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SPECTRALLY ARBITRARY COMPLEX SIGN PATTERN MATRICES*

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Abstract. An $n \times n$ complex sign pattern matrix $\mathcal{S}$ is said to be spectrally arbitrary if for every monic nth degree polynomial $f(\lambda)$ with coefficients from $\mathbb{C}$, there is a complex matrix in the complex sign pattern class of $\mathcal{S}$ such that its characteristic polynomial is $f(\lambda)$. If $\mathcal{S}$ is a spectrally arbitrary complex sign pattern matrix, and no proper subpattern of $\mathcal{S}$ is spectrally arbitrary, then $\mathcal{S}$ is a minimal spectrally arbitrary complex sign pattern matrix. This paper extends the Nilpotent-Jacobian method for sign pattern matrices to complex sign pattern matrices, establishing a means to show that an irreducible complex sign pattern matrix and all its superpatterns are spectrally arbitrary. This method is then applied to prove that for every $n \geq 2$ there exists an $n \times n$ irreducible, spectrally arbitrary complex sign pattern with exactly $3n$ nonzero entries. In addition, it is shown that every $n \times n$ irreducible, spectrally arbitrary complex sign pattern matrix has at least $3n-1$ nonzero entries.

Key words. Complex sign pattern, Spectrally arbitrary pattern, Nilpotent.

AMS subject classifications. 15A18, 05C15.

1. Introduction.

The sign of a real number a, denoted by $sgn(a)$, is defined to be $1, -1$ or 0, according to $a > 0, a < 0$ or $a = 0$. A *sign pattern matrix* $\mathcal{A}$ is a matrix whose entries are in the set $\{1, -1, 0\}$. The sign pattern of a real matrix B, denoted by $sgn(B)$, is the matrix obtained from B by replacing each entry by its sign.

Associated with each $n \times n$ sign pattern matrix $\mathcal{A}$ is a class of real matrices, called the *sign pattern class* of $\mathcal{A}$, defined by

$$Q(\mathcal{A}) = \{A \mid A \text{ is an } n \times n \text{ real matrix, and } sgn(A) = \mathcal{A}\}.$$

For two $n \times n$ sign pattern matrices $\mathcal{A} = (a_{kl})$ and $\mathcal{B} = (b_{kl})$, if $a_{kl} = b_{kl}$ whenever $b_{kl} \neq 0$, then $\mathcal{A}$ is a *superpattern* of $\mathcal{B}$, and $\mathcal{B}$ is a *subpattern* of $\mathcal{A}$. Note that each sign pattern matrix is a superpattern and a subpattern of itself. For a subpattern $\mathcal{B}$ of $\mathcal{A}$, if $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a proper subpattern of $\mathcal{A}$.

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Let $\mathcal{A}$ be a sign pattern matrix of order $n \geq 2$. If for any given real monic polynomial $f(\lambda)$ of degree n, there is a real matrix $A \in Q(\mathcal{A})$ having characteristic polynomial $f(\lambda)$, then $\mathcal{A}$ is a *spectrally arbitrary sign pattern matrix.*

The problem of classifying the spectrally arbitrary sign pattern matrices was introduced in [1] by Drew et al. In their article, they developed the Nilpotent-Jacobian method for showing that a sign pattern matrix and all its superpatterns are spectrally arbitrary. Work on spectrally arbitrary sign pattern matrices has continued in several articles including [1–9], where families of spectrally arbitrary sign pattern matrices have been presented. In particular, in [3], Britz et al. showed that every $n \times n$ irreducible, spectrally arbitrary sign pattern matrix must have at least $2n-1$ nonzero entries and they provided families of sign pattern matrices that have exactly $2n$ nonzero entries. Recently this work has extended to zero-nonzero patterns and ray patterns, respectively ([10, 11]).

Now we introduce some concepts on complex sign pattern matrices.

For $n \times n$ sign pattern matrices $\mathcal{A} = (a_{kl})$ and $\mathcal{B} = (b_{kl})$, the matrix $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ is called a complex sign pattern matrix of order n, where $i^2 = -1$ ([12]). Clearly, the (k, l)-entry of $\mathcal{S}$ is $a_{kl} + ib_{kl}$ for $k, l = 1, 2, \ldots, n$. Associated with an $n \times n$ complex sign pattern matrix $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ is a class of complex matrices, called the *complex sign pattern class* of $\mathcal{S}$, defined by

$$Q_c(\mathcal{S}) = \{C = A + iB \mid A \text{ and } B \text{ are } n \times n \text{ real matrices}, sgn(A) = \mathcal{A}, sgn(B) = \mathcal{B}\}.$$

For two $n \times n$ complex sign pattern matrices $\mathcal{S}_1 = \mathcal{A}_1 + i\mathcal{B}_1$ and $\mathcal{S}_2 = \mathcal{A}_2 + i\mathcal{B}_2$, if $\mathcal{A}_1$ is a subpattern of $\mathcal{A}_2$, and $\mathcal{B}_1$ is a subpattern of $\mathcal{B}_2$, then $\mathcal{S}_1$ is a *subpattern* of $\mathcal{S}_2$, and $\mathcal{S}_2$ is a *superpattern* of $\mathcal{S}_1$. If $\mathcal{S}_1$ is a subpattern of $\mathcal{S}_2$ and $\mathcal{S}_1 \neq \mathcal{S}_2$, then $\mathcal{S}_1$ is a proper subpattern of $\mathcal{S}_2$.

For a complex sign pattern matrix $\mathcal{S} = \mathcal{A}+i\mathcal{B}$ of order n, the sign pattern matrices $\mathcal{A}$ and $\mathcal{B}$ are the real part and complex part of $\mathcal{S}$, respectively, and the number of nonzero entries of both $\mathcal{A}$ and $\mathcal{B}$ is the number of nonzero entries of $\mathcal{S}$.

It is clear that complex sign pattern matrix and ray pattern are different generalization of sign pattern matrix. For a complex sign pattern matrix $\mathcal{S} = \mathcal{A} + i\mathcal{B}$, if $\mathcal{B} = 0$, then $\mathcal{S} = \mathcal{A}$ is a sign pattern matrix.

Let $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ be a complex sign pattern matrix of order $n \geq 2$. If there is a complex matrix $C \in Q_c(\mathcal{S})$ having characteristic polynomial $f(\lambda) = \lambda^n$, then $\mathcal{S}$ is *potentially nilpotent*, and C is a *nilpotent complex matrix.* If for every monic nth degree polynomial $f(\lambda)$ with coefficients from $\mathbb{C}$, there is a complex matrix in $Q_c(\mathcal{S})$ such that its characteristic polynomial is $f(\lambda)$, then $\mathcal{S}$ is said to be a *spectrally arbitrary complex sign pattern matrix.* If $\mathcal{S}$ is a spectrally arbitrary complex sign

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pattern matrix, and no proper subpattern of $\mathcal{S}$ is spectrally arbitrary, then $\mathcal{S}$ is a *minimal spectrally arbitrary complex sign pattern matrix.*

Let $\mathcal{SA}_n$ represent the set of all $n \times n$ spectrally arbitrary complex sign pattern matrices. Then the following result holds.

