Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

ELA

TWO INVERSE EIGENPROBLEMS FOR SYMMETRIC DOUBLY ARROW MATRICES*

HUBERT PICKMANN†, JUAN C. EGAÑA†, AND RICARDO L. SOTO†

Abstract. In this paper, the problem of constructing a real symmetric doubly arrow matrix A from two special kinds of spectral information is considered. The first kind is the minimal and maximal eigenvalues of all leading principal submatrices of A, and the second kind is one eigenvalue of each leading principal submatrix of A together with one eigenpair of A. Sufficient conditions for both eigenproblems to have a solution and sufficient conditions for both eigenproblems have a nonnegative solution are given in this paper. The results are constructive in the sense that they generate algorithmic procedures to compute the solution matrix.

Key words. Symmetric doubly arrow matrices, Inverse eigenproblem.

AMS subject classifications. 65F15, 65F18, 15A18.

1. Introduction. In this paper we consider two inverse eigenproblems for a special kind of real symmetric matrices: the real symmetric doubly arrow matrices. That is, matrices which look like two arrow matrices, forward and backward, with heads against each other at the (p,p) position, $1 \leq p \leq n$. They are matrices of the form:

$$A = \begin{pmatrix} a_1 & & & b_1 & & & \\ & \ddots & & \vdots & & & \\ & & a_{p-1} & b_{p-1} & & & \\ b_1 & \cdots & b_{p-1} & a_p & b_p & \cdots & b_{n-1} \\ & & & b_p & a_{p+1} & & \\ & & & \vdots & & \ddots & \\ & & & b_{n-1} & & & a_n \end{pmatrix}, \; a_j, b_j \in \mathbb{R}. \tag{1.1}$$

*Received by the editors October 29, 2008. Accepted for publication October 14, 2009. Handling Editor: Daniel Szyld.

†Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Casilla 1280, Chile (hpickmann@ucn.cl, jegana@ucn.cl and rsoto@ucn.cl). Supported by Fondecyt 1085125 and Project DGIP-UCN, Chile.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

ELA

Matrices of the form (1.1) generalize the well known real symmetric arrow matrices (also called real symmetric bordered diagonal matrices):

$$
\begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & & & \\ b_2 & & a_3 & & \\ \vdots & & & \ddots & \\ b_{n-1} & & & & a_n \end{pmatrix}, \ a_j, b_j \in \mathbb{R}. \tag{1.2}
$$

Arrow matrices arise in many areas of science and engineering [1]-[6]. In this paper, we construct matrices A of the form (1.1), from a special kind of spectral information, which only recently is being considered. Since this type of matrix structure generalizes the well known arrow matrices, we think that it will also become of interest in applications. For the first eigenproblem, we have as initial spectral information the minimal and maximal eigenvalues of all leading principal submatrices of A; and for the second one, the initial information is an eigenvalue of each leading principal submatrix of A together with one eigenpair of A.

Both eigenproblems considered in this paper were introduced by Peng et al. in [7], for real symmetric bordered diagonal matrices. However, as it has been shown in [8] and [9], the formulae given in [7], to compute the entries a_j, b_j of the matrix in (1.2), may lead us to some wrong solutions. In this work we study the following eigenproblems:

Problem 1. *Given the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, find necessary and sufficient conditions for the existence of an $n \times n$ matrix A of the form (1.1), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are respectively, the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of $A = A_n$, $j = 1, 2, ..., n$.*

Problem 2. *Given the real numbers $\lambda^{(j)}$, $j = 1, 2, \ldots, n$ and a real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, find necessary and sufficient conditions for the existence of an $n \times n$ matrix A of the form (1.1), such that $\lambda^{(j)}$ is an eigenvalue of the $j \times j$ leading principal submatrix A_j of A, $j = 1, 2, ..., n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of $A = A_n$.*

In the sequel, we denote I_j the $j \times j$ identity matrix, A_j the $j \times j$ leading principal submatrix of A which is in the form of (1.1), $P_j(\lambda)$ the characteristic polynomial of A_j, $\lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \ldots \leq \lambda_j^{(j)}$ the eigenvalues of A_j, and $\sigma(A_j)$ the spectrum of A_j. In the case that we consider only one eigenvalue of A_j, it will be denote by $\lambda^{(j)}$.

The following lemmas will be used to prove the results in the next sections:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

Lemma 1.1. *Let A be an $n \times n$ matrix of the form* (1.1)*. Then the sequence of characteristic polynomials $\{P_j(\lambda)\}_{j=1}^{n}$ satisfies the recurrence relation:*

$$P_j(\lambda) = \prod_{i=1}^{j} (\lambda - a_i); \qquad j = 1, \ldots, p-1. \tag{1.3}$$

$$P_j(\lambda) = (\lambda - a_j) P_{j-1}(\lambda) - \sum_{k=1}^{j-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{j-1} (\lambda - a_i); \qquad j = p. \tag{1.4}$$

$$P_j(\lambda) = (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}^2 \prod_{\substack{i=1 \\ i \neq p}}^{j-1} (\lambda - a_i); \qquad j = p+1, ..., n, \tag{1.5}$$

where $P_0(\lambda) = 1$.

Proof. The result follows by expanding the determinants $\det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$. □

Lemma 1.2. [8] *Let $P(\lambda)$ be a monic polynomial of degree n, with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and the maximal zero of $P(\lambda)$, then*

1. *If $\mu < \lambda_1$, we have that $(-1)^n P(\mu) > 0$.*
2. *If $\mu > \lambda_n$, we have that $P(\mu) > 0$.*

From Lemma 1.2, it is clear that if $\mu < \lambda_1^{(j)}$ then $(-1)^j P_j(\mu) > 0$ and if $\mu > \lambda_j^{(j)}$ then $P_j(\mu) > 0$. The minimal and maximal eigenvalues of A_j will be called *extremal eigenvalues.*

The paper is organized as follows: In Section 2, we discuss Problem 1 and give a sufficient condition for the existence of a solution. We also show that if the first $p-1$ entries b_i are equal then the solution is unique. We also give conditions under which the solution matrix A of the form (1.1) is nonnegative. In Section 3, we study Problem 2 and give a sufficient condition for the existence of a solution and a sufficient condition for the existence of a nonnegative solution. In Section 4, we show some examples to illustrate the results. All results are constructive in the sense that they generate algorithmic procedures to compute a solution matrix.

