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PARTIAL ISOMETRIES AND EP ELEMENTS IN RINGS WITH INVOLUTION*

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Abstract. If $\mathcal{R}$ is a ring with involution, and $a^\dagger$ is the Moore-Penrose inverse of $a \in \mathcal{R}$, then the element a is called: EP, if $aa^\dagger = a^\dagger a$; partial isometry, if $a^* = a^\dagger$; star-dagger, if $a^*a^\dagger = a^\dagger a^*$. In this paper, characterizations of partial isometries, EP elements and star-dagger elements in rings with involution are given. Thus, some well-known results are extended to more general settings.

Key words. Partial isometry, Moore-Penrose inverse, Group inverse, EP elements, Star-dagger elements, Ring with involution.

AMS subject classifications. 16B99, 16W10, 15A09, 46L05.

1. Introduction. Let $\mathcal{R}$ be an associative ring with the unit 1, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a.$$

Recall that $a^\#$ is uniquely determined by these equations [2]. We use $\mathcal{R}^\#$ to denote the set of all group invertible elements of $\mathcal{R}$. If a is invertible, then $a^\#$ coincides with the ordinary inverse of a.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ satisfying $aa^* = a^*a$ is called *normal*. An element $a \in \mathcal{R}$ satisfying $a = a^*$ is called *Hermitian* (or *symmetric*). In the rest of the paper, we assume that $\mathcal{R}$ is a ring with involution.

We say that $b = a^\dagger$ is the *Moore–Penrose inverse* (or *MP-inverse*) of a, if the following hold [10]:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

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There is at most one b such that above conditions hold (see [5, 7, 10]), and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

An element $a \in \mathcal{R}^\dagger$ satisfying $a^* = a^\dagger$ is called *a partial isometry*. An element $a \in \mathcal{R}^\dagger$ satisfying $a^*a^\dagger = a^\dagger a^*$ is called *star–dagger* [6].

Definition 1.1. [8] An element $a \in \mathcal{R}$ is **-cancelable* if

$$a^*ax = 0 \Rightarrow ax = 0 \quad \text{and} \quad xaa^* = 0 \Rightarrow xa = 0. \tag{1.1}$$

Applying the involution to (1.1), we observe that a is *-cancelable if and only if a^* is *-cancelable. In C^*-algebras, all elements are *-cancelable.

Theorem 1.2. [8] *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^\dagger$ if and only if a is *-cancelable and a^*a is group invertible.*

Theorem 1.3. [4, 9] *For any $a \in \mathcal{R}^\dagger$, the following are satisfied:*

(a) $(a^\dagger)^\dagger = a$;
(b) $(a^*)^\dagger = (a^\dagger)^*$;
(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;
(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;
(f) $a^* = a^\dagger aa^* = a^*aa^\dagger$;
(g) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger = (a^*a)^\# a^* = a^*(aa^*)^\#$;
(h) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

In this paper, we will use the following definition of EP elements [8].

Definition 1.4. An element a of a ring $\mathcal{R}$ with involution is said to be *EP* if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$.

Lemma 1.5. *An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$.*

We observe that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ if and only if $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ (see [8]) and a is EP if and only if a^* is EP. In [8], the equality $(a^*)^\# = (a^\#)^*$ is proved.

Theorem 1.6. [8] *An element $a \in \mathcal{R}$ is EP if and only if a is group invertible and $a^\# a$ is symmetric.*

In particular, if $a \in \mathcal{R}^\dagger$, then $(aa^*)^\dagger = (aa^*)^\#$ and aa^* is EP.

Previous results are also contained in [4].

Lemma 1.7. [9] *If $a \in \mathcal{R}^\dagger$ is normal, then a is EP.*

Theorem 1.8. [9] *Suppose that $a \in \mathcal{R}^\dagger$. Then a is normal if and only if $a \in \mathcal{R}^\#$*

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and one of the following equivalent conditions holds:

(i) $aa^*a^\# = a^\# aa^*$;
(ii) $aaa^* = aa^*a$.