LEMMA 1.1. *The set $\mathcal{SA}_n$ is closed under the following operations:*

(i) *Negation,*

(ii) *Transposition,*

(iii) *Permutational similarity,*

(iv) *Signature similarity, and*

(v) *Conjugation.*

Proof. The results are clear for cases (i)–(iv). We only prove the case (v). Note that for any $n \times n$ complex matrix C and its conjugate complex matrix $\overline{C}$, the corresponding coefficients of the characteristic polynomials of C and $\overline{C}$ are conjugate, that is, if the characteristic polynomial of C is

$$|\lambda I - C| = \lambda^n + (f_1 + ig_1)\lambda^{n-1} + \cdots + (f_{n-1} + ig_{n-1})\lambda + (f_n + ig_n),$$

where f_i, g_i, $i = 1, 2, \ldots, n$, are real, then the characteristic polynomial of $\overline{C}$ is

$$|\lambda I - \overline{C}| = \lambda^n + (f_1 - ig_1)\lambda^{n-1} + \cdots + (f_{n-1} - ig_{n-1})\lambda + (f_n - ig_n).$$

By the definition of spectrally arbitrary complex sign pattern matrix, the result holds for the case (v). □

We note that, if a complex sign pattern matrix $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ is spectrally arbitrary, then sign pattern matrices $\mathcal{A}$ and $\mathcal{B}$ are not necessarily spectrally arbitrary. For example,

$$\mathcal{S}_3 = \begin{bmatrix} 1-i & 1 & 0 \\ 1+i & 0 & -1 \\ 1 & 0 & -1+i \end{bmatrix}$$

is a spectrally arbitrary complex sign pattern matrix (This fact will be proved in Section 3), but both sign pattern matrices

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not spectrally arbitrary. On the other hand, if both $\mathcal{A}$ and $\mathcal{B}$ are spectrally arbitrary, then the complex sign pattern matrix $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ is not necessarily spectrally

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arbitrary. For example, let

$$\mathcal{A} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

From [1], both $\mathcal{A}$ and $\mathcal{B}$ are spectrally arbitrary sign pattern matrices. Consider the complex sign pattern matrix

$$\mathcal{S} = \mathcal{A} + i\mathcal{B} = \begin{bmatrix} -1-i & 1-i \\ -1+i & 1+i \end{bmatrix}.$$

Note that for any

$$C = \begin{bmatrix} -a_1 - ib_1 & a_2 - ib_2 \\ -a_3 + ib_3 & a_4 + ib_4 \end{bmatrix} \in Q_c(\mathcal{S}),$$

where $a_j > 0$ and $b_j > 0$ for $j = 1, 2, 3, 4$, the characteristic polynomial of C is

$$|\lambda I - C| = \lambda^2 + ((a_1 - a_4) + i(b_1 - b_4))\lambda + (a_2a_3 - a_1a_4 - b_2b_3 + b_1b_4)$$

$$-i(a_4b_1 + a_3b_2 + a_2b_3 + a_1b_4).$$

Since $-(a_4b_1 + a_3b_2 + a_2b_3 + a_1b_4) < 0$, $\mathcal{S}$ is not spectrally arbitrary.

In Section 2 we extend the Nilpotent-Jacobian method for sign pattern matrices to complex sign pattern matrices, establishing a means to show that an irreducible complex sign pattern matrix and all its superpatterns are spectrally arbitrary. In Section 3 we give an $n \times n$ $(n \geq 2)$ irreducible spectrally arbitrary complex sign pattern matrix $\mathcal{S}_n$ with exactly $3n$ nonzero entries. In Section 4 we prove that every $n \times n$ $(n \geq 2)$ irreducible spectrally arbitrary complex sign pattern matrix has at least $3n - 1$ nonzero entries, and conjecture that for $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix has at least $3n$ nonzero entries.

2. The Nilpotent-Jacobian method. In this section, we extend the Nilpotent -Jacobian method on sign pattern matrices in [1] to the case of complex sign pattern matrices.

Let $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ be a complex sign pattern matrix of order $n \geq 2$ with at least $2n$ nonzero entries.

- Find a nilpotent complex matrix $C = A + iB$ in the complex sign pattern class $Q_c(\mathcal{S})$, where both A and B are real matrices, and $A \in Q(\mathcal{A})$ and $B \in Q(\mathcal{B})$.
- Change the $2n$ nonzero entries (denoted $r_1, r_2, \ldots, r_{2n}$) in A and B to variables $x_1, x_2, \ldots, x_{2n}$. Denote the resulting matrix by X.

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- Express the characteristic polynomial of X as:

$$|\lambda I - X| = \lambda^n + (f_1(x_1,x_2,\ldots,x_{2n}) + ig_1(x_1,x_2,\ldots,x_{2n}))\lambda^{n-1} + \cdots$$
$$+(f_{n-1}(x_1,x_2,\ldots,x_{2n}) + ig_{n-1}(x_1,x_2,\ldots,x_{2n}))\lambda$$
$$+(f_n(x_1,x_2,\ldots,x_{2n}) + ig_n(x_1,x_2,\ldots,x_{2n})).$$

- Find the Jacobian matrix

$$J = \frac{\partial(f_1,\ldots,f_n,g_1,\ldots,g_n)}{\partial(x_1,x_2,\ldots,x_{2n})}.$$

- If the determinant of J, evaluated at $(x_1,x_2,\ldots,x_{2n}) = (r_1,r_2,\ldots,r_{2n})$ is nonzero, then by continuity of the determinant in the entries of a matrix, there is a neighborhood U of $(r_1,r_2,\ldots,r_{2n})$ such that all the vectors in U are strictly positive and the determinant of J evaluated at any of these vectors is nonzero. Moreover, by the Implicit Function Theorem, there is a neighborhood $V \subseteq U$ of $(r_1,r_2,\ldots,r_{2n}) \subseteq \mathbb{R}^{2n}$, a neighborhood W of $(0,0,\ldots,0) \subseteq \mathbb{R}^{2n}$, and a function $(h_1,\ldots,h_{2n})$ from W into V such that for any $(y_1,\ldots,y_n,z_1,\ldots,z_n) \in W$, there exists a strictly positive vector $(s_1,s_2,\ldots,s_{2n}) = (h_1,\ldots,h_{2n})(y_1,\ldots,y_n,z_1,\ldots,z_n) \in V$ where $f_k(s_1,s_2,\ldots,s_{2n}) = y_k$ and $g_k(s_1,s_2,\ldots,s_{2n}) = z_k$ for $k = 1,2,\ldots,n$. Taking positive scalar multiples of the corresponding matrices, we see that each monic nth degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in the complex sign pattern class $Q_c(\mathcal{S})$. That is, $\mathcal{S}$ is a spectrally arbitrary complex sign pattern matrix.

Next consider a superpattern of the complex sign pattern matrix $\mathcal{S}$. Represent the new nonzero entries of A by $p_1,\ldots,p_{m_1}$, and the new nonzero entries of B by $q_1,\ldots,q_{m_2}$, Let $\hat{f}_k(x_1,x_2,\ldots,x_{2n},p_1,\ldots,p_{m_1},q_1,\ldots,q_{m_2})$ and $\hat{g}_k(x_1,x_2,\ldots,x_{2n},p_1,\ldots,p_{m_1},q_1,\ldots,q_{m_2})$ represent the new functions in the characteristic polynomial, and $\hat{J} = \frac{\partial(\hat{f}_1,\ldots,\hat{f}_n,\hat{g}_1,\ldots,\hat{g}_n)}{\partial(x_1,x_2,\ldots,x_{2n})}$ the new Jacobian matrix. As above, let $(y_1,\ldots,y_n,z_1,\ldots,z_n) \in W$ and $(s_1,s_2,\ldots,s_{2n}) = (h_1,\ldots,h_{2n})\ (y_1,\ldots,y_n,z_1,\ldots,z_n)$. Then $y_k = f_k(s_1,s_2,\ldots,s_{2n}) = \hat{f}_k(s_1, s_2,\ldots,s_{2n},0,\ldots,0)$, $z_k = g_k(s_1,s_2,\ldots,s_{2n}) = \hat{g}_k(s_1,s_2,\ldots,s_{2n},0,\ldots,0)$, and the determinant of $\hat{J}$ evaluated at $(x_1,\ldots,x_{2n},p_1,\ldots,p_{m_1},q_1,\ldots,q_{m_2}) = (s_1,\ldots,s_{2n},0,0,\ldots,0)$ is equal to the determinant of J evaluated at $(x_1,x_2, \ldots,x_{2n}) = (s_1,s_2,\ldots,s_{2n})$ and hence is nonzero. By the Implicit Function Theorem, there exists a neighborhood $\hat{V} \subseteq V$ of $(s_1,s_2,\ldots,s_{2n})$, a neighborhood T of $(0,0,\ldots,0) \in \mathbb{R}^{m_1+m_2}$ and a function $(\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_{2n})$ from T into $\hat{V}$ such that for any vector $(d_1,\ldots,d_{m_1+m_2}) \in T$ there exists a strictly positive vector $(e_1,e_2,\ldots,e_{2n}) = (\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_{2n})(d_1,\ldots,d_{m_1+m_2}) \in \hat{V}$ where

