

ON THE CHARACTERIZATION OF GRAPHS WITH PENDENT VERTICES AND GIVEN NULLITY*

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Abstract. Let G be a graph with n vertices. The nullity of G , denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in its spectrum. In this paper, we characterize the graphs (resp. bipartite graphs) with pendent vertices and nullity η , where $0 < \eta \leq n$. Moreover, the minimum (resp. maximum) number of edges for all (connected) graphs with pendent vertices and nullity η are determined, and the extremal graphs are characterized.

Key words. Eigenvalue, Nullity, Pendent vertex.

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1. Introduction. Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, the degree and neighborhood of v are denoted by $d(v)$ and $N(v)$, respectively. If W is a nonempty subset of $V(G)$, then the subgraph induced by W is the subgraph of G obtained by taking the vertices in W and joining those pairs of vertices in W which are joined in G . We write $G - \{v_1, v_2, \dots, v_k\}$ for the graph obtained from G by removing the vertices $v_1, v_2, \dots, v_k$ and all edges incident to any of them.

The *disjoint union* of two graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. The disjoint union of k copies of G is often written by kG . The *null graph* of order n is the graph with n vertices and no edges. As usual, the complete graph, the cycle, the path, and the star of order n are denoted by K_n , C_n , P_n and S_n , respectively. An isolated vertex is sometimes denoted by K_1 .

Let t (≥ 2) be an integer. A graph G is called *t-partite* if $V(G)$ admits a partition into t classes $X_1, X_2, \dots, X_t$ such that every edge has its ends in different classes; vertices in the same partition must not be adjacent. Such a partition $(X_1, X_2, \dots, X_t)$ is called a *t-partition* of G . A *complete t-partite graph* is a simple *t-partite* graph with partition $(X_1, X_2, \dots, X_t)$ in which each vertex of X_i is joined to each vertex of $G - X_i$ ($1 \leq i \leq t$). If $|X_i| = n_i$ ($1 \leq i \leq t$), such a graph is denoted by $K_{n_1, n_2, \dots, n_t}$.

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Instead of “2-partite” (resp. “3-partite”) one usually says *bipartite* (resp. *tripartite*).

The *adjacency matrix* $A(G)$ of a graph G of order n , with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, is $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. A graph is said to be *singular* (resp. *nonsingular*) if its adjacency matrix is a singular (resp. nonsingular) matrix. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$ are said to be the *eigenvalues* of G , and to form the *spectrum* of this graph. The number of zero eigenvalues in the spectrum of a graph G is called its *nullity* and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$. Obviously, $\eta(G) = n - r(A(G))$. The rank of a graph G is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then $\eta(G) = n - r(G)$. Clearly, if G is a simple connected graph, then $0 \leq r(G) \leq |V(G)| \leq |E(G)| + 1$.

The problem of characterizing all graphs G with $\eta(G) > 0$ was posed in [1] and [10]. This problem is relevant in many disciplines of science (see [2, 3]), and is very difficult. At present, only some particular cases are known (see [3-9, 11-12]). On the other hand, this problem is of great interest in chemistry, because, for a bipartite graph G (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates that the molecule which such a graph represents is unstable (see [8]). The nullity of a graph G is also meaningful in linear algebra, since it is related to the singularity and the rank of $A(G)$.

It is known that $0 \leq \eta(G) \leq n - 2$ if G is a simple graph on n vertices and G is not isomorphic to nK_1 . In [4], B. Cheng and B. Liu characterized the extremal graphs attaining the upper bound $n - 2$ and the second upper bound $n - 3$.

LEMMA 1.1. ([4]) Suppose that G is a simple graph of order n . Then

(1) $\eta(G) = n - 2$ if and only if G is isomorphic to $K_{n_1, n_2} \cup kK_1$, where $n_1 + n_2 + k = n$ (≥ 2) and $n_1, n_2 > 0, k \geq 0$.

(2) $\eta(G) = n - 3$ if and only if G is isomorphic to $K_{n_1, n_2, n_3} \cup kK_1$, where $n_1 + n_2 + n_3 + k = n$ (≥ 3) and $n_1, n_2, n_3 > 0, k \geq 0$.

As a continuation, S. Li ([9]) determined the extremal graphs with pendent vertices which achieve the third upper bound $n - 4$ and fourth upper bound $n - 5$, respectively. Recently, Y. Fan and K. Qian ([6]) characterized all bipartite graphs of order n with nullity $n - 4$.

DEFINITION 1.2. ([6]) Let $P_n = v_1v_2 \cdots v_n$ ($n \geq 2$) be a path. Replacing each vertex v_i by an empty graph O_{m_i} of order m_i for $i = 1, 2, \dots, n$ and joining edges between each vertex of O_i and each vertex of O_{i+1} for $i = 1, 2, \dots, n - 1$, we get a graph G of order $(m_1 + m_2 + \cdots + m_n)$, denoted by $O_{m_1}O_{m_2} \cdots O_{m_n}$. Such graph is called an expanded path of length n , and the empty graph O_{m_i} is called an expanded

vertex of order m_i for $i = 1, 2, \dots, n$.

LEMMA 1.3. ([6]) *Let G be a bipartite graph of order $n \geq 4$. Then $\eta(G) = n - 4$ if and only if G is isomorphic to a graph H possibly adding some isolated vertices, where H is one of the following graphs: a union of two disjoint expanded paths both of length 2, an expanded path of length 4 or 5.*

In Section 2 of this paper, we give a characterization of the graphs (resp. connected graphs) with pendent vertices and nullity η ($0 < \eta \leq n$). As corollaries of this characterization, some results in [9] can be obtained immediately. Moreover, all bipartite graphs (resp. bipartite connected graphs) with pendent vertices and nullity $\eta = n - 2k$ are characterized. (It is known from [6] that the nullity set of all bipartite graphs of order n is $\{n - 2k \mid k = 0, 1, \dots, \lfloor n/2 \rfloor\}$.)

Let $\Gamma(n, e)$ be the set of all simple graphs with n vertices and e edges. In [4], the maximum nullity number of graphs with n vertices and e edges, $M(n, e) = \max\{\eta(A) \mid A \in \Gamma(n, e)\}$, was studied, where $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$. Conversely, we shall study the number of edges for the graphs with pendent vertices and nullity η ($0 < \eta \leq n$). Let $e_{\min}^{(\eta)}$ and $e_{\max}^{(\eta)}$ ($\tilde{e}_{\min}^{(\eta)}$ and $\tilde{e}_{\max}^{(\eta)}$) denote the minimum and maximum number of edges for all (connected) graphs with pendent vertices and nullity η . Let $G_{\min}^{(\eta)}$ (resp. $\tilde{G}_{\min}^{(\eta)}$) denote the graphs (resp. connected graphs) of nullity η with pendent vertices and $e_{\min}^{(\eta)}$ (resp. $\tilde{e}_{\min}^{(\eta)}$) edges. We call $G_{\min}^{(\eta)}$ (resp. $\tilde{G}_{\min}^{(\eta)}$) the minimum graphs (resp. connected graphs) with pendent vertices and nullity η . Similarly, we can define $G_{\max}^{(\eta)}$ (resp. $\tilde{G}_{\max}^{(\eta)}$), the maximum graphs (resp. connected graphs) with pendent vertices and nullity η . In Section 3, we determine the number $e_{\min}^{(\eta)}$, $e_{\max}^{(\eta)}$, $\tilde{e}_{\min}^{(\eta)}$, $\tilde{e}_{\max}^{(\eta)}$ and characterize the graphs $G_{\min}^{(\eta)}$, $G_{\max}^{(\eta)}$, $\tilde{G}_{\min}^{(\eta)}$, $\tilde{G}_{\max}^{(\eta)}$, respectively. Now we list some known results needed in this paper.