In paper [1], O.M. Baksalary, G.P.H. Styan and G. Trenkler used the representation of complex matrices provided in [6] to explore various classes of matrices, such as partial isometries, EP and star-dagger elements. Inspired by [1], in this paper we use a different approach, exploiting the structure of rings with involution to investigate partial isometries, EP and star-dagger elements. We give several characterizations, and the proofs are based on ring theory only. The paper is organized as follows: In Section 2, characterizations of MP-invertible or both MP-invertible and group invertible partial isometries in rings with involution are given. In Section 3, star-dagger, group invertible and EP elements in rings with involution are investigated.

2. Characterizations of partial isometries. In the following theorem, we present some equivalent conditions for the Moore-Penrose invertible element a of a ring with involution to be a partial isometry.

THEOREM 2.1. *Suppose that $a \in \mathcal{R}^\dagger$. The following statements are equivalent:*

(i) *a is a partial isometry;*
(ii) $aa^* = aa^\dagger$;
(iii) $a^*a = a^\dagger a$.

Proof. (i) ⇒ (ii): If a is a partial isometry, then $a^* = a^\dagger$. So, $aa^* = aa^\dagger$ and the condition (ii) holds.

(ii) ⇒ (iii): Suppose that $aa^* = aa^\dagger$. Then we get the following:

$$a^*a = a^\dagger(aa^*)a = a^\dagger aa^\dagger a = a^\dagger a.$$

Hence, the condition (iii) is satisfied.

(iii) ⇒ (i): Applying the equality $a^*a = a^\dagger a$, we obtain

$$a^* = a^*aa^\dagger = a^\dagger aa^\dagger = a^\dagger.$$

Thus, the element a is a partial isometry. □

Since for $a \in \mathcal{R}^\dagger$, the equalities $a^* = a^*aa^\dagger = a^\dagger aa^*$ hold, we deduce that a is a partial isometry if and only if $a^*aa^\dagger = a^\dagger$, or if and only if $a^\dagger aa^* = a^\dagger$.

In the following theorem, we assume that the element a is both Moore–Penrose invertible and group invertible. Then, we study the conditions involving $a^\dagger$, $a^\#$ and a^* to ensure that a is a partial isometry. Theorems 2.1 and 2.2 are inspired by Theorem 1 in [1].

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THEOREM 2.2. *Suppose that $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$. Then a is a partial isometry if and only if one of the following equivalent conditions holds:*

(i) $a^*a^\# = a^\dagger a^\#$;
(ii) $a^\# a^* = a^\# a^\dagger$;
(iii) $aa^*a^\# = a^\#$;
(iv) $a^\# a^* a = a^\#$.

Proof. If a is a partial isometry, then $a^* = a^\dagger$. It is not difficult to verify that conditions (i)-(iv) hold.

Conversely, to conclude that a is a partial isometry, we show that either the condition $a^* = a^\dagger$ is satisfied, or one of the preceding already established condition of this theorem holds.

(i) By the equality $a^*a^\# = a^\dagger a^\#$, we get

$$a^* = a^*aa^\dagger = a^*aa^\# aa^\dagger = a^*a^\# aaa^\dagger = a^\dagger a^\# aaa^\dagger = a^\dagger aa^\dagger = a^\dagger.$$

(ii) The equality $a^\# a^* = a^\# a^\dagger$ gives

$$a^* = a^\dagger aa^* = a^\dagger aaa^\# a^* = a^\dagger aaa^\# a^\dagger = a^\dagger aa^\dagger = a^\dagger.$$

(iii) Multiplying $aa^*a^\# = a^\#$ by $a^\dagger$ from the left side, we obtain

$$a^*a^\# = a^\dagger a^\#.$$

Thus, the condition (i) is satisfied, so a is a partial isometry.

