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SIGN PATTERNS OF RATIONAL MATRICES WITH LARGE RANK II*

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Abstract. It is known that, for any real m-by-n matrix A of rank $n-2$, there is a rational m-by-n matrix which has rank $n-2$ and sign pattern equal to that of A. We prove a more general result conjectured in the recent literature.

Key words. Matrix rank, Sign pattern matrices.

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The *sign pattern* of an $m \times n$ real matrix A is the $m \times n$ matrix $\mathcal{S}(A)$ defined as $S_{ij} = +$ if $A_{ij} > 0$, $S_{ij} = -$ if $A_{ij} < 0$, and $S_{ij} = 0$ if $A_{ij} = 0$. The following result was proved in [2] and later in [1].

THEOREM 1. *For any real m-by-n matrix A of rank $n-2$, there is a rational m-by-n matrix which has rank $n-2$ and sign pattern equal to that of A.*

REMARK 2. In [1, p. 370], it is stated, without any further explanation, that the proof of Lemma 3.5 in [2] (which is Theorem 1 above) contains a logical gap. After careful scrutiny, neither the current Editors in Chief of ELA nor the author of this note have found such a gap.

The proof in [2] is based on the following technical statement, appearing as [2, Lemma 3.4]. We recall that $[x]$ is the integer part of a given real number x.

LEMMA 3. *Assume that a vector $a = (a_1, \dots, a_n)$ and a matrix $B \in \mathbb{R}^{n\times 2}$ satisfy $aB = (0\,0)$. Assume that every entry of the first column of B is either 0 or 1. Define, for integer $N > 0$, the n-by-2 matrix $C = C(N)$ by $C_{ij} = [B_{ij}N]$. Then, for any sufficiently large N, there is a rational vector $x = x(N)$ satisfying $xC = (0\,0)$ and $x(N) \to a$ as $N \to \infty$.*

The goal of this note is to present a further illustration of the technique of [2]. Namely, we can prove a generalization of Theorem 1 conjectured in [1].

THEOREM 4. *Let A, B, and E be real matrices such that $AB = E$. If all the zero entries of E are contained in the first two columns, then there exist rational matrices A', B', E' such that $A'B' = E'$ and the sign patterns of A', B', E' are equal to those of A, B, E, respectively.*

Proof. We denote by $B_{(i)}$ and $B^{(i)}$ the ith row and column of B, respectively, and the scaling allows us to assume that $B^{(1)}$ consists of zeros and ones. For any positive integer N, we define the matrix $C = C(N)$ of the same size as B by the formula $C_{ij} = [NB_{ij}]$.

We define the matrix X of the same size as A as follows. The (i, j) entry of X is set to zero if $A_{ij} = 0$, and we take X_{ij} to be a variable if A_{ij} is nonzero. For every i, a row index of A, we want to assign rational values to the variables in the vector $X_{(i)}$ and get a sequence of rational vectors $X_{(i)}(N)$ such that

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$X_{(i)}(N) \cdot C^{(j)}(N) = 0$ whenever $E_{ij} = 0$ and $X_{(i)}(N) \to A_{(i)}$ as N goes to infinity. To this end, we apply Lemma 3 to the linear system

$$X_{(i)} \left(C^{(1)} \; C^{(2)} \right) = (0 \; 0)\,, \tag{1}$$

whenever $E_{i1} = E_{i2} = 0$. If $E_{ik} \neq 0$ with $k \in \{1, 2\}$, we do the same thing but with $C^{(k)}$ replaced in (1) by the zero vector. So we get the matrices $X(N)$, $C(N)$ such that $A_{ij} = 0$ implies $X(N)_{ij} = 0$, and $B_{ij} = 0$ implies $C(N)_{ij} = 0$, and also $E_{ij} = 0$ implies that the (i, j) entry of $X(N)C(N)$ is zero, and, finally, we have the limits $X(N) \to A$, $C(N)/N \to B$ as $N \to \infty$. Therefore, for any sufficiently large N, the rational matrices $X(N), C(N), X(N)C(N)$ have the same sign patterns as A, B, E, respectively. ∎

This proves Conjecture 2.16 in [1], which has a formulation similar to Theorem 4 but allows the word "columns" to be replaced by the word "rows." This is not essential because we can apply Theorem 4 to the matrices $B^\top, A^\top, E^\top$ to cover this case.

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