LEMMA 1.4. ([12]) *Let G be a simple graph of order n . Then*

(1) $\eta(G) = n$ if and only if G is a null graph.

(2) If $G = G_1 \cup G_2 \cup \dots \cup G_t$, where $G_1, G_2, \dots, G_t$ are the connected components of G , then $\eta(G) = \sum_{i=1}^t \eta(G_i)$.

LEMMA 1.5. ([9]) *Let v be a pendent vertex of a graph G and u be the vertex in G adjacent to v . Then $\eta(G) = \eta(G - \{u, v\})$.*

LEMMA 1.6. ([4])

$$r(P_n) = \begin{cases} n-1, & n \text{ is odd;} \\ n, & \text{otherwise.} \end{cases} \quad r(C_n) = \begin{cases} n-2, & n \equiv 0 \pmod{4}; \\ n, & \text{otherwise.} \end{cases}$$

2. The graphs with pendent vertices and nullity η . Let η be an integer with $0 < \eta \leq n$. Now the graphs with pendent vertices and nullity η are characterized

as follows, where $n - 3 \leq \eta \leq n$.

LEMMA 2.1. *Let G be a simple graph of order n with pendent vertices. Then*

- (1) *There exists no such graph G with nullity $\eta(G) = n, n - 1$ or $n - 3$;*
- (2) *$\eta(G) = n - 2$ if and only if G is isomorphic to $S_{n-k} \cup kK_1$ ($0 \leq k \leq n - 2$).*

Proof. (1) Obviously, there exists no such graph G with nullity $\eta(G) = n - 1$. Moreover, by Lemmas 1.1 and 1.4, the graph G of nullity $\eta(G) = n$ (resp. $n - 3$) contains no pendent vertices. This leads to the desired results.

(2) Since the graph G has pendent vertices, combining this with Lemma 1.1, $\eta(G) = n - 2$ if and only if G is isomorphic to $K_1, n_2 \cup kK_1$, where $1 + n_2 + k = n$ and $n_2 > 0, k \geq 0$. This completes the proof. $\square$

Now we give a characterization of the graphs with pendent vertices and nullity η for $0 < \eta \leq n - 4$. Let $\tilde{\Upsilon}_n^{(\eta)}$ be the set of all connected graphs of order n with nullity η ($0 \leq \eta \leq n$). Then it follows from Lemmas 1.1 and 1.4 that $\tilde{\Upsilon}_n^{(n)} = \tilde{\Upsilon}_n^{(n-1)} = \emptyset$, $\tilde{\Upsilon}_n^{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, \text{ and } n_1, n_2 > 0\}$, $\tilde{\Upsilon}_n^{(n-3)} = \{K_{n_1, n_2, n_3} \mid n_1 + n_2 + n_3 = n, \text{ and } n_1, n_2, n_3 > 0\}$.

Let n, k, t be positive integers with $4 \leq k < n$ and $1 \leq t \leq \lfloor \frac{k}{2} \rfloor - 1$, and let p, n_j, p_j ($1 \leq j \leq t$) be integers with $n_j \geq p_j > 1$ ($1 \leq j \leq t$), $\sum_{j=1}^t p_j + 2 = k$, $\sum_{j=1}^t n_j + p + 2 = n$. Let $H_{n, k}$ be any graph of order n created from $H_j \in \tilde{\Upsilon}_{n_j}^{(n_j-p_j)}$ ($j = 1, 2, \dots, t$), pK_1 and K_2 (suppose $V(K_2) = \{u, v\}$) by connecting v to all vertices of pK_1 and H_j ($j = 1, 2, \dots, t$) (see Figure 1.). Suppose that E^* is a subset of $E(G)$. Let $G\{E^*\}$ (resp. $\tilde{G}\{E^*\}$) denote the (resp. connected) spanning subgraph of G which contains the edges in E^* .

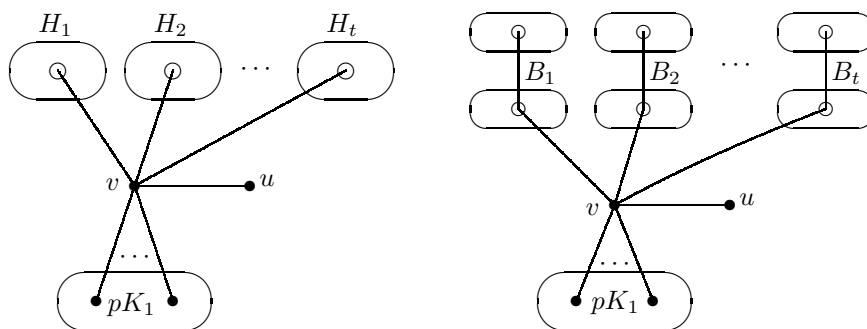


Figure 1. $H_{n, k}$ and $B_{n, k}$

THEOREM 2.2. *Let G be a graph (resp. connected graph) of order n with pendent vertices. Then $\eta(G) = n - k$ ($4 \leq k < n$) if and only if G is isomorphic to $H_{n, k}\{E^*\}$ (resp. $\widetilde{H_{n, k}\{E^*\}}$), where $E^* = \cup_{j=1}^t E(H_j) \cup \{uv\}$.*

Proof. To begin with, we need to check that $\eta(H_{n, k}\{E^*\}) = \eta(\widetilde{H_{n, k}\{E^*\}}) = n - k$ ($4 \leq k < n$). Note that u is a pendent vertex of $H_{n, k}\{E^*\}$ (resp. $\widetilde{H_{n, k}\{E^*\}}$) and $N(u) = \{v\}$. Delete u, v from $H_{n, k}\{E^*\}$ (resp. $\widetilde{H_{n, k}\{E^*\}}$), then the resultant graph is $(\cup_{j=1}^t H_j) \cup pK_1$. Since $H_j \in \widetilde{\Upsilon}_{n_j}^{(n_j - p_j)}$, we have $\eta(H_j) = n_j - p_j$ ($j = 1, 2, \dots, t$). Hence by Lemmas 1.4 and 1.5,

$$\begin{aligned} \eta(H_{n, k}\{E^*\}) &= \eta(\widetilde{H_{n, k}\{E^*\}}) = \eta((\cup_{j=1}^t H_j) \cup pK_1) = \sum_{j=1}^t \eta(H_j) + p \cdot \eta(K_1) \\ &= \sum_{j=1}^t (n_j - p_j) + p = (\sum_{j=1}^t n_j + p + 2) - (\sum_{j=1}^t p_j + 2) = n - k. \end{aligned}$$

On the other hand, assume that $\eta(G) = n - k$. Choose a pendent vertex, say x , in G . Let $N(x) = \{y\}$. Delete x, y from G , and let the resultant graph be $G_1 = G_{11} \cup G_{12} \cup \dots \cup G_{1q}$, where $G_{11}, G_{12}, \dots, G_{1q}$ are connected components of G_1 . Some of these components may be trivial, i.e. K_1 . We conclude that there exist t nontrivial connected components, where $1 \leq t \leq \lfloor \frac{k}{2} \rfloor - 1$. Without loss of generality, assume that $G_{11}, G_{12}, \dots, G_{1t}$ be nontrivial. By contradiction, suppose that $t = 0$ or $t \geq \lfloor \frac{k}{2} \rfloor$.