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$\hat{f}_k(e_1,\ldots,e_{2n},\ d_1,\ldots,d_{m_1+m_2}) = y_k$ and $\hat{g}_k(e_1,\ldots,e_{2n},d_1,\ldots,d_{m_1+m_2}) = z_k$. Choosing $(d_1,\ldots,d_{m_1+m_2}) \in T$ strictly positive we see that there are also matrices in the superpattern's class with every characteristic polynomial corresponding to a vector in W. Taking positive scalar multiples of the corresponding matrices, we see that each monic nth degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in this superpattern's class. Thus each superpattern of $\mathcal{S}$ is a spectrally arbitrary complex sign pattern matrix.

THEOREM 2.1. *Let $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ be a complex sign pattern matrix of order $n \geq 2$, and suppose that there exists some nilpotent complex matrix $C = A + iB \in Q_c(\mathcal{S})$, where $A \in Q(\mathcal{A})$, $B \in Q(\mathcal{B})$, and A and B have at least $2n$ nonzero entries, say $a_{i_1j_1},\ldots,a_{i_{n_1}j_{n_1}}, b_{i_{n_1+1}j_{n_1+1}},\ldots,b_{i_{2n}j_{2n}}$. Let X be the complex matrix obtained by replacing these entries in C by variables $x_1,\ldots,x_{2n}$, and let the characteristic polynomial of X be*

$$|\lambda I - X| = \lambda^n + (f_1(x_1,x_2,\ldots,x_{2n}) + ig_1(x_1,x_2,\ldots,x_{2n}))\lambda^{n-1} + \cdots$$

$$+(f_{n-1}(x_1,x_2,\ldots,x_{2n}) + ig_{n-1}(x_1,x_2,\ldots,x_{2n}))\lambda$$

$$+(f_n(x_1,x_2,\ldots,x_{2n}) + ig_n(x_1,x_2,\ldots,x_{2n})).$$

If the Jacobian matrix $J = \dfrac{\partial(f_1,\ldots,f_n,g_1,\ldots,g_n)}{\partial(x_1,x_2,\ldots,x_{2n})}$ is nonsingular at $(x_1,\ldots,x_{2n}) = (a_{i_1j_1},\ldots,a_{i_{n_1}j_{n_1}},\ b_{i_{n_1+1}j_{n_1+1}},\ldots,b_{i_{2n}j_{2n}})$, then the complex sign pattern matrix $\mathcal{S}$ is spectrally arbitrary, and every superpattern of $\mathcal{S}$ is a spectrally arbitrary complex sign pattern matrix.

3. Minimal spectrally arbitrary complex sign pattern matrices.

In this section we first consider the following $n \times n$ ($n \geq 7$) complex sign pattern matrix

$$(3.1)\ \mathcal{S}_n = \mathcal{A}_n + i\mathcal{B}_n = \begin{bmatrix} 1+i & 1 & & & & & & \\ 1-i & 0 & -1 & & & & & \\ 1+i & & 0 & 1 & & & & \\ 1-i & & & \ddots & -1 & & & \\ \vdots & & & & \ddots & \ddots & & \\ \vdots & & & & & 0 & \ddots & \\ 1+(-1)^n i & & & & & & -i & (-1)^n \\ 0 & 0 & 0 & (-1)^{\lceil \frac{n+1}{2} \rceil} & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

We will prove that $\mathcal{S}_n$ is a minimal spectrally arbitrary complex sign pattern matrix, and every superpattern of $\mathcal{S}_n$ is a spectrally arbitrary complex sign pattern matrix.

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Take an $n \times n$ complex matrix

$$C = \left[\begin{array}{cccccccc}
a_1 + ib_1 & 1 & & & & & & \\
a_2 - ib_2 & 0 & -1 & & & & & \\
a_3 + ib_3 & & 0 & 1 & & & & \\
a_4 - ib_4 & & & \ddots & -1 & & & \\
\vdots & & & & \ddots & \ddots & & \\
\vdots & & & & & 0 & \ddots & \\
a_{n-1} + (-1)^n ib_{n-1} & & & & & & -ib_n & (-1)^n \\
0 & 0 & 0 & (-1)^{\lceil \frac{n+1}{2} \rceil} a_n & 0 & \cdots & 0 & -1
\end{array}\right], \tag{3.2}$$

where $a_k > 0$ and $b_k > 0$ for $k = 1, 2, \ldots, n$. Then $C \in Q_c(\mathcal{S}_n)$. Denote

$$|\lambda I - C| = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_k \lambda^{n-k} + \cdots + \alpha_{n-1}\lambda + \alpha_n,$$

and $\alpha_k = f_k + ig_k$, $k = 1, 2, \ldots, n$.

Lemma 3.1. *Let $a_0 = 1$ and $b_0 = 0$. Then*

$$\begin{aligned}
&f_1 = 1 - a_1,\\
&f_k = (-1)^{\lceil \frac{3k+3}{2} \rceil} a_k + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} + (-1)^{\lceil \frac{5k+2}{2} \rceil} b_{k-1} b_n + (-1)^{\lceil \frac{5k-3}{2} \rceil} b_{k-2} b_n,\\
&\qquad k = 2, 3, \ldots, n-4,\\
&f_{n-3} = (-1)^n a_n + (-1)^{\lceil \frac{3n-6}{2} \rceil} a_{n-3} + (-1)^{\lceil \frac{3n-9}{2} \rceil} a_{n-4} + (-1)^{\lceil \frac{5n-13}{2} \rceil} b_{n-4} b_n\\
&\qquad + (-1)^{\lceil \frac{5n-18}{2} \rceil} b_{n-5} b_n,\\
&f_{n-2} = (-1)^{n+1} a_1 a_n + (-1)^{\lceil \frac{3n-3}{2} \rceil} a_{n-2} + (-1)^{\lceil \frac{3n-6}{2} \rceil} a_{n-3} + (-1)^{\lceil \frac{5n-8}{2} \rceil} b_{n-3} b_n\\
&\qquad + (-1)^{\lceil \frac{5n-13}{2} \rceil} b_{n-4} b_n,\\
&f_{n-1} = (-1)^{n+1} a_2 a_n + (-1)^{\lceil \frac{3n}{2} \rceil} a_{n-1} + (-1)^{\lceil \frac{3n-3}{2} \rceil} a_{n-2} + (-1)^{\lceil \frac{5n-3}{2} \rceil} b_{n-2} b_n\\
&\qquad + (-1)^{\lceil \frac{5n-8}{2} \rceil} b_{n-3} b_n,\\
&f_n = (-1)^n a_3 a_n + (-1)^{\lceil \frac{3n}{2} \rceil} a_{n-1} + (-1)^{\lceil \frac{5n-3}{2} \rceil} b_{n-2} b_n,
\end{aligned}$$

and

$$\begin{aligned}
&g_1 = -b_1 + b_n,\\
&g_k = (-1)^{\lceil \frac{5(k+1)}{2} \rceil} b_k + (-1)^{\lceil \frac{5k}{2} \rceil} b_{k-1} + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} b_n + (-1)^{\lceil \frac{3(k-1)}{2} \rceil} a_{k-2} b_n,\\
&\qquad k = 2, 3, \ldots, n-3,\\
&g_{n-2} = (-1)^{n+1} a_n b_1 + (-1)^{\lceil \frac{5(n-1)}{2} \rceil} b_{n-2} + (-1)^{\lceil \frac{5(n-2)}{2} \rceil} b_{n-3} + (-1)^{\lceil \frac{3(n-2)}{2} \rceil} a_{n-3} b_n\\
&\qquad + (-1)^{\lceil \frac{3(n-3)}{2} \rceil} a_{n-4} b_n,\\
&g_{n-1} = (-1)^{n+2} a_n b_2 + (-1)^{\lceil \frac{5n}{2} \rceil} b_{n-1} + (-1)^{\lceil \frac{5(n-1)}{2} \rceil} b_{n-2} + (-1)^{\lceil \frac{3(n-1)}{2} \rceil} a_{n-2} b_n\\
&\qquad + (-1)^{\lceil \frac{3(n-2)}{2} \rceil} a_{n-3} b_n,\\
&g_n = (-1)^n a_n b_3 + (-1)^{\lceil \frac{5n}{2} \rceil} b_{n-1} + (-1)^{\lceil \frac{3n-3}{2} \rceil} a_{n-2} b_n.
\end{aligned}$$