(iii) Multiplying $a^\# a^* a = a^\#$ by $a^\dagger$ from the right side, we get

$$a^\# a^* = a^\# a^\dagger.$$

Hence, the equality (ii) holds, and a is a partial isometry. □

In the following theorem, we give necessary and sufficient conditions for an element a of a ring with involution to be a partial isometry and EP. It should be mentioned that the following result generalizes Theorem 2 in [1].

THEOREM 2.3. *Suppose that $a \in \mathcal{R}^\dagger$. Then a is a partial isometry and EP if and only if $a \in \mathcal{R}^\#$ and one of the following equivalent conditions holds:*

(i) *a is a partial isometry and normal;*
(ii) $a^* = a^\#$;
(iii) $aa^* = a^\dagger a$;
(iv) $a^*a = aa^\dagger$;

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(v) $aa^* = aa^\#$;
(vi) $a^*a = aa^\#$;
(vii) $a^*a^\dagger = a^\dagger a^\#$;
(viii) $a^\dagger a^* = a^\# a^\dagger$;
(ix) $a^\dagger a^* = a^\dagger a^\#$;
(x) $a^*a^\dagger = a^\# a^\dagger$;
(xi) $a^*a^\# = a^\# a^\dagger$;
(xii) $a^*a^\dagger = a^\# a^\#$;
(xiii) $a^*a^\# = a^\dagger a^\dagger$;
(xiv) $a^*a^\# = a^\# a^\#$;
(xv) $aa^*a^\dagger = a^\dagger$;
(xvi) $aa^*a^\dagger = a^\#$;
(xvii) $aa^*a^\# = a^\dagger$;
(xviii) $aa^\dagger a^* = a^\dagger$;
(xix) $a^*a^2 = a$;
(xx) $a^2a^* = a$;
(xxi) $aa^\dagger a^* = a^\#$;
(xxii) $a^*a^\dagger a = a^\#$.

Proof. If $a \in \mathcal{R}^\dagger$ is a partial isometry and EP, then $a \in \mathcal{R}^\#$ and $a^* = a^\dagger = a^\#$. It is not difficult to verify that conditions (i)-(xxii) hold.

Conversely, we assume that $a \in \mathcal{R}^\#$. We known that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ if and only if $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$, and a is EP if and only if a^* is EP. We will prove that a is a partial isometry and EP, or we will show that the element a or a^* satisfies one of the preceding already established conditions of this theorem. If a^* satisfies one of the preceding already established conditions of the theorem, then a^* is a partial isometry and EP and so a is a partial isometry and EP.

(i) If a is a partial isometry and normal, then by Lemma 1.7, a is a partial isometry and EP.

(ii) From the condition $a^* = a^\#$, we obtain

$$aa^* = aa^\# = a^\# a = a^*a.$$

So, element a is normal. Then, by Lemma 1.7, a is EP and, by definition, $a^\# = a^\dagger$. Hence, $a^* = a^\# = a^\dagger$, i.e., a is a partial isometry.

(iii) Suppose that $aa^* = a^\dagger a$. Then

$$a^\# aa^* = a^\# a^\dagger a = (a^\#)^2 aa^\dagger a = (a^\#)^2 a = a^\#, \tag{2.1}$$

which implies

$$aa^*a^\# = a(a^\# aa^*)a^\# = aa^\# a^\# = a^\#. \tag{2.2}$$

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From the equalities (2.1) and (2.2), we get $aa^*a^\# = a^\# aa^*$. Now, by Theorem 1.8, a is normal. Then a is EP by Lemma 1.7, and

$$aa^\dagger = a^\dagger a = aa^*,$$

by (iii). Thus, by the condition (ii) of Theorem 2.1, a is a partial isometry.

(iv) Applying the involution to $a^*a = aa^\dagger$, we obtain

$$a^*(a^*)^* = (a^\dagger)^* a^* = (a^*)^\dagger a^*,$$

by Theorem 1.3. Hence, a^* satisfies the condition (iii).