2. Solution to Problem 1.

If $p = 1$, the matrix A in (1.1) becomes the matrix of the form (1.2). In this case, the conditions of Theorem 2.2 below reduce to condition (2.3), which is necessary and sufficient for the existence of an arrow matrix (with the required spectral properties), as it was shown in [8]. In this sense, Theorem 2.2 generalizes similar results for real symmetric arrow matrices. We start by recalling an important property, which establishes relations between the eigenvalues of a sym-

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

metric matrix and the eigenvalues of its principal submatrices, that is, the *Cauchy interlacing property*:

LEMMA 2.1. *Let $A = A_n$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}$. Let A_r, with eigenvalues $\lambda_1^{(r)} \leq \lambda_2^{(r)} \leq \cdots \leq \lambda_{n-1}^{(r)}$, the principal submatrix of A, obtained by deleting the $r-th$ row and $r-th$ column of A. Then*

$$\lambda_1^{(n)} \leq \lambda_1^{(r)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_{n-1}^{(n)} \leq \lambda_{n-1}^{(r)} \leq \lambda_n^{(n)}.$$

Observe that if $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalues of the leading principal submatrix A_j of A, then Lemma 2.1 implies

$$\lambda_1^{(n)} \leq \cdots \leq \lambda_1^{(3)} \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(3)} \leq \cdots \leq \lambda_n^{(n)}.$$

Since A_j, $j = 1, 2, \ldots, p-1$, is a diagonal matrix, then

$$\lambda_1^{(p)} \leq a_i \leq \lambda_p^{(p)}, i = 1, 2, \ldots, p-1. \tag{2.1}$$

Now, suppose that $a_p < \lambda_1^{(p)}$. Then from Lemma 1.2, $(-1)^p P_p(a_p) > 0$ and from (2.1), $a_p - a_i < 0$, $i = 1, 2, \ldots, p-1$. Therefore, from (1.4)

$$\begin{aligned}
(-1)^p P_p(a_p) &= (-1)^p \left[(a_p - a_p) P_{j-1}(a_p) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{p-1} (a_p - a_i) \right] \\
&= (-1)^{2p-1} \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{p-1} (a_i - a_p) \leq 0,
\end{aligned}$$

which is a contradiction. The same occurs if we assume that $\lambda_p^{(p)} < a_p$. Thus, $\lambda_1^{(p)} \leq a_i \leq \lambda_p^{(p)}$, $i = 1, 2, \ldots, p$. Finally, for $j = p+1, \ldots, n$, we obtain, from Lemma 2.1,

$$\lambda_1^{(j)} \leq a_i \leq \lambda_j^{(j)}, \quad j = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, j. \tag{2.2}$$

The following result gives a sufficient condition for the Problem 1 to have a solution.

THEOREM 2.2. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. If*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(p-1)} = \cdots = \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}. \tag{2.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

and there exist real solutions $a_p, b_k > 0$, $k = 1, \ldots, p-1$ *of the system of equations*

$$P_p\left(\lambda_j^{(p)}\right) = \left(\lambda_j^{(p)} - a_p\right) P_{p-1}\left(\lambda_j^{(p)}\right) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1\\ i\neq k}}^{p-1}\left(\lambda_j^{(p)} - \lambda_i^{(i)}\right) = 0, \quad j = 1, p, \tag{2.4}$$

then there exists an $n \times n$ *real symmetric doubly arrow matrix* A*, of the form* (1.1)*, with positive entries* b_i*,* $i = 1, 2, \ldots, n-1$*, and such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are the extremal eigenvalues of the leading principal submatrix* A_j*,* $j = 1, 2, \ldots, n$*.*

Before the proof of Theorem 2.2, we must observe two facts:

The first one is that the leading principal submatrix A_{p-1} is diagonal. Then each extremal eigenvalue of A_j, $j = 1, 2, \ldots, p-1$, is a diagonal entry of A_{p-1}. Hence, at most $p-1$ of all $2(p-1)-1$ extremal eigenvalues of the matrices $A_1, A_2, \ldots, A_{p-1}$, can be distinct, and then, for this problem, we only dispose, at most, of $2n - p + 1$ independent pieces of information.

The second fact is that a matrix A of the form (1.1) is permutationally similar to single arrow matrix of the form (1.2). However, the problem would not be simpler if A is permuted to an arrow matrix $B = P^T AP$ by a permutation matrix P. The reason is that if $B = P^T AP$ would be a single arrow matrix (of the form (1.2)) with all its b_i entries positive, then as it was shown in [8], its $2n-1$ extremal eigenvalues are all distinct, while in this case we may have, as initial information, at most $2n - p + 1$ distinct extremal eigenvalues.

Proof. Of Theorem 2.2. Assume that the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, satisfy condition (2.3). Then there exists a $j \times j$ matrix
$A_j = diag\{\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}\}$, $j = 1, \ldots, p-1$, with extremal eigenvalues $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$. To prove the existence of a $j \times j$ matrix A_j, $j = p, \ldots, n$, of the form (1.1), with the desired spectral properties is equivalent to show that the system of equations

$$\left.\begin{aligned} P_j\left(\lambda_1^{(j)}\right) &= 0 \\ P_j\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\} \tag{2.5}$$

has real solutions a_j, $j = p+1, \ldots, n$ and $b_{j-1} > 0, j = 1, \ldots, n$.