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Proof. Denote $c_0 = 1$, and $c_k = a_k + (-1)^{k+1} i b_k$ for $k = 1, 2, \ldots, n-1$. Then

$$|\lambda I - C| = \left| \begin{array}{cccccccc} \lambda - c_1 & -1 & & & & & & \\ -c_2 & \lambda & 1 & & & & & \\ -c_3 & & \ddots & -1 & & & & \\ -c_4 & & & \ddots & 1 & & & \\ \vdots & & & & \ddots & \ddots & & \\ \vdots & & & & & \lambda & \ddots & \\ -c_{n-1} & & & & & & \lambda + i b_n & (-1)^{n-1} \\ 0 & 0 & 0 & (-1)^{\lceil \frac{n+3}{2} \rceil} a_n & 0 & \cdots & 0 & \lambda + 1 \end{array} \right|$$

$$= (\lambda + 1) \left| \begin{array}{ccccccc} \lambda - c_1 & -1 & & & & & \\ -c_2 & \lambda & 1 & & & & \\ -c_3 & & \ddots & -1 & & & \\ -c_4 & & & \ddots & 1 & & \\ \vdots & & & & \ddots & \ddots & \\ \vdots & & & & & \lambda & (-1)^{n-2} \\ -c_{n-1} & & & & & & \lambda + i b_n \end{array} \right|_{n-1}$$

$$+(-1)^{\lceil \frac{n+3}{2} \rceil + n + 4 + \lceil \frac{n-5}{2} \rceil} a_n \left| \begin{array}{ccc} \lambda - c_1 & -1 & 0 \\ -c_2 & \lambda & 1 \\ -c_3 & 0 & \lambda \end{array} \right|$$

$$= (-1)^n a_n (\lambda^3 - c_1 \lambda^2 - c_2 \lambda + c_3) + (\lambda + 1)(-1)^{\lceil \frac{3n}{2} \rceil} c_{n-1} + (\lambda + 1)(\lambda + i b_n) \Delta_{n-2},$$

where

$$\Delta_{n-2} = \left| \begin{array}{cccccc} \lambda - c_1 & -1 & & & & \\ -c_2 & \lambda & 1 & & & \\ -c_3 & & \ddots & -1 & & \\ -c_4 & & & \ddots & \ddots & \\ \vdots & & & & \ddots & (-1)^{n-3} \\ -c_{n-2} & & & & & \lambda \end{array} \right|_{n-2}$$

$$= (-1)^{\lceil \frac{3n-3}{2} \rceil} c_{n-2} + \lambda \Delta_{n-3}$$

$$= (-1)^{\lceil \frac{3n-3}{2} \rceil} c_{n-2} + (-1)^{\lceil \frac{3n-6}{2} \rceil} c_{n-3} \lambda + \lambda^2 \Delta_{n-4}$$

$$= \cdots\cdots$$

$$= (-1)^{\lceil \frac{3n-3}{2} \rceil} c_{n-2} + (-1)^{\lceil \frac{3n-6}{2} \rceil} c_{n-3} \lambda + (-1)^{\lceil \frac{3n-9}{2} \rceil} c_{n-4} \lambda^2 + \cdots$$

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$$+(-1)^{\lceil\frac{3(n-k-1)}{2}\rceil}c_{n-k-2}\lambda^k+\cdots-c_2\lambda^{n-4}-c_1\lambda^{n-3}+\lambda^{n-2}$$
$$=\sum_{k=0}^{n-2}(-1)^{\lceil\frac{3(k+1)}{2}\rceil}c_k\lambda^{n-k-2}.$$

So

$$|\lambda I-C|=(-1)^na_n(\lambda^3-c_1\lambda^2-c_2\lambda+c_3)+(\lambda+1)(-1)^{\lceil\frac{3n}{2}\rceil}c_{n-1}$$
$$+(\lambda^2+(1+ib_n)\lambda+ib_n)\sum_{k=0}^{n-2}(-1)^{\lceil\frac{3(k+1)}{2}\rceil}c_k\lambda^{n-k-2}.$$

Thus

$$\alpha_1=-c_1+(1+ib_n),$$
$$\alpha_k=(-1)^{\lceil\frac{3(k+1)}{2}\rceil}c_k+(-1)^{\lceil\frac{3k}{2}\rceil}c_{k-1}(1+ib_n)+(-1)^{\lceil\frac{3(k-1)}{2}\rceil}ic_{k-2}b_n,$$
$$k=2,3,\dots,n-4,$$
$$\alpha_{n-3}=(-1)^na_n+(-1)^{\lceil\frac{3n-6}{2}\rceil}c_{n-3}+(-1)^{\lceil\frac{3n-9}{2}\rceil}c_{n-4}(1+ib_n)+(-1)^{\lceil\frac{3n-12}{2}\rceil}ic_{n-5}b_n,$$
$$\alpha_{n-2}=(-1)^{n+1}a_nc_1+(-1)^{\lceil\frac{3n-3}{2}\rceil}c_{n-2}+(-1)^{\lceil\frac{3n-6}{2}\rceil}c_{n-3}(1+ib_n)$$
$$+(-1)^{\lceil\frac{3n-9}{2}\rceil}ic_{n-4}b_n,$$
$$\alpha_{n-1}=(-1)^{n+1}a_nc_2+(-1)^{\lceil\frac{3n}{2}\rceil}c_{n-1}+(-1)^{\lceil\frac{3n-3}{2}\rceil}c_{n-2}(1+ib_n)$$
$$+(-1)^{\lceil\frac{3n-6}{2}\rceil}ic_{n-3}b_n,$$
$$\alpha_n=(-1)^na_nc_3+(-1)^{\lceil\frac{3n}{2}\rceil}c_{n-1}+(-1)^{\lceil\frac{3n-3}{2}\rceil}ic_{n-2}b_n.$$

Noticing that $c_k=a_k+(-1)^{k+1}ib_k$ for $k=1,2,\dots,n-1$, the lemma holds. ◻

LEMMA 3.2. *There are unique positive integers $\hat{a}_k$ and $\hat{b}_k$, $k=1,2,\dots,n$, such that when $a_k=\hat{a}_k$ and $b_k=\hat{b}_k$ for $k=1,2,\dots,n$, the complex matrix C having the form (3.2) is nilpotent. Further,* $\det(\dfrac{\partial(f_1,\dots,f_n,g_1,\dots,g_n)}{\partial(a_1,\dots,a_n,b_1,\dots,b_n)})|_{a_k=\hat{a}_k,b_k=\hat{b}_k,k=1,\dots,n}$ $=(-1)^{\lceil\frac{n+2}{2}\rceil}6.$

Proof. We prove the lemma according to the four cases $n=4m$, $n=4m+1$, $n=4m+2$, and $n=4m+3$.