(v) The equality $aa^* = aa^\#$ yields

$$aaa^* = aaa^\# = aa^\# a = aa^*a.$$

Therefore, a is normal by Theorem 1.8. From Lemma 1.7, a is EP and, by definition, $a^\# = a^\dagger$. Now, by (v), $aa^* = aa^\dagger$ and, by the condition (ii) of Theorem 2.1, a is a partial isometry.

(vi) Applying the involution to $a^*a = aa^\#$, we get

$$a^*(a^*)^* = (a^\#)^* a^* = (a^*)^\# a^* = a^*(a^*)^\#,$$

by the equality $(a^\#)^* = (a^*)^\#$ [8]. Thus, a^* satisfies the equality (v).

(vii) Assume that $a^*a^\dagger = a^\dagger a^\#$. Then

$$\begin{aligned} aa^\# &= aa(a^\#)^2 = aaa^\dagger a(a^\#)^2 = a^2(a^\dagger a^\#) = a^2a^*a^\dagger \\ &= a^2(a^*a^\dagger)aa^\dagger = a^2a^\dagger a^\# aa^\dagger = aaa^\dagger aa^\# a^\dagger \\ &= a^2a^\# a^\dagger = aa^\dagger. \end{aligned}$$

Since $aa^\dagger$ is symmetric, $aa^\#$ is symmetric too. By Theorem 1.6, a is EP and $a^\# = a^\dagger$. Then, by (vii), $a^*a^\# = a^\dagger a^\#$, i.e., the condition (i) of Theorem 2.2 is satisfied. Hence, a is a partial isometry.

(viii) Applying the involution to $a^\dagger a^* = a^\# a^\dagger$, we have

$$(a^*)^*(a^\dagger)^* = (a^\dagger)^*(a^\#)^*,$$

i.e.,

$$(a^*)^*(a^*)^\dagger = (a^*)^\dagger(a^*)^\#.$$

So, a^* satisfies the condition (vii).

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(ix) The condition $a^\dagger a^* = a^\dagger a^\#$ implies

$$\begin{aligned} aa^\# &= aa(a^\#)^2 = aaa^\dagger a(a^\#)^2 = a^2(a^\dagger a^\#) = a^2 a^\dagger a^* \\ &= a^2(a^\dagger a^*)aa^\dagger = a^2 a^\dagger a^\# aa^\dagger = aaa^\dagger aa^\# a^\dagger \\ &= a^2 a^\# a^\dagger = aa^\dagger. \end{aligned}$$

Thus, $aa^\#$ is symmetric, and by Theorem 1.6 a is EP. From $a^\dagger = a^\#$ and (ix) we get $a^\dagger a^* = a^\# a^\dagger$, i.e., the equality (viii) holds.

(x) Applying the involution to $a^* a^\dagger = a^\# a^\dagger$, we get

$$(a^\dagger)^*(a^*)^* = (a^\dagger)^*(a^\#)^*,$$

which gives

$$(a^*)^\dagger(a^*)^* = (a^*)^\dagger(a^*)^\#,$$

i.e., a^* satisfies the condition (ix).

(xi) Using the assumption $a^* a^\# = a^\# a^\dagger$, we have

$$a^*a = (a^*a^\#)a^2 = a^\# a^\dagger a^2 = (a^\#)^2 aa^\dagger a^2 = (a^\#)^2 a^2 = a^\# a = aa^\#.$$

Hence, the condition (vi) is satisfied.

(xii) If $a^* a^\dagger = a^\# a^\#$, then (x) holds, since

$$a^*a^\dagger = (a^*a^\dagger)aa^\dagger = a^\# a^\# aa^\dagger = a^\# a^\dagger.$$