Case 1. $t = 0$. Then all the connected components are trivial, adding x, y to G_1 gives a star with some isolated vertices, which contradicts to Lemma 2.1.

Case 2. $t \geq \lfloor \frac{k}{2} \rfloor$. By Lemmas 1.1, 1.4 and 1.5, $\eta(G) = \sum_{j=1}^t \eta(G_{1j}) + z\eta(K_1) \leq \sum_{j=1}^t (|V(G_{1j})| - 2) + z$, where z is the number of isolated vertices in G_1 . The above equality holds iff $G_{11}, \dots, G_{1t}$ are all complete bipartite graphs.

Therefore, $\eta(G) \leq \sum_{j=1}^t |V(G_{1j})| - 2t + z = (n - 2 - z) - 2t + z = n - 2t - 2 < n - k$ for $t \geq \lfloor \frac{k}{2} \rfloor$, contradicting that $\eta(G) = n - k$.

Hence $1 \leq t \leq \lfloor \frac{k}{2} \rfloor - 1$. Let $|V(G_{1j})| = n_j$ ($j = 1, 2, \dots, t$). Then $G_1 = (\cup_{j=1}^t G_{1j}) \cup (n - \sum_{j=1}^t n_j - 2)K_1$. It follows from Lemmas 1.4 and 1.5 that

$$n - k = \eta(G) = \eta(G_1) = \eta(\cup_{j=1}^t G_{1j}) + \eta((n - \sum_{j=1}^t n_j - 2)K_1).$$

Since G_{1j} ($j = 1, 2, \dots, t$) are nontrivial connected components, suppose that $\eta(G_{1j}) = n_j - p_j$, where $1 < p_j \leq n_j$ ($j = 1, 2, \dots, t$). Thus we have

$$n - k = \sum_{j=1}^t (n_j - p_j) + (n - \sum_{j=1}^t n_j - 2).$$

Hence $\sum_{j=1}^t p_j + 2 = k$ and $G_{1j} \in \widetilde{\Upsilon}_{n_j}^{(n_j - p_j)}$ ($j = 1, 2, \dots, t$).

Let $p = n - \sum_{j=1}^t n_j - 2$. In order to recover G , to add x, y to G_1 , we need

to insert edges from y to x and to some (maybe partial or all) vertices of pK_1 and G_{1j} ($j = 1, 2, \dots, t$). Thus the graph (resp. connected graph) G is isomorphic to $H_{n,k}\{E^*\}$ (resp. $\widetilde{H_{n,k}\{E^*\}}$), where $E^* = \cup_{j=1}^t E(H_j) \cup \{uv\}$. $\square$

Now we have the following corollaries of this characterization.

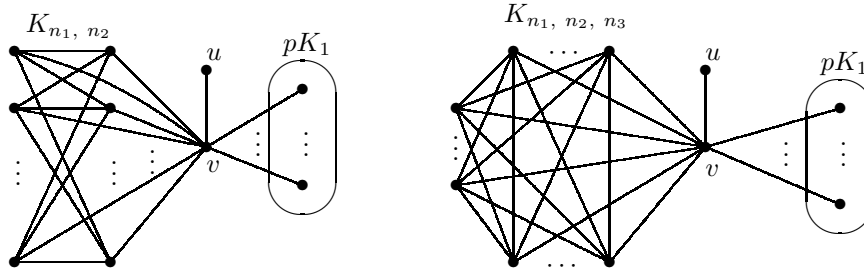


Figure 2. Q_1 and Q_2

Let Q_1 be a graph of order n created from K_{n_1, n_2} , pK_1 and K_2 (suppose $V(K_2) = \{u, v\}$) with $n_1 + n_2 + p + 2 = n$ and $n_1, n_2 > 0, p \geq 0$ by connecting v to all vertices of pK_1 and K_{n_1, n_2} . Let Q_2 be a graph of order n created from K_{n_1, n_2, n_3} , pK_1 and K_2 ($V(K_2) = \{u, v\}$) with $n_1 + n_2 + n_3 + p + 2 = n$ and $n_1, n_2, n_3 > 0, p \geq 0$ by connecting v to all vertices of pK_1 and K_{n_1, n_2, n_3} (see Figure 2.).

COROLLARY 2.3. *Let G be a graph (resp. connected graph) of order n with pendent vertices. Then*

(1) $\eta(G) = n - 4$ if and only if G is isomorphic to $Q_1\{E^*\}$ (resp. $\widetilde{Q_1\{E^*\}}$), where $E^* = E(K_{n_1, n_2}) \cup \{uv\}$.

(2) $\eta(G) = n - 5$ if and only if G is isomorphic to $Q_2\{E^*\}$ (resp. $\widetilde{Q_2\{E^*\}}$), where $E^* = E(K_{n_1, n_2, n_3}) \cup \{uv\}$.

Proof. By Theorem 2.2, $\eta(G) = n - k = n - 4$ implies $t = 1, p_1 = 2$, while $\eta(G) = n - k = n - 5$ implies $t = 1, p_1 = 3$. Besides, $\widetilde{\Upsilon}_n^{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, \text{ and } n_1, n_2 > 0\}$, $\widetilde{\Upsilon}_n^{(n-3)} = \{K_{n_1, n_2, n_3} \mid n_1 + n_2 + n_3 = n, \text{ and } n_1, n_2, n_3 > 0\}$. Then we obtain the results as desired. $\square$

Remark. If G is connected, the results of Corollary 2.3 are that in [9].

Now we shall determine all bipartite graphs with pendent vertices and nullity $\eta = n - 2k$ ($k = 0, 1, \dots, \lfloor n/2 \rfloor$). Since $S_{n-k} \cup kK_1$ ($0 \leq k \leq n - 2$) is a bipartite graph, combining Lemma 2.1, the following corollary is obvious.