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Let $n = 4m$. By Lemma 3.1, we have

$$\begin{cases} f_1 = 1 - a_1, \\ f_k = (-1)^{\lceil \frac{3k+3}{2} \rceil} a_k + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} + (-1)^{\lceil \frac{5k+2}{2} \rceil} b_{k-1} b_n + (-1)^{\lceil \frac{5k-3}{2} \rceil} b_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-4, \\ f_{n-3} = a_n - a_{n-3} + a_{n-4} + b_{n-4} b_n - b_{n-5} b_n, \\ f_{n-2} = -a_1 a_n - a_{n-2} - a_{n-3} + b_{n-3} b_n + b_{n-4} b_n, \\ f_{n-1} = -a_2 a_n + a_{n-1} - a_{n-2} - b_{n-2} b_n + b_{n-3} b_n, \\ f_n = a_3 a_n + a_{n-1} - b_{n-2} b_n, \end{cases}$$

and

$$\begin{cases} g_1 = -b_1 + b_n, \\ g_k = (-1)^{\lceil \frac{5(k+1)}{2} \rceil} b_k + (-1)^{\lceil \frac{5k}{2} \rceil} b_{k-1} + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} b_n + (-1)^{\lceil \frac{3(k-1)}{2} \rceil} a_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-3, \\ g_{n-2} = -a_n b_1 + b_{n-2} - b_{n-3} - a_{n-3} b_n + a_{n-4} b_n, \\ g_{n-1} = a_n b_2 + b_{n-1} + b_{n-2} - a_{n-2} b_n - a_{n-3} b_n, \\ g_n = a_n b_3 + b_{n-1} - a_{n-2} b_n. \end{cases}$$

Let $f_k = 0$ and $g_k = 0$ for $k = 1, 2, \ldots, n$. Then

$$\begin{cases} a_1 = 1, \\ a_{2k} = a_{2k+1}, \ k = 1, 2, \ldots, \frac{n}{2} - 3, \\ a_{n-4} = a_{n-3} - a_n, \\ a_{n-2} = a_{n-1} - 2a_n b_1^2 - a_2 a_n, \\ a_{n-1} = b_{n-2} b_n - a_3 a_n, \\ a_{2k-1} + a_{2k} = b_1^{2k}, \ k = 1, 2, \ldots, \frac{n}{2} - 2, \\ a_{n-3} + a_{n-2} = b_1^{n-2} - a_n, \end{cases}$$

and

$$\begin{cases} b_1 = b_2 = b_n, \\ b_{2k+1} = b_{2k+2}, \ k = 1, 2, \ldots, \frac{n}{2} - 3, \\ b_{n-3} = b_{n-2} - 2a_n b_1, \\ b_{n-1} = a_{n-2} b_n - a_n b_3, \\ b_{2k} + b_{2k+1} = b_1^{2k+1}, \ k = 1, 2, \ldots, \frac{n}{2} - 2, \\ b_{n-2} + b_{n-1} = b_1^{n-1} - a_n b_1 - a_n b_2. \end{cases}$$

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We have that

$$\begin{cases} a_1 = 1, \\ a_{2k} = a_{2k+1} = \displaystyle\sum_{j=0}^{k} (-1)^{k-j} b_1^{2j}, \ k = 1, 2, \ldots, \frac{n}{2} - 3, \\ a_{n-4} = \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-j} b_1^{2j}, \\ a_{n-3} = a_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-j} b_1^{2j}, \\ a_{n-2} = -2a_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} b_1^{2j}, \\ a_{n-1} = 2a_n b_1^2 + a_2 a_n - 2a_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} b_1^{2j}, \\ a_{n-1} = 2a_n b_1^2 - a_3 a_n + \displaystyle\sum_{j=1}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} b_1^{2j}, \end{cases}$$

and

$$\begin{cases} b_1 = b_2 = b_n \\ b_{2k+1} = b_{2k+2} = \displaystyle\sum_{j=0}^{k} (-1)^{k-j} b_1^{2j+1}, \ k = 1, 2, \ldots, \frac{n}{2} - 3, \\ b_{n-3} = \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-2-j} b_1^{2j+1}, \\ b_k = b_1 a_{k-1}, \ k = 3, 4, \ldots, n-3, \\ b_{n-2} = 2a_n b_1 + \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-2-j} b_1^{2j+1}, \\ b_{n-1} = -a_n b_3 - 2a_n b_1 + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} b_1^{2j+1}, \\ b_{n-1} = -4a_n b_1 + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} b_1^{2j+1}. \end{cases}$$

From the second equation and last two equations in the second set of equations, respectively, we have $b_3 = -b_1 + b_1^3$, and $a_n b_3 + 2a_n b_1 = 4a_n b_1$, so $b_1 = \sqrt{3}$. From the second equation and last two equations in the first set of equations, respectively, we have $a_2 = -1 + b_1^2$, and $2a_2 a_n - 2a_n - 1 = 0$, so $a_n = \frac{1}{2b_1^2 - 4} = \frac{1}{2}$. Thus there is

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unique solution for $f_k = 0$ and $g_k = 0$, $k = 1, 2, \ldots, n$, as follows.

$$
\begin{cases}
\hat{a}_1 = 1, \quad \hat{a}_n = \frac{1}{2}, \quad \hat{b}_1 = \hat{b}_2 = \hat{b}_n = \sqrt{3}, \\
\hat{a}_{2k} = \hat{a}_{2k+1} = \displaystyle\sum_{j=0}^{k} (-1)^{k-j} \hat{b}_1^{2j}, \ k = 1, 2, \ldots, \frac{n}{2} - 3, \\
\hat{a}_{n-4} = \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-j} \hat{b}_1^{2j}, \\
\hat{a}_{n-3} = \hat{a}_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-j} \hat{b}_1^{2j}, \\
\hat{a}_{n-2} = -2\hat{a}_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} \hat{b}_1^{2j}, \\
\hat{a}_{n-1} = 2\hat{a}_n \hat{b}_1^2 + \hat{a}_2 \hat{a}_n - 2\hat{a}_n + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} \hat{b}_1^{2j}, \\
\hat{b}_k = \hat{b}_1 \hat{a}_{k-1}, \ k = 3, 4, \ldots, n-3, \\
\hat{b}_{n-2} = 2\hat{a}_n \hat{b}_1 + \displaystyle\sum_{j=0}^{\frac{n}{2}-2} (-1)^{\frac{n}{2}-2-j} \hat{b}_1^{2j+1}, \\
\hat{b}_{n-1} = -\hat{a}_n \hat{b}_3 - 2\hat{a}_n \hat{b}_1 + \displaystyle\sum_{j=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1-j} \hat{b}_1^{2j+1}.
\end{cases}
$$

Since $\det(J) = \det\left(\dfrac{\partial(f_1, \ldots, f_n, g_1, \ldots, g_n)}{\partial(a_1, \ldots, a_n, b_1, \ldots, b_n)}\right) =$

$$
\left|\begin{array}{ccccccccc|ccccccccc}
-1 & & & & & & & & & 0 & & & & & & & & 0 \\
-1 & -1 & & & & & & & & b_n & & & & & & & & b_1 \\
 & -1 & 1 & & & & & & & b_n & -b_n & & & & & & & b_1 - b_2 \\
 & & 1 & 1 & & & & & & & -b_n & -b_n & & & & & & -b_2 - b_3 \\
 & & & 1 & \ddots & & & & & & & \ddots & \ddots & & & & & \vdots \\
 & & & & \ddots & -1 & & & 1 & & & & -b_n & b_n & & & & -b_{n-5} + b_{n-4} \\
-a_n & & & & & -1 & -1 & & -a_1 & & & & & b_n & b_n & & & b_{n-4} + b_{n-3} \\
 & -a_n & & & & & -1 & 1 & -a_2 & & & & & & b_n & -b_n & & b_{n-3} - b_{n-2} \\
 & & a_n & & & & & 1 & a_3 & & & & & & & -b_n & 0 & -b_{n-2} \\
\hline
0 & & & & & & & & & -1 & & & & & & & & 1 \\
-b_n & & & & & & & & & -1 & 1 & & & & & & & 1 - a_1 \\
-b_n & -b_n & & & & & & & & & 1 & 1 & & & & & & -a_1 - a_2 \\
 & -b_n & b_n & & & & & & & & & 1 & -1 & & & & & -a_2 + a_3 \\
 & & \ddots & \ddots & & & & & & & & & \ddots & \ddots & & & & \vdots \\
 & & & b_n & b_n & & & & & & & & & -1 & -1 & & & a_{n-5} + a_{n-4} \\
 & & & & b_n & -b_n & & & -b_1 & -a_n & & & & & -1 & 1 & & a_{n-4} - a_{n-3} \\
 & & & & & -b_n & -b_n & & b_2 & & a_n & & & & & 1 & 1 & -a_{n-3} - a_{n-2} \\
 & & & & & & -b_n & 0 & b_3 & & & a_n & & & & & 1 & -a_{n-2}
\end{array}\right|_{2n}
$$