(xiii) By the equality $a^* a^\# = a^\dagger a^\dagger$, we obtain

$$a^*aa^\# a^\dagger = (a^*a^\#)aa^\dagger = a^\dagger a^\dagger aa^\dagger = a^\dagger a^\dagger = a^*a^\# = a^*a(a^\#)^2,$$

which implies

$$a^*a(a^\# a^\dagger - (a^\#)^2) = 0. \tag{2.3}$$

Since $a \in \mathcal{R}^\dagger$, a is *-cancelable by Theorem 1.2. From (2.3) and *-cancellation, we get $a(a^\# a^\dagger - (a^\#)^2) = 0$, i.e.,

$$aa^\# a^\dagger = a^\#. \tag{2.4}$$

Multiplying (2.4) by a from the left side, we have

$$aa^\dagger = aa^\#.$$

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Therefore, $aa^{\#}$ is symmetric, so a is EP by Theorem 1.6. Now, from $a^\dagger = a^{\#}$ and (xiii) we get $a^*a^{\#} = a^{\#}a^\dagger$, i.e., the condition (xi) is satisfied.

(xiv) The assumption $a^*a^{\#} = a^{\#}a^{\#}$ gives

$$a^*a = (a^*a^{\#})aa = a^{\#}a^{\#}aa = a^{\#}a = aa^{\#}.$$

So, the equality (vi) holds.

(xv) From $aa^*a^\dagger = a^\dagger$, we get

$$\begin{aligned} aa^* &= a^{\#}aaa^* = a^{\#}(aa^*a^*)^* = a^{\#}(aa^*a^\dagger aa^*)^* \\ &= a^{\#}(a^\dagger aa^*)^* = a^{\#}(a^*)^* = a^{\#}a = aa^{\#}. \end{aligned}$$

Thus, the equality (v) is satisfied.

(xvi) Multiplying $aa^*a^\dagger = a^{\#}$ by $a^\dagger$ from the left side, we get

$$a^*a^\dagger = a^\dagger a^{\#}.$$

Therefore, the condition (vii) holds.

(xvii) Multiplying $aa^*a^{\#} = a^\dagger$ by $a^\dagger$ from the left side, we obtain the condition (xiii).

(xviii) Suppose that $aa^\dagger a^* = a^\dagger$. Then

$$\begin{aligned} aa^\dagger a^\dagger a &= aa^\dagger(a^\dagger a)^* = (aa^\dagger a^*)(a^\dagger)^* = a^\dagger(a^\dagger)^* \\ &= a^\dagger(aa^\dagger a^*)^* = a^\dagger a(aa^\dagger)^* = a^\dagger aaa^\dagger. \end{aligned}$$

Now, from this equality and (xviii), we have

$$\begin{aligned} a^{\#}a^* &= a^{\#}a^{\#}aa^* = (a^{\#})^2aa(a^{\#})^2aa^* = (a^{\#})^2a(a^\dagger aaa^\dagger)a(a^{\#})^2aa^* \\ &= a^{\#}aa^\dagger a^\dagger aa^{\#}aa^* = a^{\#}aa^\dagger a^\dagger aa^* = a^{\#}(aa^\dagger a^*) = a^{\#}a^\dagger. \end{aligned}$$

Hence, the equality (ii) of Theorem 2.2 holds and a is a partial isometry. From $a^* = a^\dagger$ and (xviii), we obtain

$$aa^*a^\dagger = aa^\dagger a^* = a^\dagger,$$

i.e., the equality (xv) is satisfied.

(xix) Multiplying $a^*a^2 = a$ by $a^{\#}$ from the right side, we get

$$a^*a = aa^{\#}.$$

So, the condition (vi) holds.

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(xx) Multiplying $a^2a^* = a$ by $a^\#$ from the left side, we have

$$aa^* = a^\# a = aa^\#.$$

Thus, the equality (v) is satisfied.

(xxi) Multiplying $aa^\dagger a^* = a^\#$ by $a^\dagger$ from the left side, we obtain

$$a^\dagger a^* = a^\dagger a^\#.$$

Hence, a satisfies the condition (ix).

(xxii) Multiplying $a^*a^\dagger a = a^\#$ by $a^\dagger$ from the right side, we get

$$a^*a^\dagger = a^\# a^\dagger.$$

Therefore, the condition (x) holds. □

The following result is well-known for complex matrices (see [1, Theorem 1]). However, we are not in a position to prove this result for elements of a ring with involution, so we state it as a conjecture.