COROLLARY 2.4. *Let G be a bipartite graph of order n with pendent vertices. Then*

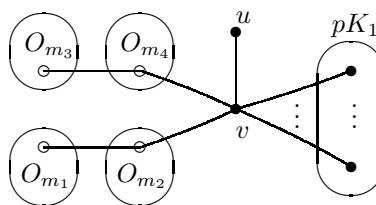
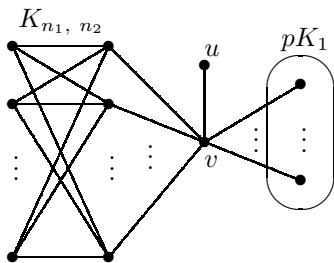
- (1) There exists no such graphs G with nullity $\eta(G) = n$;
 (2) $\eta(G) = n - 2$ if and only if G is isomorphic to $S_{n-k} \cup kK_1$ ($0 \leq k \leq n - 2$).

Let $\tilde{\Phi}_n^{(\eta)}$ be the set of all connected bipartite graphs of order n with nullity $\eta = n - 2k$ ($k = 0, 1, \dots, \lfloor n/2 \rfloor$). It is easy to see that $\tilde{\Phi}_n^{(n)} = \emptyset$, $\tilde{\Phi}_n^{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, n_1, n_2 > 0\}$. Let n, k, t be positive integers such that k is even, $4 \leq k < n$, and $1 \leq t \leq \frac{k}{2} - 1$. Let p, n_j, p_j ($1 \leq j \leq t$) be integers such that p_j is even, $n_j \geq p_j > 1$ ($1 \leq j \leq t$), $\sum_{j=1}^t p_j + 2 = k$, $\sum_{j=1}^t n_j + p + 2 = n$. Let $B_{n, k}$ be a graph of order n created from $B_j \in \tilde{\Phi}_{n_j}^{(n_j - p_j)}$ ($j = 1, 2, \dots, t$), pK_1 and K_2 (suppose $V(K_2) = \{u, v\}$) by connecting v to all vertices of pK_1 and to all vertices in one partite set of B_j ($j = 1, 2, \dots, t$) (also see Figure 1.).

THEOREM 2.5. Let G be a bipartite graph (resp. connected graph) of order n with pendent vertices. Then $\eta(G) = n - k$ (k is even and $4 \leq k < n$) if and only if G is isomorphic to $B_{n, k}\{E^*\}$ (resp. $\widetilde{B_{n, k}\{E^*\}}$), where $E^* = \cup_{j=1}^t E(B_j) \cup \{uv\}$.

Proof. Note that $B_{n, k}\{E^*\}$ (resp. $\widetilde{B_{n, k}\{E^*\}}$) is a bipartite graph. The proof is now analogous to that of Theorem 2.2. $\square$

Let Q_3 be a graph of order n created from K_{n_1, n_2} , pK_1 and K_2 (suppose $V(K_2) = \{u, v\}$) with $n_1 + n_2 + p + 2 = n$ and $n_1, n_2 > 0, p \geq 0$ by connecting v to all vertices of pK_1 and all vertices in one partite set of K_{n_1, n_2} . Let Q_4 be a graph of order n created from $O_{m_1}O_{m_2}, O_{m_3}O_{m_4}, pK_1$ and K_2 ($V(K_2) = \{u, v\}$) with $m_i > 0$ ($i = 1, \dots, 4$), $p \geq 0$ and $\sum_{i=1}^4 m_i + p + 2 = n$ by connecting v to all vertices of O_{m_1} (or O_{m_2}), O_{m_3} (or O_{m_4}) and pK_1 . Let Q_5 be a graph of order n created from $O_{m_1}O_{m_2}O_{m_3}O_{m_4}, pK_1$ and K_2 ($V(K_2) = \{u, v\}$) with $m_i > 0$ ($i = 1, \dots, 4$), $p \geq 0$ and $\sum_{i=1}^4 m_i + p + 2 = n$ by connecting v to all vertices of pK_1, O_{m_1}, O_{m_3} (or pK_1, O_{m_2}, O_{m_4}). Let Q_6 be a graph of order n created from $O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}, pK_1$ and K_2 ($V(K_2) = \{u, v\}$) with $m_i > 0$ ($i = 1, \dots, 5$), $p \geq 0$ and $\sum_{i=1}^5 m_i + p + 2 = n$ by connecting v to all vertices of $pK_1, O_{m_1}, O_{m_3}, O_{m_5}$ (or pK_1, O_{m_2}, O_{m_4}) (see Figure 3.).



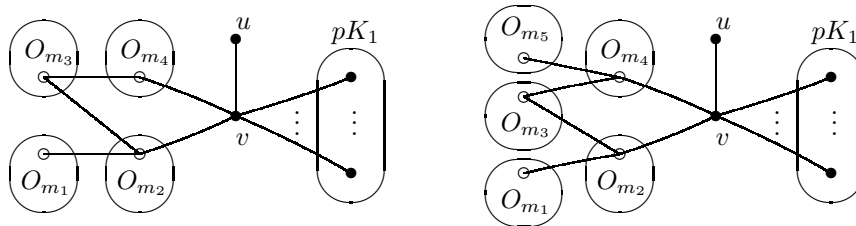


Figure 3. Q_3 , Q_4 , Q_5 and Q_6

COROLLARY 2.6. Let G be a bipartite graph (resp. connected graph) of order n with pendent vertices. Then

(1) $\eta(G) = n - 4$ if and only if G is isomorphic to $Q_3\{E^*\}$ (resp. $\widetilde{Q}_3\{E^*\}$), where $E^* = E(K_{n_1, n_2}) \cup \{uv\}$.

(2) $\eta(G) = n - 6$ if and only if G is isomorphic to $Q_4\{E_1^*\}$, $Q_5\{E_2^*\}$ or $Q_6\{E_3^*\}$ (resp. $\widetilde{Q}_4\{E_1^*\}$, $\widetilde{Q}_5\{E_2^*\}$ or $\widetilde{Q}_6\{E_3^*\}$), where $E_1^* = E(O_{m_1}O_{m_2}) \cup E(O_{m_3}O_{m_4}) \cup \{uv\}$, $E_2^* = E(O_{m_1}O_{m_2}O_{m_3}O_{m_4}) \cup \{uv\}$, $E_3^* = E(O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}) \cup \{uv\}$.

Proof. (1) Note that $\eta(G) = n - 4$ implies $t = 1$, $p_1 = 2$. Since $\widetilde{\Phi}_n^{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, \text{ and } n_1, n_2 > 0\}$, by Theorem 2.5, the result follows.

(2) Notice that $\eta(G) = n - 6$ implies the following two cases: Case 1. $t = 1$, $p_1 = 4$; Case 2. $t = 2$, $p_1 = 2$, $p_2 = 2$. By Lemma 1.3, we have $\widetilde{\Phi}_n^{(n-4)} = \{O_{m_1}O_{m_2}O_{m_3}O_{m_4}, O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}\}$, $\widetilde{\Phi}_n^{(n-2)} = \{O_{m_1}O_{m_2}\}$ (Here $\sum m_i = n$). Thus the results are obtained by applying Theorem 2.5 to Cases 1 and 2. $\square$

3. The minimum and maximum (connected) graphs with pendent vertices and nullity η . In this section, we shall determine the number $e_{min}^{(\eta)}$, $e_{max}^{(\eta)}$, $\widetilde{e}_{min}^{(\eta)}$, $\widetilde{e}_{max}^{(\eta)}$ and characterize $G_{min}^{(\eta)}$, $G_{max}^{(\eta)}$, $\widetilde{G}_{min}^{(\eta)}$, $\widetilde{G}_{max}^{(\eta)}$ for $0 < \eta \leq n$.