For $j = p$, from condition (2.4) it follows that the system of equations (2.5) has real solutions $a_p, b_k > 0$, $k = 1, \ldots, p-1$. Hence, there exists a matrix A_p with the required spectral properties.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

For $j = p+1, \ldots, n$, the system (2.5) has the form:

$$\left.\begin{array}{rcl} a_j P_{j-1}\left(\lambda_1^{(j)}\right) + b_{j-1}^2 \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right) & = & \lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \\ a_j P_{j-1}\left(\lambda_j^{(j)}\right) + b_{j-1}^2 \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) & = & \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \end{array}\right\}. \tag{2.6}$$

We claim that the determinant

$$h_j = P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right)$$

of the coefficient matrix of the system (2.6) is nonzero. In fact, we shall prove that $(-1)^{j-1} h_j > 0$. First, we observe from (2.2) and (2.3) that $\lambda_1^{(j)} < \lambda_1^{(j-1)} \leq a_i \leq \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$, $i = 1, \ldots, j-1$. Consequently $\left(\lambda_j^{(j)} - a_i\right) > 0$ and $\left(\lambda_1^{(j)} - a_i\right) < 0$, $i = 1, \ldots, p-1$. Then,

$$\prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) > 0$$

and since the product $\prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right)$ has $j-2$ factors, it follows that

$$-(-1)^{p-1} \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = (-1)^p (-1)^{p-2} \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(a_i - \lambda_1^{(p)}\right) > 0.$$

Now, from Lemma 1.2 we have $(-1)^{j-1} P_{j-1}\left(\lambda_1^{(j)}\right) > 0$ and $P_{j-1}\left(\lambda_j^{(j)}\right) > 0$. Therefore, $(-1)^{j-1} h_j > 0$, and then $h_j \neq 0$. Thus, the system (2.6) has solutions given by

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{h_j}$$

and

$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{h_j}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

Since from Lemma 1.2,

$$(-1)^{j-1}\left(\lambda_j^{(j)}-\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right)>0,$$

then b_{j-1} is a real number, which can be chosen as positive. Hence, for $j=1,\ldots,n$, there exists a matrix A_j with the required spectral properties. In particular, $A_n=A$ is the desired symmetric doubly arrow matrix of the form (1.1). □

Looking for the uniqueness of the solution to Problem 1, we study a particular case in which the matrix A of the form (1.1) has all its b_i entries positive with the first $p-1$ entries being equal. That is,

$$A=\begin{pmatrix} a_1 & & & b & & & \\ & \ddots & & \vdots & & & \\ & & a_{p-1} & b & & & \\ b & \cdots & b & a_p & b_p & \cdots & b_{n-1} \\ & & & b_p & a_{p+1} & & \\ & & & \vdots & & \ddots & \\ & & & b_{n-1} & & & a_n \end{pmatrix},\qquad \begin{array}{l} a_j\in\mathbb{R}, b>0, \\ b_p,\ldots,b_{n-1}>0. \end{array} \tag{2.7}$$

Now, in this case, we dispose of $2n-p+1$ independent pieces of information, with $2n-p+1$ unknowns. Then the formulae of Lemma 1.1 reduces to:

$$P_j(\lambda)=\prod_{i=1}^{j}(\lambda-a_i);\qquad j=1,\ldots,p-1. \tag{2.8}$$

$$P_j(\lambda)=(\lambda-a_j)P_{j-1}(\lambda)-b^2\sum_{k=1}^{j-1}\prod_{\substack{i=1\\ i\neq k}}^{j-1}(\lambda-a_i);\qquad j=p. \tag{2.9}$$

$$P_j(\lambda)=(\lambda-a_j)P_{j-1}(\lambda)-b_{j-1}^2\prod_{\substack{i=1\\ i\neq p}}^{j-1}(\lambda-a_i);\qquad j=p+1,...,n. \tag{2.10}$$

Then we have the following Corollary.

Corollary 2.3. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j=1,2,\ldots,n$, be given. If*

$$\lambda_1^{(n)}<\cdots<\lambda_1^{(p-1)}=\cdots=\lambda_1^{(1)}<\lambda_2^{(2)}<\cdots<\lambda_n^{(n)}, \tag{2.11}$$

then there exists a unique $n\times n$ matrix A of the form (2.7), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of its leading principal submatrix A_j, $j=1,2,\ldots,n$.

Proof. It is clear from (2.11) that for $j=1,2,\ldots,p-1$, there exists a unique matrix $A_j=diag\{\lambda_1^{(1)},\lambda_2^{(2)},\ldots,\lambda_j^{(j)}\}$ with extremal eigenvalues $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

As in the proof of Theorem 2.2, it is enough to show that the system of equations

$$\left.\begin{aligned} P_j\left(\lambda_1^{(j)}\right) &= 0 \\ P_j\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\} \tag{2.12}$$

has real solutions a_j and b_{j-1}, with $b_{j-1} > 0$, $j = p+1, \ldots, n$.

For $j = p$, from (2.9) the system (2.12) can be written as

$$\begin{aligned} a_j P_{j-1}\left(\lambda_1^{(j)}\right) + b^2 \sum_{k=1}^{j-1} \prod_{\substack{i=1 \\ i \neq k}}^{j-1} \left(\lambda_1^{(j)} - \lambda_i^{(i)}\right) &= \lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \\ a_j P_{j-1}\left(\lambda_j^{(j)}\right) - b^2 \sum_{k=1}^{j-1} \prod_{\substack{i=1 \\ i \neq k}}^{j-1} \left(\lambda_j^{(j)} - \lambda_i^{(i)}\right) &= \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \end{aligned} \tag{2.13}$$

The determinant

$$h_p = P_{p-1}\left(\lambda_1^{(p)}\right) \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda_p^{(p)} - \lambda_i^{(i)}\right) - P_{p-1}\left(\lambda_p^{(p)}\right) \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda_1^{(p)} - \lambda_i^{(i)}\right),$$

of the coefficients matrix in (2.13) is nonzero. To show this, we first observe from (2.2) and (2.11) that $\lambda_1^{(p)} < a_i = \lambda_i^{(i)} < \lambda_p^{(p)}$, $i = 1, \ldots, p-1$. Consequently, $\left(\lambda_p^{(p)} - a_i\right) > 0$ and $\left(\lambda_1^{(p)} - a_i\right) < 0$, $i = 1, \ldots, p-1$. Then,

$$\sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda_p^{(p)} - a_i\right) > 0.$$