Conjecture. Suppose that $a \in \mathcal{R}^\dagger$. Then a is a partial isometry if and only if one of the following equivalent conditions holds:

(i) $a^*aa^* = a^\dagger$;
(ii) $aa^*aa^*a = a$.

3. EP, star-dagger and group-invertible elements. First, we state the following result concerning sufficient conditions for Moore-Penrose invertible element a in ring with involution to be star–dagger. This result is proved for complex matrices in [1].

THEOREM 3.1. *Suppose that $a \in \mathcal{R}^\dagger$. Then each of the following conditions is sufficient for a to be star–dagger:*

(i) $a^* = a^*a^\dagger$;
(ii) $a^* = a^\dagger a^*$;
(iii) $a^\dagger = a^\dagger a^\dagger$;
(iv) $a^* = a^\dagger a^\dagger$;
(v) $a^\dagger = a^*a^*$.

Proof. (i) Using the equation $a^* = a^*a^\dagger$, we get

$$a^*aa^\dagger = a^* = a^*a^\dagger = a^*aa^\dagger a^\dagger,$$

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i.e.,

$$a^*a(a^\dagger - a^\dagger a^\dagger) = 0. \tag{3.1}$$

From $a \in \mathcal{R}^\dagger$, by Theorem 1.2, we know that a is *-cancelable. Then, by (3.1) and *-cancelation, we have

$$a(a^\dagger - a^\dagger a^\dagger) = 0,$$

which gives

$$aa^\dagger = aa^\dagger a^\dagger. \tag{3.2}$$

Now, by (i) and (3.2),

$$a^*a^\dagger = a^* = a^\dagger aa^* = a^\dagger(aa^\dagger)aa^* = a^\dagger aa^\dagger a^\dagger aa^* = a^\dagger a^*.$$

(ii) Applying the involution to $a^* = a^\dagger a^*$, we obtain

$$(a^*)^* = (a^*)^*(a^\dagger)^* = (a^*)^*(a^*)^\dagger.$$

Since the condition (i) holds for a^*, we deduce that a^* is star–dagger. Thus, $(a^*)^*(a^*)^\dagger = (a^*)^\dagger(a^*)^*$, i.e.,

$$a(a^\dagger)^* = (a^\dagger)^*a. \tag{3.3}$$

Applying the involution to (3.3), we get $a^\dagger a^* = a^*a^\dagger$.

(iii) The condition $a^\dagger = a^\dagger a^\dagger$ implies

$$a^*a^\dagger = a^*a(a^\dagger a^\dagger) = a^*aa^\dagger = a^* = a^\dagger aa^* = a^\dagger a^\dagger aa^* = a^\dagger a^*.$$

(iv) From the equality $a^* = a^\dagger a^\dagger$, we have

$$a^*a^\dagger = a^*a(a^\dagger a^\dagger) = a^*aa^* = a^\dagger a^\dagger aa^* = a^\dagger a^*.$$

(v) If $a^\dagger = a^*a^*$, then

$$a^*a^\dagger = a^*a^*a^* = a^\dagger a^*. \qquad \square$$

Now, we prove an alternative characterization of the group inverse in a ring. This result is proved for complex matrices in [1], where the authors use the rank of a matrix.

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THEOREM 3.2. *Let $\mathcal{R}$ be an associative ring with the unit* 1*, and let $a \in \mathcal{R}$. Then $b \in \mathcal{R}$ is the group inverse of a if and only if*

$$ba^2 = a, \quad a^2b = a, \quad b\mathcal{R} = ba\mathcal{R}.$$

Proof. If $b = a^{\#}$, then, by definition, $a = ba^2 = a^2b$. It is clear that $ba\mathcal{R} \subseteq b\mathcal{R}$. To show that $b\mathcal{R} \subseteq ba\mathcal{R}$, we assume that $y \in b\mathcal{R}$. Then $y = bx$ for some $x \in \mathcal{R}$. Since $bab = b$, we have $y = bx = babx \in ba\mathcal{R}$. Hence, $b\mathcal{R} = ba\mathcal{R}$.

