46), (3.47) and (3.48), we reach

$$\begin{aligned}&h\left(\nabla\left(u;A^{-1}\nabla_\alpha B^{-1}\right)^{-1},\nabla\left(v;X^{-1}\nabla_\alpha Y^{-1}\right)^{-1}\right)\\&\qquad\le\left((u\otimes v)^*\left(\alpha h(A,X)^{-1}+(1-\alpha)h(B,Y)^{-1}\right)(u\otimes v)\right)^{-1}.\end{aligned}\tag{3.49}$$

Assume that $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_m$ are the eigenvalues of $A!_\alpha B$ and $X!_\alpha Y$, respectively. As in the first case, for every $\ell=1,\cdots,nm$, there are indices i_ℓ and j_ℓ with $i_\ell\in\{1,\ldots,n\}$ and $j_\ell\in\{1,\ldots,m\}$ such that

$$\lambda_\ell^{\downarrow}\{h(A!_\alpha B,X!_\alpha Y)\}=h\left(\lambda_{i_\ell},\mu_{j_\ell}\right).\tag{3.50}$$

On the other hand, $\lambda_{i_\ell}^{-1}$ and $\mu_{j_\ell}^{-1}$ are eigenvalues of $A^{-1}\nabla_\alpha B^{-1}$ and $X^{-1}\nabla_\alpha Y^{-1}$, respectively. Therefore,

$$\lambda_{i_\ell}^{-1}=w_{i_\ell}^*(A^{-1}\nabla_\alpha B^{-1})w_{i_\ell}=\nabla\left(w_{i_\ell};A^{-1}\nabla_\alpha B^{-1}\right),$$

and

$$\mu_{j_\ell}^{-1}=z_{j_\ell}^*(X^{-1}\nabla_\alpha Y^{-1})z_{j_\ell}=\nabla\left(z_{j_\ell};X^{-1}\nabla_\alpha Y^{-1}\right),$$

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where $\{w_1, \ldots, w_n\}$ and $\{z_1, \ldots, z_m\}$ are orthonormal sets of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Therefore,

$$\begin{aligned} h\left(\lambda_{i_\ell}, \mu_{j_\ell}\right) &= h\left(\nabla\left(w_{i_\ell}; A^{-1}\nabla_\alpha B^{-1}\right)^{-1}, \nabla\left(z_{j_\ell}; X^{-1}\nabla_\alpha Y^{-1}\right)^{-1}\right) \\ &\leq \left((w_{i_\ell}\otimes z_{j_\ell})^*\left(\alpha h(A,X)^{-1} + (1-\alpha)h(B,Y)^{-1}\right)(w_{i_\ell}\otimes z_{j_\ell})\right)^{-1}, \end{aligned} \tag{3.51}$$

where the last inequality is obtained from (3.49). It follows from (3.50) and (3.51) that

$$\begin{aligned} \sum_{\ell=1}^{k} \lambda_\ell^{\downarrow}\left\{h(A!_\alpha B, X!_\alpha Y)\right\} &= h(\lambda_{i_1}, \mu_{j_1}) + \ldots + h(\lambda_{i_k}, \mu_{j_k}) \\ &\leq \left((w_{i_1}\otimes z_{j_1})^*\left(\alpha h(A,X)^{-1} + (1-\alpha)h(B,Y)^{-1}\right)(w_{i_1}\otimes z_{j_1})\right)^{-1} \\ &+ \cdots \\ &+ \left((w_{i_k}\otimes z_{j_k})^*\left(\alpha h(A,X)^{-1} + (1-\alpha)h(B,Y)^{-1}\right)(w_{i_k}\otimes z_{j_k})\right)^{-1}. \end{aligned} \tag{3.52}$$

Utilizing (2.26) and the fact that $\{(w_i \otimes z_j);\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ is an orthonormal set of vectors in $\mathbb{C}^{nm}$, we find that the right side of (3.52) is dominated by:

$$\sum_{\ell=1}^{k} \lambda_\ell^{\downarrow}\left(\left(\alpha h(A,X)^{-1} + (1-\alpha)h(B,Y)^{-1}\right)^{-1}\right).$$

This gives the desired result when $\sigma = !_\alpha$.