Note that there exists no graph G of order n with pendent vertices and nullity $\eta(G) = n$, $n - 1$, $n - 3$ by Lemma 2.1, so we exclude these three cases.

THEOREM 3.1. $G_{min}^{(n-2k)} \cong kK_2 \cup (n - 2k)K_1$, $e_{min}^{(n-2k)} = k$, where $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.

Proof. Suppose $|E(G_{min}^{(n-2k)})| = i$ and there are j nontrivial connected components $G_{11}, G_{12}, \dots, G_{1j}$ of $G_{min}^{(n-2k)}$. Then $j \leq i$.

Claim 1. $|E(G_{min}^{(n-2k)})| = k$. By contradiction, suppose $i \leq k - 1$.

Note that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ ($t = 1, 2, \dots, j$). It follows that

$$r(G_{min}^{(n-2k)}) = \sum_{t=1}^j r(G_{1t}) \leq \sum_{t=1}^j |V(G_{1t})| \leq \sum_{t=1}^j |E(G_{1t})| + j = i + j \leq 2i \leq 2k - 2.$$

Hence $\eta(G_{min}^{(n-2k)}) = n - r(G_{min}^{(n-2k)}) \geq n - 2k + 2$, a contradiction.

Hence $i \geq k$. Note that $\eta(kK_2 \cup (n-2k)K_1) = n - 2k$, and $|E(kK_2 \cup (n-2k)K_1)| = k$, then we have $|E(G_{min}^{(n-2k)})| = k$.

Claim 2. There are k nontrivial connected components of $G_{min}^{(n-2k)}$.

Since $|E(G_{min}^{(n-2k)})| = k$, we have $j \leq k$. Assume that $j \leq k - 1$.

Notice that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ ($t = 1, 2, \dots, j$), hence

$$r(G_{min}^{(n-2k)}) = \sum_{t=1}^j r(G_{1t}) \leq \sum_{t=1}^j |E(G_{1t})| + j = k + j \leq 2k - 1.$$

It is a contradiction that $n - 2k = \eta(G_{min}^{(n-2k)}) = n - r(G_{min}^{(n-2k)}) \geq n - 2k + 1$.

Hence $j = k$. Combining Claims 1 and 2, $G_{min}^{(n-2k)}$ is isomorphic to a graph with k edges and k nontrivial connected components. Clearly, $G_{min}^{(n-2k)} \cong kK_2 \cup (n-2k)K_1$, and $e_{min}^{(n-2k)} = |E(G_{min}^{(n-2k)})| = k$, where $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. $\square$

THEOREM 3.2. $G_{min}^{(n-2k-1)} \cong K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$, and $e_{min}^{(n-2k-1)} = k + 2$, where $k = 2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor$.

Proof. Suppose that $|E(G_{min}^{(n-2k-1)})| = i$ and there are j nontrivial connected components $G_{11}, G_{12}, \dots, G_{1j}$ of $G_{min}^{(n-2k-1)}$.

Claim 1. There are at most k nontrivial connected components of $G_{min}^{(n-2k-1)}$.

By contradiction, suppose $j \geq k + 1$. By Lemma 1.4, $\eta(G_{1t}) \leq |V(G_{1t})| - 2$ ($t = 1, 2, \dots, j$) and $\eta(G_{min}^{(n-2k-1)}) = \sum_{t=1}^j \eta(G_{1t}) + z$, where z is the number of isolated vertices of $G_{min}^{(n-2k-1)}$. Hence $n - 2k - 1 = \eta(G_{min}^{(n-2k-1)}) = \sum_{t=1}^j \eta(G_{1t}) + z \leq \sum_{t=1}^j (|V(G_{1t})| - 2) + z \leq n - 2j \leq n - 2k - 2$, a contradiction.

Claim 2. $|E(G_{min}^{(n-2k-1)})| = k + 2$.

Note that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ ($t = 1, 2, \dots, j$). Thus

$$r(G_{min}^{(n-2k-1)}) = \sum_{t=1}^j r(G_{1t}) \leq \sum_{t=1}^j |V(G_{1t})| \leq \sum_{t=1}^j |E(G_{1t})| + j = i + j.$$

It follows that

$$n - 2k - 1 = \eta(G_{min}^{(n-2k-1)}) = n - r(G_{min}^{(n-2k-1)}) \geq n - i - j.$$

Hence $i + j \geq 2k + 1$. Since $j \leq k$ by Claim 1, we have $i \geq k + 1$.

If $i = k + 1$, then $j = k$. Thus $G_{min}^{(n-2k-1)} \cong K_{1,2} \cup (k-1)K_2 \cup (n-2k-1)K_1$. However, $\eta(K_{1,2} \cup (k-1)K_2 \cup (n-2k-1)K_1) = n - 2k \neq n - 2k - 1$.

Thus $i \geq k + 2$. Note that $\eta(K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1) = n - 2k - 1$, and $|E(K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1)| = k + 2$. Then $|E(G_{min}^{(n-2k-1)})| = k + 2$.

By Claim 2, $|E(G_{min}^{(n-2k-1)})| = i = k + 2$, and it follows that $i + j = (k + 2) + j \geq 2k + 1$. Combining this with Claim 1, we have $j = k - 1$ or k .

Case 1. $j = k - 1$. First we show that there is no nontrivial connected components which are isomorphic to P_3 . Suppose to the contrary that $G_{11} \cong P_3$.

Note that $r(P_3) = 2$ by Lemma 1.6 and $\sum_{t=2}^j |E(G_{1t})| = k$. Hence

$$\begin{aligned} r(G_{min}^{(n-2k-1)}) &= r(P_3) + \sum_{t=2}^j r(G_{1t}) \\ &\leq r(P_3) + \sum_{t=2}^j |V(G_{1t})| \leq r(P_3) + \sum_{t=2}^j |E(G_{1t})| + (j-1) = 2k. \end{aligned}$$

Thus $n - 2k - 1 = \eta(G_{min}^{(n-2k-1)}) = n - r(G_{min}^{(n-2k-1)}) \geq n - 2k$, a contradiction.

Therefore, $G_{min}^{(n-2k-1)}$ may be isomorphic to one of the following:

- (1) $T_1 = C_4 \cup (k-2)K_2 \cup (n-2k)K_1$;
- (2) $T_2 = P_4 \cup (k-2)K_2 \cup (n-2k-1)K_1$;
- (3) $T_3 = T^* \cup (k-2)K_2 \cup (n-2k)K_1$, where T^* is a graph of order 4 created from C_3 and K_2 by identifying a vertex of C_3 with a vertex of K_2 ;
- (4) $T_4 = T^{**} \cup (k-2)K_2 \cup (n-2k-1)K_1$, where T^{**} is a graph of order 5 created from K_2 and S_3 by connecting the center of S_3 to a vertex of K_2 ;
- (5) $T_5 = S_5 \cup (k-2)K_2 \cup (n-2k-1)K_1$.