Since there are $p-2$ factors in each product $\prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda_1^{(p)} - a_i\right)$, then

$$-(-1)^{p-1} \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda_1^{(p)} - a_i\right) = (-1)^p (-1)^{p-2} \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(a_i - \lambda_1^{(p)}\right) > 0.$$

From Lemma (1.2), we have $(-1)^{p-1} P_{p-1}\left(\lambda_1^{(p)}\right) > 0$ and $P_{p-1}\left(\lambda_p^{(p)}\right) > 0$. Hence,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

$(-1)^{p-1} h_p > 0$, and then $h_p \neq 0$. Thus, the system (2.13) has unique solutions (2.14)

$$a_p = \frac{\lambda_1^{(p)} P_{p-1}\left(\lambda_1^{(p)}\right) \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i\neq k}}^{p-1} \left(\lambda_p^{(p)} - a_i\right) - \lambda_p^{(p)} P_{p-1}\left(\lambda_p^{(p)}\right) \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i\neq k}}^{p-1} \left(\lambda_1^{(p)} - a_i\right)}{h_p}$$

and

$$b^2 = \frac{\left(\lambda_p^{(p)} - \lambda_1^{(p)}\right) P_{p-1}\left(\lambda_1^{(p)}\right) P_{p-1}\left(\lambda_p^{(p)}\right)}{h_p}.$$

As

$$(-1)^{p-1}\left(\lambda_p^{(p)} - \lambda_1^{(p)}\right) P_{p-1}\left(\lambda_1^{(p)}\right) P_{p-1}\left(\lambda_p^{(p)}\right) \geq 0,$$

then $b = b_1 = \cdots = b_{p-1}$ is a real number and therefore there exists a unique matrix A_p with the desired spectral properties.

Similarly to the proof of Theorem (2.2), for $j = p+1, \ldots, n$, from (2.10), it follows that system (2.6) has unique solutions

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{\substack{i=1 \\ i\neq p}}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{\substack{i=1 \\ i\neq p}}^{j-1} \left(\lambda_1^{(j)} - a_i\right)}{h_j} \tag{2.15}$$

and

$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{h_j} > 0.$$

Hence b_{j-1} is a real number which can be chosen positive. Therefore, there exists a unique symmetric doubly arrow matrix A of the form (1.1) with the required spectral properties. $\square$

Now we discuss Problem 1 for a matrix $A = \widetilde{A} + aI$, $a \in \mathbb{R}$, where $\widetilde{A}$ is of the form

$$\widetilde{A} = \begin{pmatrix} 0 & & & b & & & \\ & \ddots & & \vdots & & & \\ & & 0 & b & & & \\ b & \cdots & b & 0 & b_p & \cdots & b_{n-1} \\ & & & b_p & 0 & & \\ & & & \vdots & & \ddots & \\ & & & b_{n-1} & & & 0 \end{pmatrix}, \quad \begin{array}{c} b, b_i \neq 0, \\ i = p, \ldots, n-1. \end{array} \tag{2.16}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

LEMMA 2.4. *Let $\widetilde{A}$ be an $n \times n$ matrix of the form (2.16). Let $\widetilde{P}_j(\lambda)$ be the characteristic polynomial of the leading principal submatrix $\widetilde{A}_j$ of $\widetilde{A}$, $j = 1, \ldots, n$. Then if j is even, $\widetilde{P}_j(\lambda)$ is an even polynomial, and if j is odd, $\widetilde{P}_j(\lambda)$ is an odd polynomial.*

Proof. If $a_j = 0$, $j = 1, 2, \ldots, n$, then the recurrence relations (2.8)-(2.10) become

$$\begin{aligned} \widetilde{P}_j(\lambda) &= \prod_{i=1}^{j} \lambda; \qquad j = 1, \ldots, p-1. \\ \widetilde{P}_j(\lambda) &= \lambda \widetilde{P}_{j-1}(\lambda) - b^2 \sum_{k=1}^{j-1} \prod_{\substack{i=1 \\ i \neq k}}^{j-1} \lambda; \qquad j = p. \qquad (2.17) \\ \widetilde{P}_j(\lambda) &= \lambda \widetilde{P}_{j-1}(\lambda) - b_{j-1}^2 \prod_{\substack{i=1 \\ i \neq p}}^{j-1} \lambda; \qquad j = p+1, ..., n. \end{aligned}$$

Clearly, $\widetilde{P}_j(\lambda)$, $j = 1, \ldots, p-1$, is an even or odd polynomial if j is even or odd, respectively. Now, suppose $j = p$ is odd. Then $\widetilde{P}_{p-1}(\lambda)$ is an even polynomial. Thus, from (2.17)

$$\begin{aligned} \widetilde{P}_p(-\lambda) &= -\lambda \widetilde{P}_{p-1}(-\lambda) - b^2 \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} (-\lambda) \\ &= -\lambda \widetilde{P}_{p-1}(\lambda) - b^2 (-1)^{p-2} \sum_{k=1}^{p-1} \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \lambda \\ &= -\widetilde{P}_{p-1}(\lambda), \end{aligned}$$

and $\widetilde{P}_p(\lambda)$ is an odd polynomial. Similarly, if $j = p$ is even, then $\widetilde{P}_p(-\lambda) = \widetilde{P}_{p-1}(\lambda)$.