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$$
= \left|\begin{array}{ccccccccc|ccccccccc}
-1 & & & & & & & & & 0 & & & & & & & & 0 \\
 & -1 & & & & & & & & b_n & & & & & & & & b_1 \\
 & & 1 & & & & & & & & -b_n & & & & & & & -b_2 \\
 & & & 1 & & & & & & & & -b_n & & & & & & -b_3 \\
 & & & & \ddots & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & -1 & & & 1 & & & & & b_n & & & & b_{n-4} \\
 & & & & & & -1 & & -a_1-1 & & & & & & b_n & & & b_{n-3} \\
0 & -a_n & & & & & & 1 & -a_2+a_1+1 & & & & & & & -b_n & & -b_{n-2} \\
0 & a_n & a_n & & & & & & a_3+a_2-a_1-1 & & & & & & & & 0 & 0 \\
\hline
0 & & & & & & & & & -1 & & & & & & & & 1 \\
0 & & & & & & & & & & 1 & & & & & & & -a_1 \\
0 & -b_n & & & & & & & & & & 1 & & & & & & -a_2 \\
 & & b_n & & & & & & & & & & -1 & & & & & a_3 \\
 & & & \ddots & & & & & & & & & & \ddots & & & & \vdots \\
 & & & & b_n & & & & & & & & & & -1 & & & a_{n-4} \\
 & & & & & -b_n & & & -b_1 & -a_n & & & & & & 1 & & -a_{n-3} \\
 & & & & & & -b_n & & b_2+b_1 & a_n & a_n & & & & & & 1 & -a_{n-2} \\
 & & & & & & & 0 & b_3-b_2-b_1 & -a_n & -a_n & a_n & & & & & & 0
\end{array}\right|_{2n}
$$

$$
= -\left|\begin{array}{ccc|cccc}
-1 & 0 & 0 & b_n & 0 & 0 & b_1 \\
0 & 1 & 0 & 0 & -b_n & 0 & -b_2 \\
a_n & a_n & a_3+a_2-a_1-1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -a_1 \\
-b_n & 0 & 0 & 0 & 0 & 1 & -a_2 \\
0 & 0 & b_3-b_2-b_1 & -a_n & -a_n & a_n & 0
\end{array}\right|,
$$

we have

$$
\det(J)|_{a_k=\hat{a}_k, b_k=\hat{b}_k, k=1,2,\ldots,n} = -\begin{vmatrix}
-1 & 0 & 0 & \sqrt{3} & 0 & 0 & \sqrt{3} \\
0 & 1 & 0 & 0 & -\sqrt{3} & 0 & -\sqrt{3} \\
\frac{1}{2} & \frac{1}{2} & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 \\
-\sqrt{3} & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0
\end{vmatrix} = -6.
$$

As for cases $n = 4m+1$, $n = 4m+2$ and $n = 4m+3$, noting that if $n = 4m+1$,

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then

$$\begin{cases} f_1 = 1 - a_1, \\ f_k = (-1)^{\lceil \frac{3k+3}{2} \rceil} a_k + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} + (-1)^{\lceil \frac{5k+2}{2} \rceil} b_{k-1} b_n + (-1)^{\lceil \frac{5k-3}{2} \rceil} b_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-4, \\ f_{n-3} = -a_n - a_{n-3} - a_{n-4} + b_{n-4} b_n + b_{n-5} b_n, \\ f_{n-2} = a_1 a_n + a_{n-2} - a_{n-3} - b_{n-3} b_n + b_{n-4} b_n, \\ f_{n-1} = a_2 a_n + a_{n-1} + a_{n-2} - b_{n-2} b_n - b_{n-3} b_n, \\ f_n = -a_3 a_n + a_{n-1} - b_{n-2} b_n, \end{cases}$$

and

$$\begin{cases} g_1 = -b_1 + b_n, \\ g_k = (-1)^{\lceil \frac{5(k+1)}{2} \rceil} b_k + (-1)^{\lceil \frac{5k}{2} \rceil} b_{k-1} + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} b_n + (-1)^{\lceil \frac{3(k-1)}{2} \rceil} a_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-3, \\ g_{n-2} = a_n b_1 + b_{n-2} + b_{n-3} - a_{n-3} b_n - a_{n-4} b_n, \\ g_{n-1} = -a_n b_2 - b_{n-1} + b_{n-2} + a_{n-2} b_n - a_{n-3} b_n, \\ g_n = -a_n b_3 - b_{n-1} + a_{n-2} b_n; \end{cases}$$

if $n = 4m + 2$, then

$$\begin{cases} f_1 = 1 - a_1, \\ f_k = (-1)^{\lceil \frac{3k+3}{2} \rceil} a_k + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} + (-1)^{\lceil \frac{5k+2}{2} \rceil} b_{k-1} b_n + (-1)^{\lceil \frac{5k-3}{2} \rceil} b_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-4, \\ f_{n-3} = a_n + a_{n-3} - a_{n-4} - b_{n-4} b_n + b_{n-5} b_n, \\ f_{n-2} = -a_1 a_n + a_{n-2} + a_{n-3} - b_{n-3} b_n - b_{n-4} b_n, \\ f_{n-1} = -a_2 a_n - a_{n-1} + a_{n-2} + b_{n-2} b_n - b_{n-3} b_n, \\ f_n = a_3 a_n - a_{n-1} + b_{n-2} b_n, \end{cases}$$

and

$$\begin{cases} g_1 = -b_1 + b_n, \\ g_k = (-1)^{\lceil \frac{5(k+1)}{2} \rceil} b_k + (-1)^{\lceil \frac{5k}{2} \rceil} b_{k-1} + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} b_n + (-1)^{\lceil \frac{3(k-1)}{2} \rceil} a_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-3, \\ g_{n-2} = -a_n b_1 - b_{n-2} + b_{n-3} + a_{n-3} b_n - a_{n-4} b_n, \\ g_{n-1} = a_n b_2 - b_{n-1} - b_{n-2} + a_{n-2} b_n + a_{n-3} b_n, \\ g_n = a_n b_3 - b_{n-1} + a_{n-2} b_n; \end{cases}$$

if $n = 4m + 3$, then

$$\begin{cases} f_1 = 1 - a_1, \\ f_k = (-1)^{\lceil \frac{3k+3}{2} \rceil} a_k + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} + (-1)^{\lceil \frac{5k+2}{2} \rceil} b_{k-1} b_n + (-1)^{\lceil \frac{5k-3}{2} \rceil} b_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-4, \\ f_{n-3} = -a_n + a_{n-3} + a_{n-4} - b_{n-4} b_n - b_{n-5} b_n, \\ f_{n-2} = a_1 a_n - a_{n-2} + a_{n-3} + b_{n-3} b_n - b_{n-4} b_n, \\ f_{n-1} = a_2 a_n - a_{n-1} - a_{n-2} + b_{n-2} b_n + b_{n-3} b_n, \\ f_n = -a_3 a_n - a_{n-1} + b_{n-2} b_n, \end{cases}$$