By Lemmas 1.4 and 1.6, we get $\eta(T_1) = \eta(T_5) = n - 2k + 2 \neq n - 2k - 1$, $\eta(T_2) = \eta(T_3) = \eta(T_4) = n - 2k \neq n - 2k - 1$. Hence $j \neq k - 1$.

Case 2. $j = k$. $G_{min}^{(n-2k-1)}$ may be isomorphic to one of the following:

- (1) $U_1 = K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$;
- (2) $U_2 = K_{1,3} \cup (k-1)K_2 \cup (n-2k-2)K_1$;
- (3) $U_3 = P_4 \cup (k-1)K_2 \cup (n-2k-2)K_1$;
- (4) $U_4 = 2K_{1,2} \cup (k-2)K_2 \cup (n-2k-2)K_1$.

It is not difficult to check that $\eta(U_1) = n - 2k - 1$, $\eta(U_2) = \eta(U_4) = n - 2k \neq n - 2k - 1$, $\eta(U_3) = n - 2k - 2 \neq n - 2k - 1$.

All in all, $G_{min}^{(n-2k-1)} \cong U_1 = K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$, and $e_{min}^{(n-2k-1)} = k+2$, where $k = 2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor$. $\square$

Let S_{n_j} be a star of order n_j , where $j = 1, 2, \dots, k$ and $\sum_{j=1}^k n_j = n$. Let $S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}$ denote a tree of order n created from S_{n_j} ($j = 1, 2, \dots, k$) by adding $k-1$ edges to connect these stars, but the connection of two non-center vertices (not the center of a star) is not permitted. It is easy to see that $S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_p}$ ($2 \leq p \leq k$) can be constructed recurrently by connecting the center of S_{n_p} to one vertex of $S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_{p-1}}$.

Now $\tilde{G}_{min}^{(n-2k)}$ can be characterized for $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ as follows.

THEOREM 3.3. $\tilde{G}_{min}^{(n-2k)} \cong S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}$, $\tilde{e}_{min}^{(n-2k)} = n-1$, where $\sum_{j=1}^k n_j = n$ and $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.

Proof. On one hand, by the definition of $S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}$, there is a pendent vertex u_{n_k} which is adjacent to the center of S_{n_k} . Then

$$\begin{aligned} \eta(S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}) &= \eta(S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_{k-1}}) + \eta((n_k-2)K_1) \\ &= \eta(S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_{k-1}}) + (n_k-2) \\ &= \dots = \eta(S_{n_1}) + \sum_{i=2}^k (n_i-2) = n-2k. \end{aligned}$$

On the other hand we prove that $\tilde{G}_{min}^{(n-2k)}$ is isomorphic to $S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}$ by induction on k , where $\sum_{j=1}^k n_j = n$ and $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$.

For $k = 1$, by Lemma 2.1, $\tilde{G}_{min}^{(n-2)} \cong S_n$. Thus, the statement holds in this case. Suppose the statement holds for $k \leq p-1$. Now we consider the case of $k = p$, where $2 \leq p \leq \lfloor \frac{n}{2} \rfloor$.

Claim 1. It's obvious that for any connected graph of order n , the minimum connected graph is a tree which has $n-1$ edges.

Claim 2. If T is a tree of order n with $\eta(T) = n-l$, then l is even.

Note that a tree T could be decomposed into t (with possibly $t = 0$) isolated vertices by deleting a pendent vertex and its adjacent vertex from T (and its resultant graph, suppose s times) recurrently. Hence $r(T) = r(tK_1) + 2s = 2s$, and then $\eta(T) = n - r(T) = n - 2s$. Therefore, $l = 2s$ is even.

Notice that $\tilde{G}_{min}^{(n-2p)}$ has pendent vertices and $\eta(\tilde{G}_{min}^{(n-2p)}) = n-2p$. Choose a pendent vertex, say x , in $\tilde{G}_{min}^{(n-2p)}$. Let $N(x) = \{y\}$. Delete x, y from $\tilde{G}_{min}^{(n-2p)}$, and

let the resultant graph be $\tilde{G}_1 = \tilde{G}_{11} \cup \tilde{G}_{12} \cup \cdots \cup \tilde{G}_{1q} \cup zK_1$, where $\tilde{G}_{1j}$ are nontrivial connected components of order n_j^* ($j = 1, 2, \dots, q$), and $\sum_{j=1}^q n_j^* + z + 2 = n$.

By the definition of $\tilde{G}_{min}^{(n-2p)}$ and Claim 1, each nontrivial connected component $\tilde{G}_{1j}$ should be a tree with $n_j^* - 1$ edges ($j = 1, 2, \dots, q$). Moreover, it follows from Claim 2 that we suppose $\eta(\tilde{G}_{1j}) = n_j^* - p_j$, where p_j is even and $0 < p_j \leq n_j^*$ ($1 \leq j \leq q$). By Theorem 2.2, we have $\sum_{j=1}^q p_j + 2 = 2p$.

Let $p_j = 2k_j$, and then $k_j = \frac{p_j}{2} \leq p - 1$ ($j = 1, 2, \dots, q$). According to the inductive assumption, since $\eta(\tilde{G}_{1j}) = n_j^* - 2k_j$, each $\tilde{G}_{1j}$ is isomorphic to $S_{n_{j1}^*} \oplus S_{n_{j2}^*} \oplus \cdots \oplus S_{n_{jk_j}^*}$, where $\sum_{i=1}^{k_j} n_{ji}^* = n_j^*$ ($1 \leq j \leq q$).

In order to recover the connected graph $\tilde{G}_{min}^{(n-2p)}$, to add x, y to $\tilde{G}_1$, we need to insert edges from y to each of z isolated vertices of $\tilde{G}_1$ and x . This gives a star $K_1, z+1 = S_{z+2}$. Moreover, we shall connect the vertex y (namely, the center of S_{z+2}) to one vertex of each $\tilde{G}_{1j}$ ($j = 1, 2, \dots, q$). So $\tilde{G}_{min}^{(n-2p)}$ is a tree of order n created from $S_{n_{ji}^*}$ ($i = 1, 2, \dots, k_j; j = 1, 2, \dots, p$) and S_{z+2} by adding $\sum_{j=1}^q k_j = p - 1$ edges to connect these stars, and any two non-center vertices are not connected since y is the center of S_{z+2} .

All in all, it follows from the induction that $\tilde{G}_{min}^{(n-2k)} \cong S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, and then $\tilde{e}_{min}^{(n-2k)} = n - 1$, where $\sum_{j=1}^k n_j = n$ and $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. $\square$

Let C_{2h+1} be a $(2h+1)$ -cycle and let S_{n_j} be a star of order n_j , where $1 \leq h < k$, $1 \leq j \leq k - h$ and $(2h+1) + \sum_{j=1}^{k-h} n_j = n$. Let $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$ denote a unicyclic connected graph of order n created from C_{2h+1} ($1 \leq h < k$) and S_{n_j} ($j = 1, 2, \dots, k-h$) by adding $k-h$ edges to connect them, but the connection of two non-center vertices is not permitted. It is easy to see that $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_p}$ ($1 \leq p \leq k-h$) can be constructed recurrently by connecting the center of S_{n_p} to one vertex of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{p-1}}$.