Let $j = p+1, \ldots, n-1$. By using induction, assume that $\widetilde{P}_j(\lambda)$ is odd for odd j. We shall prove that $\widetilde{P}_{j+1}(\lambda)$ is even for even $j+1$. In fact, suppose $j+1$ is even. Then j is odd with $\widetilde{P}_j(\lambda)$ odd and $j-1$ is even. From (2.17), we have

$$\begin{aligned} \widetilde{P}_{j+1}(-\lambda) &= -\lambda \widetilde{P}_j(-\lambda) - b_j^2 \prod_{\substack{i=1 \\ i \neq p}}^{j} (-\lambda) \\ &= \lambda \widetilde{P}_j(\lambda) - b_j^2 (-\lambda)^{j-1} \\ &= \widetilde{P}_{j+1}(\lambda). \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

Therefore, $\widetilde{P}_{j+1}(\lambda)$ is an even polynomial. In the same way we may show that $\widetilde{P}_j(\lambda)$ is odd when j is odd. It is enough to observe that $\widetilde{P}_{j+1}(-\lambda) = -\widetilde{P}_{j+1}(\lambda)$ when $j+1$ is odd. □

Definition 2.5. A vector $v = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of real numbers is said to be *skew-symmetric* if $\lambda_i = -\lambda_{n-i+1}$, with $\lambda_{\frac{n+1}{2}} = 0$ if n is odd.

If $\lambda_1^{(1)} = 0$ and $\lambda_1^{(j)} = -\lambda_j^{(j)}$, $j = 2, 3, ..., n$, then the extremal eigenvalues $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}$ of the $j \times j$ leading principal submatrices $\widetilde{A}_j$ of the matrix $\widetilde{A}$ of the form (2.16) form a skew-symmetric vector, where $\lambda_1^{(j)} = \lambda_j^{(j)} = 0$ for $j = 1, 2, \ldots, p-1$. Thus, we have the following result

Corollary 2.6. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists a unique $n \times n$ matrix $A = \widetilde{A} + aI$, $a \in \mathbb{R}$, of the form* (2.16)*, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of its $j \times j$ leading principal submatrix A_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(p-1)} = \cdots = \lambda_1^{(1)} = \cdots = \lambda_{p-1}^{(p-1)} < \cdots < \lambda_n^{(n)} \tag{2.18}$$

and

$$\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}, \quad j = 1, 2, \ldots, n. \tag{2.19}$$

Proof. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, ..., n$, satisfying (2.18) and (2.19). It is enough to prove the result for a skew-symmetric vector $(\lambda_1^{(n)}, \lambda_1^{(n-1)}, \ldots, \lambda_{n-1}^{(n-1)}, \lambda_n^{(n)})$, with $\lambda_1^{(1)} = 0$. Otherwise, if $\lambda_1^{(1)} \neq 0$, then we define $\mu_i^{(j)} = \lambda_i^{(j)} - \lambda_1^{(1)}$, $j = 1, 2, \ldots, n$, $i = 1, j$ to obtain $\mu_1^{(1)} = 0$, $\mu_1^{(j)} = -\mu_j^{(j)}$, $j = 2, \ldots, n$. Then, if there exists $\widetilde{A}$ with $\mu_1^{(j)}$ and $\mu_j^{(j)}$ being the extremal eigenvalues of $\widetilde{A}_j$, $j = 1, 2, \ldots, n$, then $A = \widetilde{A} + \lambda_1^{(1)} I$ is the matrix with the required spectral properties.

Let $\lambda_1^{(1)} = 0$ and $\lambda_1^{(j)} = -\lambda_j^{(j)}$, $j = 2, ..., n$. From (2.18) it is clear that $a_j = \lambda_1^{(1)} = 0$, $j = 1, 2, \ldots, p-1$. Now, from the proof of Corollary 2.3 and (2.18) the matrices A_j, $j = p, p+1, \ldots, n$ exist, are unique and satisfy the required spectral properties.

It only remain to show that $a_j = 0$, $j = p, ..., n$. Let $j = p$ be even. Then from

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

Lemma 2.4, $P_{p-1}(\lambda)$ is odd and the numerator in (2.14) is given by

$$\lambda_1^{(p)} P_{p-1}\left(\lambda_1^{(p)}\right)(p-1)\left(\lambda_p^{(p)}\right)^{p-2} - \lambda_p^{(p)} P_{p-1}\left(\lambda_p^{(p)}\right)(p-1)\left(\lambda_1^{(p)}\right)^{p-2}$$

$$= -\lambda_p^{(p)} P_{p-1}\left(-\lambda_p^{(p)}\right)(p-1)\left(\lambda_p^{(p)}\right)^{p-2} - \lambda_p^{(p)} P_{p-1}\left(\lambda_p^{(p)}\right)(p-1)\left(-\lambda_p^{(p)}\right)^{p-2}$$

$$= \lambda_p^{(p)} P_{p-1}\left(\lambda_p^{(p)}\right)(p-1)\left(\lambda_p^{(p)}\right)^{p-2} - \lambda_p^{(p)} P_{p-1}\left(\lambda_p^{(p)}\right)(p-1)\left(\lambda_p^{(p)}\right)^{p-2}$$

$$= 0,$$

which implies $a_p = 0$. Now, suppose $a_j = 0$, $j = 1, 2, \ldots, k$; $p \le k < n$. Let $k+1$ even. Then, from Lemma 2.4, $P_k(\lambda)$ is odd and the numerator in (2.15) is given by

$$\lambda_1^{(k+1)} P_k\left(\lambda_1^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_1^{(k+1)}\right)^{k-1}$$

$$= -\lambda_{k+1}^{(k+1)} P_k\left(-\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(-\lambda_{k+1}^{(k+1)}\right)^{k-1}$$

$$= \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1} - \lambda_{k+1}^{(k+1)} P_k\left(\lambda_{k+1}^{(k+1)}\right)\left(\lambda_{k+1}^{(k+1)}\right)^{k-1}$$

$$= 0,$$

which implies that $a_{k+1} = 0$. Similarly it can be proved that $a_{k+1} = 0$ when $k+1$ is odd.