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and

$$\begin{cases} g_1 = -b_1 + b_n, \\ g_k = (-1)^{\lceil \frac{5(k+1)}{2} \rceil} b_k + (-1)^{\lceil \frac{5k}{2} \rceil} b_{k-1} + (-1)^{\lceil \frac{3k}{2} \rceil} a_{k-1} b_n + (-1)^{\lceil \frac{3(k-1)}{2} \rceil} a_{k-2} b_n, \\ \qquad k = 2, 3, \ldots, n-3, \\ g_{n-2} = a_n b_1 - b_{n-2} - b_{n-3} + a_{n-3} b_n + a_{n-4} b_n, \\ g_{n-1} = -a_n b_2 + b_{n-1} - b_{n-2} - a_{n-2} b_n + a_{n-3} b_n, \\ g_n = -a_n b_3 + b_{n-1} - a_{n-2} b_n, \end{cases}$$

the proof methods are similar to the case $n = 4m$, and we omit them. ☐

By Theorem 2.1 and Lemma 3.2, the following theorem is immediately.

Theorem 3.3. *For $n \geq 7$, the $n \times n$ complex sign pattern matrix $\mathcal{S}_n$ having the form* (3.1) *is spectrally arbitrary, and every superpattern of $\mathcal{S}_n$ is a spectrally arbitrary complex sign pattern matrix.*

Theorem 3.4. *For $n \geq 7$, the $n \times n$ complex sign pattern matrix $\mathcal{S}_n$ having the form* (3.1) *is a minimal spectrally arbitrary complex sign pattern matrix.*

Proof. Let $\mathcal{S}_n = (s_{kl})$, $\mathcal{T} = (t_{kl})$ be a subpattern of $\mathcal{S}_n$ and $\mathcal{T}$ be spectrally arbitrary.

Firstly, it is easy to see that $t_{kk} = s_{kk}$ for $k = 1, n-1, n$.

Secondly, note that if all matrices in $Q_c(\mathcal{T})$ are singular, or all matrices in $Q_c(\mathcal{T})$ are nonsingular, then $\mathcal{T}$ is not spectrally arbitrary. Thus $t_{k,k+1} = s_{k,k+1}$ for $k = 1, 2, \ldots, n-1$.

Finally, since $\mathcal{T}$ is spectrally arbitrary, there is a complex matrix $C \in Q_c(\mathcal{T})$ which is nilpotent. We may assume C has been scaled so that the (n, n) entry of C is -1. We can also assume that the $(k, k+1)$ entry of C is 1 or -1 for $k = 1, 2, \ldots, n-1$ (otherwise they can be adjusted to be 1 or -1 by suitable similarities). Thus, without loss of generality, suppose that C has the form (3.2). From $f_k = 0$ and $g_k = 0$ for $k = 1, 2, \ldots, n$, as in Lemma 3.1, we can conclude that $a_k \neq 0$ for $k = 2, \ldots, n$, and $b_k \neq 0$ for $k = 2, \ldots, n-1$.

Then $\mathcal{T} = \mathcal{S}_n$, and so $\mathcal{S}_n$ is a minimal spectrally arbitrary complex sign pattern matrix. ☐

Lemma 3.5. *Let complex sign pattern matrices*

$$\mathcal{S}_2 = \begin{bmatrix} 1-i & 1 \\ i & -1+i \end{bmatrix}, \mathcal{S}_3 = \begin{bmatrix} 1-i & 1 & 0 \\ 1+i & 0 & -1 \\ 1 & 0 & -1+i \end{bmatrix}, \mathcal{S}_4 = \begin{bmatrix} 1+i & 1 & 0 & 0 \\ 1+i & 0 & -1 & 0 \\ -1 & i & -i & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

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$$\mathcal{S}_5 = \begin{bmatrix} 1+i & 1 & 0 & 0 & 0 \\ 1-i & 0 & -1 & 0 & 0 \\ 1+i & 0 & 0 & 1 & 0 \\ 1-i & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \ \mathcal{S}_6 = \begin{bmatrix} 1+i & 1 & 0 & 0 & 0 & 0 \\ -1-i & 0 & -1 & 0 & 0 & 0 \\ 1+i & 0 & 0 & 1 & 0 & 0 \\ -1 & -i & 0 & 0 & -1 & 0 \\ -1 & i & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}.$$

Then $\mathcal{S}_j$, $j = 2, 3, 4, 5, 6$ *are minimal spectrally arbitrary complex sign pattern matrices.*

Proof. First, we prove that each $\mathcal{S}_j$ is spectrally arbitrary. For $\mathcal{S}_2$, we are able to obtain a nilpotent complex matrix

$$C_2 = \begin{bmatrix} a_1 - ib_1 & 1 \\ ia_2 & -1 + ib_2 \end{bmatrix} \in Q_c(\mathcal{S}_2),$$

where $a_2 = 2, a_1 = b_1 = b_2 = 1$. Replacing the entries a_1, b_1, a_2, b_2 of C_2 by variables in using Theorem 2.1, it can be verified that $\mathcal{S}_2$ is spectrally arbitrary.

For $\mathcal{S}_3$, we are able to obtain a nilpotent complex matrix

$$C_3 = \begin{bmatrix} a_1 - ib_1 & 1 & 0 \\ a_2 + ib_2 & 0 & -1 \\ a_3 & 0 & -1 + ib_3 \end{bmatrix} \in Q_c(\mathcal{S}_3),$$

where $a_1 = 1, a_2 = 2, a_3 = 8, b_1 = b_3 = \sqrt{3}, b_2 = 2\sqrt{3}$. Replacing the entries $a_1, b_1, a_2, b_2, a_3, b_3$ of C_3 by variables in using Theorem 2.1, it can be verified that $\mathcal{S}_3$ is spectrally arbitrary.

For $\mathcal{S}_4$, we are able to obtain a nilpotent complex matrix

$$C_4 = \begin{bmatrix} a_1 + ib_1 & 1 & 0 & 0 \\ a_2 + ib_2 & 0 & -1 & 0 \\ -a_3 & ib_3 & -ib_4 & 1 \\ 0 & 0 & a_4 & -1 \end{bmatrix} \in Q_c(\mathcal{S}_4),$$

where $a_1 = 1, , a_2 = \sqrt{5}, a_3 = 2(7+4\sqrt{5}), a_4 = 2+\sqrt{5}, b_1 = b_2 = b_4 = \sqrt{3+2\sqrt{5}}, b_3 = 2\sqrt{3+2\sqrt{5}}$. Replacing the entries $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$ of C_4 by variables in using Theorem 2.1, it can be verified that $\mathcal{S}_4$ is spectrally arbitrary.

For $\mathcal{S}_5$, we are able to obtain a nilpotent complex matrix

$$C_5 = \begin{bmatrix} a_1 + ib_1 & 1 & 0 & 0 & 0 \\ a_2 - ib_2 & 0 & -1 & 0 & 0 \\ a_3 + ib_3 & 0 & 0 & 1 & 0 \\ a_4 - ib_4 & 0 & 0 & -ib_5 & -1 \\ 0 & 0 & 0 & -a_5 & -1 \end{bmatrix} \in Q_c(\mathcal{S}_5),$$

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where $a_1 = 1, a_2 = 1+\sqrt{2}, a_3 = 2, a_4 = 6\sqrt{2}, a_5 = \sqrt{2}-1, b_1 = b_2 = b_5 = \sqrt{1+2\sqrt{2}}, b_3 = 2\sqrt{1+2\sqrt{2}}, b_4 = 2(2\sqrt{1+2\sqrt{2}} - \sqrt{2(1+2\sqrt{2})})$. Replacing the entries $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5$ of C_5 by variables in using Theorem 2.1, it can be verified that $\mathcal{S}_5$ is spectrally arbitrary.