Case III: Next suppose that $\sigma = \sharp_\alpha$. It is easily seen that h is jointly $\sharp\sharp$-convex if and only if the two-variable function $(t,s) \mapsto \log h(e^t, e^s)$ is jointly convex on $\mathbb{R}^2$. It follows from the first part that

$$\lambda^{\downarrow}\left[\log h\left(e^{A\nabla_\alpha B}, e^{X\nabla_\alpha Y}\right)\right] \prec_w \lambda^{\downarrow}\left[\log h(e^A, e^X)\nabla_\alpha \log h(e^B, e^Y)\right].$$

Note that the matrix:

$$\log h(e^A, e^X)\nabla_\alpha \log h(e^B, e^Y) = \alpha \log h(e^A, e^X) + (1-\alpha)\log h(e^B, e^Y),$$

is majorized by $\log\left(h(e^A, e^X)^{\alpha/2} h(e^B, e^Y)^{1-\alpha} h(e^A, e^X)^{\alpha/2}\right)$, see [3]. It follows that

$$\lambda^{\downarrow}\left[\log h\left(e^{A\nabla_\alpha B}, e^{X\nabla_\alpha Y}\right)\right] \prec_w \lambda^{\downarrow}\left[\log\left(h(e^A, e^X)^{\alpha/2} h(e^B, e^Y)^{1-\alpha} h(e^A, e^X)^{\alpha/2}\right)\right],$$

and so (3.44) holds true. □

Remark 3.4. It should be remarked that when $\sigma = \nabla$, the assertion of Theorem 3.3 has been proved in [9].

It was shown in [7, Theorem 1] that if a real function f is $\sigma\sigma$-convex, then the associated perspective function $g(t,s) = sf(t/s)$ is jointly $\sigma\sigma$-convex. For example, the function $f(t) = t/\log t$ is !!-convex on $(0,\infty)$ and so $g(t,s) = \frac{t}{\log t - \log s}$ is jointly !!-convex. Hence, Theorem 3.3 concludes that

$$\frac{A!_\alpha B}{\log A!_\alpha B - \log X!_\alpha Y} \prec_w \left(\frac{A}{\log A - \log X}\right)!_\alpha \left(\frac{B}{\log B - \log Y}\right),$$

holds for all $A, B \in \mathbb{P}_n, X, Y \in \mathbb{P}_m$. As another example, it was shown in [7] that the *Hellinger distance* $H^2(\mathbf{p},\mathbf{q}) := 2\sum_{j=1}^{n}\left(\sqrt{p_j} - \sqrt{q_j}\right)^2$ is jointly $\sharp\sharp$-convex. Theorem 3.3 gives

$$\left(\left(e^{A\nabla_\alpha B}\right)^{1/2} - \left(e^{X\nabla_\alpha Y}\right)^{1/2}\right)^2 \prec_{w\log} \left(\left(e^A\right)^{1/2} - \left(e^X\right)^{1/2}\right)^{2\alpha}\left(\left(e^B\right)^{1/2} - \left(e^Y\right)^{1/2}\right)^{2(1-\alpha)}.$$

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REFERENCES

[1] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen. Generalized convexity and inequalities. *J. Math. Anal. Appl.*, 335:1294–1308, 2007.
[2] J.S. Aujla and J.-C. Bourin. Eigenvalue inequalities for convex and log-convex functions. *Linear Algebra Appl.*, 424:25–35, 2007.
[3] J.S. Aujla and F.C. Silva. Weak majorization inequalities and convex functions. *Linear Algebra Appl.*, 369:217–233, 2003.
[4] R. Bhatia. *Matrix Analysis*. Springer-Verlag, New York, 1997.
[5] J.-C. Bourin. Convexity or concavity inequalities for Hermitian operators. *Math. Inequal. Appl.*, 7:607–620, 2004.
[6] T. Furuta, J. Mićić, J. Pećarić, and Y. Seo. *Mond–Pećarić Method in Operator Inequalities*, Element, Zagreb, 2005.
[7] M. Kian. How type of convexity of the core function affects the Csiszár f-divergence functional. *Math. Ineq. Appl.*, arXiv:2101.02934.
[8] M. Kian and M. Bakherad. A new estimation for eigenvalues of matrix power functions. *Analysis Math.*, 45:527–534, 2019.
[9] J.S. Matharu and J.S. Aujla. Some majorization inequalities for convex functions of several variables. *Math. Inequal. Appl.*, 14:947–956, 2011.
[10] C.P. Niculescu and L.-E. Persson. *Convex Functions and Their Applications. A Contemporary Approach*, 2nd edition. CMS Books in Mathematics, Vol. 23. Springer-Verlag, New York, 2018.