For the necessity, assume that $A = \widetilde{A} + aI$, $a \in \mathbb{R}$ is the unique $n \times n$ matrix such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the extremal eigenvalues of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Either $\widetilde{P}_j(\lambda)$ is even or $\widetilde{P}_j(\lambda)$ is odd, we have that $\widetilde{P}_j(\lambda) = 0$ implies $\widetilde{P}_j(-\lambda) = 0$. Then the eigenvalues $\widetilde{\lambda}_1^{(j)}, \widetilde{\lambda}_2^{(j)}, ..., \lambda_j^{(j)}$ of $\widetilde{A}_j$ satisfy $\widetilde{\lambda}_i^{(j)} + \widetilde{\lambda}_{j-i+1}^{(j)} = 0$ and $\widetilde{\lambda}_i^{(j)} = \widetilde{\lambda}_{j-i+1}^{(j)} = 0$, $j = 1, \ldots, p-1$. It is clear that the extremal eigenvalues of $\widetilde{A}_j$ are $\lambda_1^{(j)} - \lambda_1^{(1)}$ and $\lambda_j^{(j)} - \lambda_1^{(1)}$, $j = 1, 2, \ldots, n$, and satisfy (2.18). Moreover, $\left(\lambda_1^{(j)} - \lambda_1^{(1)}\right) + \left(\lambda_j^{(j)} - \lambda_1^{(1)}\right) = 0$, and consequently $\lambda_1^{(j)} + \lambda_j^{(j)} = 2\lambda_1^{(1)}$, $j = 1, 2, \ldots, n$. This completes the proof. □

2.1. Nonnegative realization.

Corollary 2.7. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Let*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(p-1)} = \cdots = \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}. \tag{2.20}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

If there exist nonnegative solutions $a_p, b_1, b_2, \ldots, b_{p-1}$ *of the system of equations*

$$P_p\left(\lambda_j^{(p)}\right) = \left(\lambda_j^{(p)} - a_p\right) P_{p-1}\left(\lambda_j^{(p)}\right) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1\\ i\neq k}}^{p-1}\left(\lambda_j^{(p)} - a_i\right) = 0, \quad j = 1, p, \tag{2.21}$$

$$\lambda_1^{(1)} \geq 0, \tag{2.22}$$

and

$$\frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geq \frac{P_{j-1}\left(\lambda_j^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{P_{j-1}\left(\lambda_1^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}, \quad j = p+1, \ldots, n, \tag{2.23}$$

then there exists an $n \times n$ *nonnegative matrix* A *of the form* (1.1)*, such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are the extremal eigenvalues of its leading principal submatrix* A_j*,* $j = 1, 2, \ldots, n$*.*

Proof. If conditions (2.20) and (2.21) hold, then Theorem 2.2 guarantees the existence of a matrix A of the form (1.1) with $b_i \geq 0$, $i = 1, \ldots, n-1$. Moreover, from condition (2.21) it follows that $a_p \geq 0$. Only remain to show that the remaining diagonal elements a_i are nonnegative. From (2.20) and (2.22), we have $a_j = \lambda_j^{(j)} \geq 0$, $j = 1, \ldots, p-1$. Finally, for $j = p+1, \ldots, n$, from (2.23) we have

$$\frac{\lambda_1^{(j)}}{\lambda_j^{(j)}} \geq \frac{(-1)^{j-1} P_{j-1}\left(\lambda_j^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{(-1)^{j-1} P_{j-1}\left(\lambda_1^{(j)}\right) \prod\limits_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right)}.$$

Now, from (2.2) and (2.20), $\lambda_1^{(j)} < \lambda_1^{(j-1)} \leq a_i \leq \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$, $i = 1, \ldots, j-1$. Besides, from Lemma 1.2, $(-1)^{j-1} P_{j-1}\left(\lambda_1^{(j)}\right) > 0$. Then

$$(-1)^{j-1} P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) > 0,$$

Since $0 \leq \lambda_1^{(1)} < \lambda_j^{(j)}$, it follows that

$$\lambda_1^{(j)} (-1)^{j-1} P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \geq \lambda_j^{(j)} (-1)^{j-1} P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{\substack{i=1\\ i\neq p}}^{j-1}\left(\lambda_1^{(j)} - a_i\right),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

or

$$(-1)^{j-1}\left[\lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)\prod_{\substack{i=1\\i\neq p}}^{j-1}\left(\lambda_j^{(j)}-a_i\right)-\lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)\prod_{\substack{i=1\\i\neq p}}^{j-1}\left(\lambda_1^{(j)}-a_i\right)\right]$$
$$=\widetilde{g}_j\geq 0.$$

Therefore, from (2.20) and the proof of Corollary 2.3, we obtain

$$a_j=\frac{\widetilde{g}_j}{\widetilde{h}_j}\geq 0.$$

The proof is complete. □

3. Solution to Problem 2. In this section, we discuss a solution to Problem 2 and construct an $n\times n$ symmetric doubly arrow matrix A, of the form (1.1), from a list $\{\lambda^{(j)}\}_{j=1}^{n}$, and a vector $\mathbf{x}$, where $\lambda^{(j)}$ is an eigenvalue of the leading principal submatrix A_j of A and $(\lambda^{(n)},\mathbf{x})$ is an eigenpair of $A_n=A$.

THEOREM 3.1. *Let the real numbers $\lambda^{(j)}$, $j=1,2,\ldots,n$ and a real vector $\mathbf{x}=(x_1,\ldots,x_n)^T$, be given. Let*

$$x_i\neq 0,\quad i=1,\ldots,n, \tag{3.1}$$

and

$$P_{p-1}\left(\lambda^{(p)}\right)\neq 0. \tag{3.2}$$

If there exists a real solution b_{j-1} of the equation

$$b_{j-1}^2\prod_{\substack{i=1\\i\neq p}}^{j-1}\left(\lambda^{(j)}-a_i\right)-b_{j-1}\frac{x_p}{x_j}P_{j-1}\left(\lambda^{(j)}\right)+\left(\lambda^{(n)}-\lambda^{(j)}\right)P_{j-1}\left(\lambda^{(j)}\right)=0, \tag{3.3}$$

$j=p+1,\ldots,n-1$, then there exists an $n\times n$ matrix A of the form (1.1), such that $\lambda^{(j)}$ is an eigenvalue of its leading principal submatrix A_j, $j=1,\ldots,n$ and $(\lambda^{(n)},\mathbf{x})$ is an eigenpair of A.