For $\mathcal{S}_6$, we are able to obtain a nilpotent complex matrix

$$C_6 = \begin{bmatrix} a_1+ib_1 & 1 & 0 & 0 & 0 & 0 \\ -a_2-ib_2 & 0 & -1 & 0 & 0 & 0 \\ a_3+ib_3 & 0 & 0 & 1 & 0 & 0 \\ -a_4 & -ib_4 & 0 & 0 & -1 & 0 \\ -a_5 & ib_5 & 0 & 0 & -ib_6 & 1 \\ 0 & 0 & 0 & -a_6 & 0 & -1 \end{bmatrix} \in Q_c(\mathcal{S}_6),$$

where $a_1 = 1, a_2 = \frac{4}{3} - \frac{\sqrt{37}}{6}, a_3 = \frac{1}{6}(2\sqrt{37}-1), a_4 = 2, a_5 = \frac{1}{12}(4+19\sqrt{37}), a_6 = \frac{1}{6}(7+\sqrt{37}), b_1 = b_2 = b_6 = \sqrt{\frac{\sqrt{37}}{6}-\frac{1}{3}}, b_3 = 2\sqrt{\frac{\sqrt{37}}{6}-\frac{1}{3}}, b_4 = \frac{10}{3}\sqrt{\frac{\sqrt{37}}{6}-\frac{1}{3}} - \frac{1}{6}\sqrt{37(\frac{\sqrt{37}}{6}-\frac{1}{3})}, b_5 = \frac{13}{3}\sqrt{\frac{\sqrt{37}}{6}-\frac{1}{3}} + \frac{1}{3}\sqrt{37(\frac{\sqrt{37}}{6}-\frac{1}{3})}$. Replacing the entries $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6$ of C_6 by variables in using Theorem 2.1, it can be verified that $\mathcal{S}_6$ is spectrally arbitrary.

Next, by the same argument as in Theorem 3.4, we see that each $\mathcal{S}_j$ is minimal spectrally arbitrary. □

Theorem 3.4 and Lemma 3.5 immediately yield the following.

THEOREM 3.6. *For $n \geq 2$, there exists an $n \times n$ minimal, irreducible, spectrally arbitrary complex sign pattern matrix.*

4. The minimum number of nonzero entries in a spectrally arbitrary complex sign pattern matrix. Recall that the number of nonzero entries of a complex sign pattern matrix $\mathcal{S}$ is the number of nonzero entries of both the real and imaginary parts of $\mathcal{S}$. In this section we will study the minimum number of nonzero entries in a irreducible spectrally arbitrary complex sign pattern matrix.

Given a sign pattern $\mathcal{A}$, let $D(\mathcal{A})$ be its associated digraph. For any digraph D, let $G(D)$ denote the underlying multigraph of D, i.e., the graph obtained from D by ignoring the direction of each arc.

LEMMA 4.1. ([3]) *Let $\mathcal{A}$ be an $n \times n$ sign pattern and let $A \in Q(\mathcal{A})$. If T is a subdigraph of $D(\mathcal{A})$ such that $G(T)$ is a forest, then $\mathcal{A}$ has a realization that is positive diagonally similar to A such that each entry corresponding to an arc of T has magnitude 1. In particular, if $\mathcal{A}$ is irreducible, then $G(D(\mathcal{A}))$ contains a spanning tree, and $\mathcal{A}$ must therefore have a realization with at least $n-1$ off-diagonal entries in $\{-1, 1\}$ that is positive diagonally similar to A.*

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We easily extend Lemma 4.1 to complex sign pattern matrices.

LEMMA 4.2. *Let $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ be an $n \times n$ irreducible complex sign pattern matrix, and let $C = A + iB \in Q_c(\mathcal{S})$. Then there is a complex matrix $\hat{C} = \hat{A} + i\hat{B} \in Q_c(\mathcal{S})$ (where $\hat{A}$ and $\hat{B}$ are real matrices, $\hat{A} \in Q(\mathcal{A})$ and $\hat{B} \in Q(\mathcal{B})$) such that the following two conditions hold.*

(1) $\hat{C}$ has at least $n-1$ off-diagonal entries in which either the real part or complex part of each entry is in $\{-1, 1\}$;

(2) $\hat{C}$ is positive diagonally similar to C.

Let $\mathbb{Q}[X]$ be the set of polynomials with rational coefficients and finite degree. A set $H \subseteq \mathbb{R}$ is algebraically independent if, for all $h_1, h_2, \ldots, h_n \in H$ and each nonzero polynomial $p(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[X]$, $p(h_1, h_2, \ldots, h_n) \neq 0$ (see [13, p.316] for further details). Let $\mathbb{Q}(H)$ denote the field of rational expressions

$$\{\frac{p(h_1, h_2, \ldots, h_m)}{q(t_1, t_2, \ldots, t_n)} \mid p(x_1, x_2, \ldots, x_m), q(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[X],$$

$$h_1, h_2, \ldots, h_m, t_1, t_2, \ldots, t_n \in H\},$$

and let the *transcendental degree* of H be

$$tr.d.H = \sup\{|T| \mid T \subseteq H,\ T \text{ is algebraically independent}\}.$$

In [3] it was shown that every $n \times n$ irreducible spectrally arbitrary sign pattern matrix contains at least $2n-1$ nonzero entries. We adapt that proof to the complex sign pattern matrix case to obtain:

THEOREM 4.3. *For $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix must have at least $3n-1$ nonzero entries.*

Proof. Let $\mathcal{S} = \mathcal{A} + i\mathcal{B}$ be an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix with m nonzero entries. Choose a set $V = \{f_1, g_1, \cdots, f_n, g_n\} \subseteq \mathbb{R}$ that $tr.d.V = 2n$. By Lemma 4.2, there is a complex matrix $\hat{C} = \hat{A} + i\hat{B} \in Q_c(\mathcal{S})$ (where $\hat{A}$ and $\hat{B}$ are real matrices, $\hat{A} \in Q(\mathcal{A})$ and $\hat{B} \in Q(\mathcal{B})$) with characteristic polynomial

$$\lambda^n + (f_1 + ig_1)\lambda^{n-1} + \cdots + (f_{n-1} + ig_{n-1})\lambda + (f_n + ig_n)$$

such that $\hat{C}$ satisfies the two conditions in Lemma 4.2.

Denote $\hat{A} = (\hat{a}_{kl})$, $\hat{B} = (\hat{b}_{kl})$, and $H = \{\hat{a}_{kl} \mid 1 \leq k, l \leq n\} \cup \{\hat{b}_{kl} \mid 1 \leq k, l \leq n\}$. Since for each $1 \leq k \leq n$, f_k and g_k are polynomials in the entries of H with rational coefficients, it follows that $\mathbb{Q}(V) \subseteq \mathbb{Q}(H)$. Then

$$2n = tr.d.\mathbb{Q}(V) \leq tr.d.\mathbb{Q}(H) \leq m - (n-1).$$

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Thus $m \geq 3n - 1$. □

Note that the spectrally arbitrary complex sign pattern $\mathcal{S}_n$ $(n \geq 2)$ in Section 3 is irreducible, and has exactly $3n$ nonzero entries. Then for every $n \geq 2$ there exists an $n \times n$ irreducible, spectrally arbitrary complex sign pattern with exactly $3n$ nonzero entries. By Theorem 4.3 the minimum number of nonzero entries in an $n \times n$ irreducible, spectrally arbitrary complex sign pattern must be either $3n$ or $3n - 1$.

A well known conjecture in [3] is that for $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary sign pattern matrix has at least $2n$ nonzero entries. Here, we extend the conjecture to complex sign pattern matrix case.

COROLLARY 4.4. *For $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix has at least $3n$ nonzero entries.*

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