Suppose that $ba^2 = a, a^2b = a, b\mathcal{R} = ba\mathcal{R}$. Now, $ab = ba^2b = ba$ and $aba = baa = a$. Since $b = b1 \in b\mathcal{R} = ba\mathcal{R}$, then $b = bax$ for some $x \in \mathcal{R}$. Thus, $b = bax = ba^2bx = ba(bax) = bab$ and $b = a^{\#}$. □

Finally, we prove the result involving EP elements in a ring.

THEOREM 3.3. *Suppose that $a, b \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *$aba = a$ and a is EP;*
(ii) *$a \in \mathcal{R}^\dagger \cap \mathcal{R}^{\#}$ and $a^\dagger = a^\dagger ba$;*
(iii) *$a \in \mathcal{R}^\dagger \cap \mathcal{R}^{\#}$ and $a^* = a^* ba$;*
(iv) *$a \in \mathcal{R}^\dagger \cap \mathcal{R}^{\#}$ and $a^* = aba^*$;*
(v) *$a \in \mathcal{R}^\dagger \cap \mathcal{R}^{\#}$ and $a^\dagger = aba^\dagger$.*

Proof. (i) ⇒ (ii): Let $aba = a$ and let a be EP. We get

$$a^\dagger = a^{\#} = (a^{\#})^2 a = (a^{\#})^2 aba = a^{\#} ba = a^\dagger ba,$$

i.e., the condition (ii) holds.

(ii) ⇒ (iii): From $a^\dagger = a^\dagger ba$, we get

$$a^* = a^* aa^\dagger = a^* aa^\dagger ba = a^* ba.$$

Therefore, the condition (iii) is satisfied.

(iii) ⇒ (ii): The condition $a^* = a^* ba$ is equivalent to

$$a^* aa^\dagger = a^* aa^\dagger ba,$$

which implies

$$a^* a(a^\dagger - a^\dagger ba) = 0. \tag{3.4}$$

From $a \in \mathcal{R}^\dagger$ and Theorem 1.2, it follows that a is *-cancelable. Thus, by (3.4) and *-cancelation, $a(a^\dagger - a^\dagger ba) = 0$ which yields

$$aa^\dagger = aa^\dagger ba. \tag{3.5}$$

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Multiplying (3.5) by $a^\dagger$ from the left side, we obtain $a^\dagger = a^\dagger ba$. So, the condition (ii) holds.

(ii) $\Rightarrow$ (i): If $a^\dagger = a^\dagger ba$, then

$$aa^\# = aa^\dagger aa^\# = aa^\dagger baaa^\# = aa^\dagger ba = aa^\dagger.$$

Hence, $aa^\#$ is symmetric. By Theorem 1.6, a is EP and $a^\# = a^\dagger$. Now, by (ii) we get $a^\# = a^\# ba$, and consequently, $a = a^2a^\# = a^2a^\# ba = aba$. Thus, the condition (i) is satisfied.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i): These implications can be proved analogously. □

Notice that in the case of complex matrices, the equivalencies (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) are proved in [3], and the equivalencies (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(v) are proved in [1].

4. Conclusions. In this paper we consider Moore-Penrose invertible or both Moore-Penrose invertible and group invertible elements in rings with involution to characterize partial isometries, EP and star-dagger elements in terms of equations involving their adjoints, Moore-Penrose and group inverses. All of these results are already known for complex matrices. However, we demonstrated a new technique in proving the results. In the theory of complex matrices, various authors used an elegant representation of complex matrices and the matrix rank to characterize partial isometries, EP elements and star-dagger. In this paper, we applied a purely algebraic technique, involving different characterizations of the Moore-Penrose inverse.

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