THEOREM 3.4. $\tilde{G}_{min}^{(n-2k-1)} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$, $\tilde{e}_{min}^{(n-2k-1)} = n$, where $1 \leq h < k$, $(2h+1) + \sum_{j=1}^{k-h} n_j = n$ and $k = 2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor$.

Proof. By the definition of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$,

$$\begin{aligned} \eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}) &= \eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h-1}}) + \eta((n_{k-h} - 2)K_1) \\ &= \cdots = \eta(C_{2h+1}) + \sum_{i=1}^{k-h} (n_i - 2) = 0 + (\sum_{i=1}^{k-h} n_i - 2k + 2h) = n - 2k - 1. \end{aligned}$$

On the other hand, we show that $\tilde{G}_{min}^{(n-2k-1)}$ is isomorphic to $C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}$ by induction on k , where $1 \leq h < k$ and $(2h+1) + \sum_{j=1}^{k-h} n_j = n$.

For $k = 2$, we have $h = 1$, and it follows from Corollary 2.3 (2) that $\tilde{G}_{min}^{(n-5)} \cong$

$C_3 \oplus S_{n-3}$. Therefore, the statement holds in this case. Suppose the statement holds for $k \leq p-1$. We consider the case of $k = p$, where $3 \leq p \leq \lfloor \frac{n-1}{2} \rfloor$.

Note that $\tilde{G}_{min}^{(n-2p-1)}$ has pendent vertices and $\eta(\tilde{G}_{min}^{(n-2p-1)}) = n-2p-1$. Choose a pendent vertex, say x , in $\tilde{G}_{min}^{(n-2p-1)}$. Let $N(x) = \{y\}$. Delete x, y from $\tilde{G}_{min}^{(n-2p-1)}$, and let the resultant graph be $\tilde{G}_1 = \tilde{G}_{11} \cup \dots \cup \tilde{G}_{1q} \cup zK_1$, where $\tilde{G}_{1j}$ are nontrivial connected components of order n_j^* ($j = 1, 2, \dots, q$), and $\sum_{j=1}^q n_j^* + z + 2 = n$.

Assume that $\eta(\tilde{G}_{1j}) = n_j^* - l_j^*$ ($0 < l_j^* \leq n_j^*$) for $j = 1, 2, \dots, q$.

Claim 1. One of the nontrivial connected components (suppose $\tilde{G}_{11}$) is an unicyclic connected graph, and others are trees.

If all $\tilde{G}_{1j}$ are trees, then l_j^* ($j = 1, 2, \dots, q$) is even by Theorem 3.3 Claim 2, and

$$2p+1 = n - \eta(\tilde{G}_{min}^{(n-2p-1)}) = n - [\sum_{j=1}^q \eta(\tilde{G}_{1j}) + z] = 2 + \sum_{j=1}^q l_j^*,$$

a contradiction. Since the number of edges for $\tilde{G}_{min}^{(n-2p-1)}$ should be as least as possible, and $C_{2h+1} \oplus S_{n_1} \oplus \dots \oplus S_{n_{p-h}}$ with nullity $n-2p-1$ which satisfies this claim, it follows that Claim 1 holds.

Claim 2. l_1^* is odd. Otherwise, we get a similar contradiction as Claim 1.

Claim 3. Let $l_1^* = 2t^* + 1$. Then $\tilde{G}_{11} \cong C_{2t^*+1}$ ($n_1^* = 2t^* + 1$), or $\tilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n_{1,1}^*} \oplus \dots \oplus S_{n_{1,t^*-h_1}^*}$, where $1 \leq h_1 < t^*$, $(2h_1+1) + \sum_{j=1}^{t^*-h_1} n_{1j}^* = n_1^*$.

Case 1. If $\tilde{G}_{11}$ has pendent vertices, since $t^* = \frac{l_1^*-1}{2} \leq p-1$ (note that $\sum_{j=1}^q l_j^* = 2p-1$) and $\eta(\tilde{G}_{11}) = n_1^* - 2t^* - 1$, according to the inductive assumption, $\tilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n_{1,1}^*} \oplus \dots \oplus S_{n_{1,t^*-h_1}^*}$, where $1 \leq h_1 < t^*$, $(2h_1+1) + \sum_{j=1}^{t^*-h_1} n_{1j}^* = n_1^*$.

Case 2. If $\tilde{G}_{11}$ has no pendent vertex, since $\tilde{G}_{11}$ is an unicyclic connected graph, $\tilde{G}_{11}$ is an odd cycle of order n_1^* . Hence $\tilde{G}_{11} \cong C_{2t^*+1}$ and $l_1^* = 2t^* + 1 = n_1^*$.

Claim 4. Combining Claim 1 with Theorem 3.3, each $\tilde{G}_{1j}$ ($2 \leq j \leq q$) is isomorphic to $S_{n_{j,1}^*} \oplus S_{n_{j,2}^*} \oplus \dots \oplus S_{n_{j,k_j}^*}$, where $\sum_{i=1}^{k_j} n_{j,i}^* = n_j^*$ and $l_j^* = 2k_j$.

In order to recover the connected graph $\tilde{G}_{min}^{(n-2p-1)}$, to add x, y to $\tilde{G}_1$, we insert edges from y to each of z isolated vertices of $\tilde{G}_1$ and x . This gives a star $K_{1, z+1} = S_{z+2}$. Moreover, we shall connect the vertex y (namely, the center of S_{z+2}) to one vertex of each $\tilde{G}_{1j}$ ($j = 1, 2, \dots, q$). Let $t^* - h_1 = k_1$. Then $\tilde{G}_{min}^{(n-2p-1)}$ is an unicyclic connected graph of order n created from C_{2h_1+1} , $S_{n_{j,i}^*}$ ($i = 1, 2, \dots, k_j$; $j = 1, 2, \dots, p$) and S_{z+2} by adding $\sum_{j=1}^q k_j + 1 = p - h_1$

($1 \leq h_1 < p$) edges to connect these graphs, and any two non-center vertices are not connected since y is the center of S_{z+2} .

In conclusion,

$$\tilde{G}_{min}^{(n-2k-1)} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}},$$

and then $\tilde{e}_{min}^{(n-2k-1)} = n$, where $1 \leq h < k$, $(2h+1) + \sum_{j=1}^{k-h} n_j = n$ and $k = 2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor$. $\square$

The following lemma describes the relationship between $G_{max}^{(\eta)}$ and $\tilde{G}_{max}^{(\eta)}$.

LEMMA 3.5. $G_{max}^{(\eta)} \cong \tilde{G}_{max}^{(\eta)}$, $e_{max}^{(\eta)} = \tilde{e}_{max}^{(\eta)}$, where $0 < \eta \leq n$.