Proof. To show the existence of the required matrix A is equivalent to show that the system of equations

$$P_j\left(\lambda^{(j)}\right)=0,\quad j=1,2,\ldots,n \tag{3.4}$$

$$A\mathbf{x}=\lambda^{(n)}\mathbf{x} \tag{3.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

has real solutions a_j and b_{j-1}. Next we rewrite (3.5) as

$$\left.\begin{array}{ll} a_j x_j + b_j x_p & = \lambda^{(n)} x_j, \quad j = 1, \ldots, p-1, \\ \sum_{k=1}^{p-1} b_k x_k + a_p x_p + \sum_{k=p}^{n-1} b_k x_{k+1} & = \lambda^{(n)} x_p \\ b_{j-1} x_p + a_j x_j & = \lambda^{(n)} x_j, \quad j = p+1, \ldots, n. \end{array}\right\} \tag{3.6}$$

From relation (1.3) of Lemma 1.1 we have that

$$a_j = \lambda^{(j)}, \quad j = 1, \ldots, p-1 \tag{3.7}$$

is a solution of (3.4). Then, from (3.1) and (3.6),

$$b_j = \left(\lambda^{(n)} - a_j\right) \frac{x_j}{x_p} \quad j = 1, \ldots, p-1. \tag{3.8}$$

From relation (1.4) of Lemma 1.1 and (3.4), we obtain

$$P_p\left(\lambda^{(p)}\right) = \left(\lambda^{(p)} - a_p\right) P_{p-1}\left(\lambda^{(p)}\right) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda^{(p)} - a_i\right) = 0.$$

Thus, by assuming that condition (3.2) holds, we have

$$a_p = \frac{\lambda^{(p)} P_{p-1}\left(\lambda^{(p)}\right) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda^{(p)} - a_i\right)}{P_{p-1}\left(\lambda^{(p)}\right)}. \tag{3.9}$$

From (3.6)

$$a_j = \lambda^{(n)} - b_{j-1} \frac{x_p}{x_j} \quad j = p+1, \ldots, n. \tag{3.10}$$

Besides, from (1.5) of Lemma 1.1 and (3.4), we obtain for $j = p+1, ..., n$,

$$P_j\left(\lambda^{(j)}\right) = \left(\lambda^{(j)} - a_j\right) P_{j-1}\left(\lambda^{(j)}\right) - b_{j-1}^2 \prod_{\substack{i=1 \\ i \neq p}}^{j-1} \left(\lambda^{(j)} - a_i\right) = 0. \tag{3.11}$$

By substituting a_j in (3.10) into (3.11) for $j = p+1, \ldots, n-1$, we obtain the quadratic equation

$$b_{j-1}^2 \prod_{\substack{i=1 \\ i \neq p}}^{j-1} \left(\lambda^{(j)} - a_i\right) - b_{j-1} \frac{x_p}{x_j} P_{j-1}\left(\lambda^{(j)}\right) + \left(\lambda^{(n)} - \lambda^{(j)}\right) P_{j-1}\left(\lambda^{(j)}\right) = 0,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

which, because of condition (3.3) has real solutions b_{j-1}.

Finally, from (3.6), it follows that

$$b_{n-1} = \frac{1}{x_n}\left\{\left(\lambda^{(n)} - a_p\right)x_p - \sum_{k=1}^{p-1} b_k x_k - \sum_{k=p}^{n-2} b_k x_{k+1}\right\}. \square \tag{3.12}$$

3.1. Nonnegative realization.

Corollary 3.2. *Let the real numbers $\lambda^{(j)}$, $j = 1, 2, \ldots, n$, and the real vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ be given. Let*

$$x_i > 0, \quad i = 1, \ldots, n, \tag{3.13}$$

and

$$P_{p-1}\left(\lambda^{(p)}\right) \neq 0. \tag{3.14}$$

If there exists a nonnegative solution b_{j-1} of the equation

$$b_{j-1}^2 \prod_{\substack{i=1 \\ i \neq p}}^{j-1}\left(\lambda^{(j)} - a_i\right) - b_{j-1}\frac{x_p}{x_j}P_{j-1}\left(\lambda^{(j)}\right) + \left(\lambda^{(n)} - \lambda^{(j)}\right)P_{j-1}\left(\lambda^{(j)}\right) = 0, \tag{3.15}$$

$j = p+1, \ldots, n-1$, with

$$\lambda^{(n)} \geq \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, p-1, \tag{3.16}$$

$$\lambda^{(n)} \geq a_p + \frac{1}{x_p}\left\{\sum_{k=1}^{p-1} b_k x_k + \sum_{k=p}^{n-2} b_k x_{k+1}\right\}, \tag{3.17}$$

$$\lambda^{(p)} \geq \frac{\sum\limits_{k=1}^{p-1} b_k^2 \prod\limits_{\substack{i=1 \\ i \neq k}}^{p-1}\left(\lambda^{(p)} - a_i\right)}{P_{p-1}\left(\lambda^{(p)}\right)}, \tag{3.18}$$

and

$$\lambda^{(n)} \geq b_{j-1}\frac{x_p}{x_j}, \quad j = p+1, \ldots, n, \tag{3.19}$$

then there exists an $n \times n$ nonnegative matrix A of the form (1.1), such that $\lambda^{(j)}$ is an eigenvalue of its leading principal submatrix A_j, $j = 1, 2, ..., n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of $A = A_n$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

Proof. From Theorem 3.1, conditions (3.13), (3.14) and (3.15) guarantee the existence of an $n \times n$ matrix A of the form (1.1) such that $\lambda^{(j)}$ is an eigenvalue of its leading principal submatrix A_j, $j = 1, 2, ..., n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of A. From (3.7) and (3.16), for $j = 1, 2, \ldots, p-1$, it follows that

$$a_j = \lambda^{(j)} \geq 0.$$

From (3.8) and conditions (3.13) and (3.16), it follows that

$$b_j = \left(\lambda^{(n)} - a_j\right) \frac{x_j}{x_p} \geq 0 \quad j = 1, \ldots, p-1.$$

From condition (3.15), we see that $b_{j-1} \geq 0$, $j = p+1, p+2, \ldots, n-1$. From (3.12) and conditions (3.13) and (3.17) we obtain

$$\left(\lambda^{(n)} - a_p\right) x_p \geq \frac{x_n}{x_n} \left\{ \sum_{k=1}^{p-1} b_k x_k + \sum_{k=p}^{n-1} b_k x_{k+1} \right\}.$$

Hence,

$$\left(\lambda^{(n)} - a_p\right) \frac{x_p}{x_n} - \frac{1}{x_n} \left\{ \sum_{k=1}^{p-1} b_k x_k + \sum_{k=p}^{n-1} b_k x_{k+1} \right\} \geq 0,$$

which implies $b_{n-1} \geq 0$.