Proof. Since we want to insert edges as many as possible, by Lemma 2.1 and Theorem 2.2, this lemma is proved. $\square$

Now $G_{max}^{(\eta)}$ (namely, $\tilde{G}_{max}^{(\eta)}$) is characterized for $\eta = n-2, n-4, n-5$.

THEOREM 3.6. $G_{max}^{(n-2)} \cong \tilde{G}_{max}^{(n-2)} \cong S_n$, $e_{max}^{(n-2)} = \tilde{e}_{max}^{(n-2)} = n-1$.

Proof. By Lemma 2.1 (2), we obtain the results as desired. $\square$

Let $U_{max}^{(n-4)}$ be a graph of order n created from $K_{\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1}$ and K_2 by connecting a vertex v of K_2 to all vertices of $K_{\lceil \frac{n}{2} \rceil - 1, \lfloor \frac{n}{2} \rfloor - 1}$.

THEOREM 3.7. $G_{max}^{(n-4)} \cong \tilde{G}_{max}^{(n-4)} \cong U_{max}^{(n-4)}$, $e_{max}^{(n-4)} = \tilde{e}_{max}^{(n-4)} = \lfloor \frac{n^2}{4} \rfloor$.

Proof. By Corollary 2.3 (1), $G_{max}^{(n-4)}$ should be a graph Q_{max} of order n created from K_{n_1, n_2} , pK_1 and K_2 such that $n_1 + n_2 + p + 2 = n$ and $n_1, n_2 > 0, p \geq 0$ by connecting a vertex v of K_2 to all vertices of pK_1 and K_{n_1, n_2} .

Since $n_2 = n - n_1 - p - 2$ and $n_1, n_2 > 0, p \geq 0$, we have

$$\begin{aligned} |E(Q_{max})| &= n_1 n_2 + n - 1 = -n_1^2 + (n - p - 2)n_1 + (n - 1) \\ &\leq -n_1^2 + (n - 2)n_1 + (n - 1) \\ &= -(n_1 - \frac{n}{2} + 1)^2 + \frac{n^2}{4} \\ &\leq \begin{cases} \frac{n^2}{4}, & n \text{ is even;} \\ \frac{n^2-1}{4}, & n \text{ is odd.} \end{cases} \end{aligned}$$

where the first equality holds if and only if $p = 0$, and the second equality holds if and only if $n_1 = \frac{n}{2} - 1$ (n is even); $n_1 = \frac{n-1}{2} - 1$ or $\frac{n+1}{2} - 1$ (n is odd), which implies that $n_2 = \frac{n}{2} - 1$ (n is even); $n_2 = \frac{n+1}{2} - 1$ or $\frac{n-1}{2} - 1$ (n is odd).

Combining Lemma 3.5, it follows that $G_{max}^{(n-4)} \cong \tilde{G}_{max}^{(n-4)} \cong U_{max}^{(n-4)}$.

Moreover, $e_{max}^{(n-4)} = \tilde{e}_{max}^{(n-4)} = \begin{cases} \frac{n^2}{4}, & n \text{ is even}; \\ \frac{n^2-1}{4}, & n \text{ is odd}. \end{cases} \square$

Let $U_{max}^{(n-5)}$ be a graph of order n created from

$$U^* = \begin{cases} K_{\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3}}, & n \equiv 2 \pmod{3} \\ K_{\frac{n}{3}, \frac{n-3}{3}, \frac{n-3}{3}}, & n \equiv 0 \pmod{3} \\ K_{\frac{n-4}{3}, \frac{n-1}{3}, \frac{n-1}{3}}, & n \equiv 1 \pmod{3} \end{cases}$$

and K_2 by connecting a vertex v of K_2 to all vertices of U^* .

THEOREM 3.8. $G_{max}^{(n-5)} \cong \tilde{G}_{max}^{(n-5)} \cong U_{max}^{(n-5)}$, $e_{max}^{(n-5)} = \tilde{e}_{max}^{(n-5)} = \lfloor \frac{n^2-n+1}{3} \rfloor$.

Proof. By Corollary 2.3 (2), $G_{max}^{(n-5)}$ is isomorphic to a graph C_{max} of order n created from K_{n_1}, n_2, n_3 , pK_1 and K_2 satisfying $n_1 + n_2 + n_3 + p + 2 = n$ and $n_1, n_2, n_3 > 0, p \geq 0$ by connecting a vertex v of K_2 to all vertices of pK_1 and K_{n_1}, n_2, n_3 .

Since $n_3 = n - n_1 - n_2 - p - 2$ and $n_1, n_2, n_3 > 0, p \geq 0$, we have

$$\begin{aligned} |E(C_{max})| &= n_1n_2 + n_2n_3 + n_3n_1 + n - 1 \\ &= -(n_1 + n_2)^2 + (n - 2 - p)(n_1 + n_2) + (n - 1) + n_1n_2 \\ &\leq -(n_1 + n_2)^2 + (n - 2 - p)(n_1 + n_2) + (n - 1) + \frac{(n_1 + n_2)^2}{4} \\ &= -\frac{3}{4}(n - n_3 - p - 2)^2 + (n - 2 - p)(n - n_3 - p - 2) + (n - 1) \\ &= \frac{1}{4}[-3n_3^2 + 2(n - p - 2)n_3 + (n - p - 2)^2] + (n - 1) \\ &\leq \frac{1}{4}[-3n_3^2 + 2(n - 2)n_3 + (n - 2)^2] + (n - 1) \\ &= -\frac{3}{4}(n_3 - \frac{n-2}{3})^2 + \frac{n^2-n+1}{3} \leq \begin{cases} \frac{n^2-n+1}{3}, & n-2 \equiv 0 \pmod{3}; \\ \frac{n^2-n}{3}, & n-2 \not\equiv 0 \pmod{3}, \end{cases} \end{aligned}$$

where the first equality holds if and only if $n_1 = n_2$, the second equality holds if and only if $p = 0$, and the third equality holds if and only if

$$n_3 = \begin{cases} \frac{n-2}{3}, & n-2 \equiv 0 \pmod{3}; \\ \frac{n}{3}, & n-2 \equiv 1 \pmod{3}; \\ \frac{n-4}{3}, & n-2 \equiv 2 \pmod{3}. \end{cases}$$

$$\text{Thus } n_1 = n_2 = \begin{cases} \frac{n-2}{3}, & n-2 \equiv 0 \pmod{3}; \\ \frac{n-3}{3}, & n-2 \equiv 1 \pmod{3}; \\ \frac{n-1}{3}, & n-2 \equiv 2 \pmod{3}. \end{cases}$$

Hence $G_{max}^{(n-5)} \cong U_{max}^{(n-5)}$ and then $e_{max}^{(n-5)} = \begin{cases} \frac{n^2-n+1}{3}, & n-2 \equiv 0 \pmod{3}; \\ \frac{n^2-n}{3}, & n-2 \not\equiv 0 \pmod{3}. \end{cases}$

Combining this with Lemma 3.5 gives the desired results. $\square$

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