It only remains to show the nonnegativity of the diagonal entries a_j, $j = p$, $p+1, \ldots, n.$ From (3.9) and condition (3.18), we obtain

$$a_p = \frac{\lambda^{(p)} P_{p-1}\left(\lambda^{(p)}\right) - \sum_{k=1}^{p-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{p-1} \left(\lambda^{(p)} - a_i\right)}{P_{p-1}\left(\lambda^{(p)}\right)} \geq 0.$$

Finally, from (3.10) and (3.19), for $j = p+1, \ldots, n$, we have

$$a_j = \lambda^{(n)} - b_{j-1} \frac{x_p}{x_j} \geq 0. \quad \square$$

4. Examples.

Example 4.1. The given numbers

$\lambda_1^{(6)}$	$\lambda_1^{(5)}$	$\lambda_1^{(4)}$	$\lambda_1^{(3)}$	$\lambda_1^{(2)}$	$\lambda_1^{(1)}$
−11.8001	−11.7127	−11.3826	−9.2151	−5.8980	−4.5178

$\lambda_2^{(2)}$	$\lambda_3^{(3)}$	$\lambda_4^{(4)}$	$\lambda_5^{(5)}$	$\lambda_6^{(6)}$
−3.1377	0.1794	2.3469	2.6770	2.7644

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

satisfy conditions (2.18) and (2.19) of Corollary 2.6, with $p = 2$. The resultant matrix, with constant diagonal entries, is

$$A = \begin{pmatrix} -4.5178 & 1.3802 & & & & \\ 1.3802 & -4.5178 & 4.4899 & 5.0061 & 2.1542 & 1.1250 \\ & 4.4899 & -4.5178 & & & \\ & 5.0061 & & -4.5178 & & \\ & 2.1542 & & & -4.5178 & \\ & 1.1250 & & & & -4.5178 \end{pmatrix}.$$

EXAMPLE 4.2. The given numbers

$$\begin{array}{cccccc} \lambda_1^{(6)} & \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} \\ -5.2702 & -5.0130 & -2.7101 & 0.8992 & 0.8992 & 0.8992 \end{array}$$

$$\begin{array}{ccccc} \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & \lambda_6^{(6)} \\ 1.1538 & 1.3960 & 4.1261 & 6.8664 & 8.1710 \end{array},$$

satisfy conditions (2.20)-(2.23) of Corollary 2.7. Then we may construct the symmetric nonnegative doubly arrow matrix A, with $p = 4$,

$$A = \begin{pmatrix} 0.8992 & & & 0.8546 & & \\ & 1.1538 & & 0.8909 & & \\ & & 1.3960 & 3.1197 & & \\ 0.8546 & 0.8909 & 3.1197 & 0.0679 & 4.8156 & 2.2621 \\ & & & 4.8156 & 1.9926 & \\ & & & 2.2621 & & 6.2755 \end{pmatrix}$$

EXAMPLE 4.3. Let the numbers

$$\begin{array}{ccccc} \lambda^{(1)} & \lambda^{(2)} & \lambda^{(3)} & \lambda^{(4)} & \lambda^{(5)} \\ 2.2630 & 2.2291 & 6.4834 & 5.6679 & 7.3089 \end{array}$$

and the vector

$$\mathbf{x} = \begin{bmatrix} 0.2071 & 0.6072 & 0.6299 & 0.3705 & 0.5751 \end{bmatrix}^T,$$

be given and satisfying Corollary 3.2, for $p = 3$. Then we compute the nonnegative matrix

$$A = \begin{pmatrix} 2.2630 & & 1.6593 & & \\ & 2.2291 & 4.8968 & & \\ 1.6593 & 4.8968 & 0.1946 & 0.8015 & 1.5079 \\ & & 0.8015 & 5.9462 & \\ & & 1.5079 & & 05.6575 \end{pmatrix}$$

with the required spectral properties.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 700-718, November 2009

REFERENCES

[1] Daniel Boley and Gene H. Golub. *A survey of matrix inverse eigenvalue problems*. Inverse Problems, 3:595–622, 1987.
[2] Moody T. Chu and Gene H. Golub. *Inverse Eigenvalue Problems: Theory, Algorithms and Applications*. Oxford University Press, New York, 2005.
[3] J.C. Egaña and R.L. Soto. On the numerical reconstruction of a spring-mass system from its natural frequencies. *Proyecciones*, 19:27–41, 2000.
[4] W. B. Gao. *Introduction to the Nonlinear Control Systems*. Science Press, Beijing, 1988.
[5] G. M. L. Gladwell. *Inverse Problems in Vibration*, second edition. Kluwer Academic Publishers, Dordrecht, 2004.
[6] G. M. L. Gladwell and N. B. Willms. The reconstruction of a tridiagonal system from its frecuency response at an interior point. *Inverse Problems*, 4:1013–1024, 1988.
[7] J. H. Peng, X.-Y. Hu, and L. Zhang. Two inverse eigenvalue problems for a special kind of matrices. *Linear Algebra Appl.*, 416:336–347, 2006.
[8] H. Pickmann, J. Egaña, and R. L. Soto. Extremal inverse eigenvalue problem for bordered diagonal matrices. *Linear Algebra Appl.*, 427:256–271, 3007.
[9] H. Pickmann, J. Egaña, and R. L. Soto. Constructing real symmetric arrow and real symmetric tridiagonal matrices from spectral information. Preprint.