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GROUP INVERSES OF MATRICES ASSOCIATED WITH CERTAIN GRAPH CLASSES*

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Abstract. We obtain formulae for group inverses of matrices that are associated with a new class of digraphs obtained from stars. This new class contains both bipartite and non-bipartite graphs. Expressions for the group inverse of matrices corresponding to double star digraphs and the adjacency matrix of certain undirected multi-star graphs are also proven. A blockwise representation of the inverse or group inverse of the adjacency matrix of the Dutch windmill graph is presented.

Key words. Inverse, Group inverse, Adjacency matrix, Star, Double star, Dutch windmill graph, Multi-star graph.

AMS subject classifications. 05C20, 05C50, 15A09.

1. Introduction. An interesting challenge in matrix theory is to provide a succinct formula for the inverse or the group inverse of a matrix, based on its graph structure. The results in the literature focus on determining the inverse ([17], [5], [1], [4]) and group inverse ([10], [11], [8], [19], [9], [12]) of the matrix associated with graphs. In this article, we include results relating to digraphs and to undirected graphs. Also, we give a new blockwise representation for the group inverse of a matrix associated with double star digraphs and the adjacency matrix of certain undirected multi-star graphs. As another contribution, we present a formula for the inverse or group inverse of the adjacency matrix of a Dutch windmill graph.

We begin with some definitions. Let G be a *simple undirected* graph on n vertices $\{v_1, v_2, \ldots, v_n\}$. Define the *adjacency* matrix of G to be the $n \times n$ matrix $A = (a_{ij})$ given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

If A is non-singular (singular), then we say that the graph G is non-singular (singular). Note that the adjacency matrix, by definition, is always symmetric and this fact will be useful in the last two sections of this article.

Let $A = (a_{ij})$ be an $n \times n$ matrix. The *digraph* corresponding to the matrix A is $D(A) = (V, E)$ having vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, where $(v_i, v_j) \in E$ if and only if $a_{ij} \neq 0$. The digraph $D(A)$ corresponding to a matrix A is called a *tree graph* if it is strongly connected and all of its cycles have length 2. If the digraph $D(A)$ of a matrix A is a tree graph, then A is referred to as a *combinatorially symmetric matrix*.

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Next, let us recall that for a real $n \times n$ matrix A, the group inverse, if it exists, is the unique matrix $A^{\#}$ that satisfies the equations $AA^{\#}A = A, A^{\#}AA^{\#} = A^{\#}$ and $AA^{\#} = A^{\#}A$. For a symmetric matrix A, the group inverse $A^{\#}$ exists. Further, (as was observed in [19]) it follows that, if A is symmetric and X satisfies the equations $AXA = A$ and $AX = XA$, then $X = A^{\#}$. Moreover, if A, X are symmetric and the product AX is symmetric, then $AX = (AX)^T = X^TA^T = XA$. This means that if A, X are symmetric, the product AX is symmetric and $AXA = A$, then we have $X = A^{\#}$. This fact will be used in one of the proofs. Let us recall that for a real rectangular matrix A, the Moore–Penrose inverse of A is the unique matrix $A^{\dagger}$ that satisfies the equations $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$, $(AA^{\dagger})^T = AA^{\dagger}$ and $(A^{\dagger}A)^T = A^{\dagger}A$. We refer the reader to [6] for more details on these notions of generalized inverses and Moore–Penrose inverses.

In what follows, we present a brief survey of the literature on the topic that we have considered here.

A formula for determining the inverse of the adjacency matrix of an undirected tree with a perfect matching in terms of alternating paths appeared in [17] and was extended to bipartite graphs with a unique perfect matching in [5, 1, 4].

We recall some well-known results on the role of group inverses in graph theory. In [10], a formula for the group inverse of a 2×2 block matrix corresponding to a bipartite digraph, as well as a graphical description for the entries of group inverse of a matrix A with path digraph $D(A)$, is presented. In the work [11], a necessary and sufficient condition for the existence of the group inverse of a special bipartite matrix is given and a formula is obtained for the group inverse in terms of block submatrices. Also, in [11], a graphical description for the entries of the group inverse of a matrix A such that $D(A)$ is a broom tree is presented.

A formula for the entries of the group inverse of a symmetric matrix can be derived from a result of [8] stated in terms of undirected bipartite graphs associated with arbitrary matrices (with the vertex set being the union of row and column indices of a matrix), in the special case when the graph corresponding to the given symmetric matrix is acyclic. The authors of [19] derived a formula for the entries of the group inverse of the adjacency matrix of an undirected weighted tree. The entries are given in terms of graph notions relative to the given graph. In a recent work, a group inverse formula for the adjacency matrix of undirected singular cycle is given in [18].

In this last part of the introduction, we recall some notation and introduce the terminology. Let J_{kl} denote the all ones matrix of order $k \times l$. When $k = l$, we denote J_{kl} by J_k. I will stand for the identity matrix of an order that will be clear from the context. We denote the i-th column of I by e_i. For a real square matrix A, $\rho(A)$ denote the spectral radius of A and $R(A)$ denote the range space of A. We denote the cycle on n vertices by C_n and the path on n vertices by P_n.

2. Group inverses of matrices with digraph linked stars.

DEFINITION 2.1. *Let D be any digraph on n vertices $v_1, v_2, \ldots, v_n$ and let K_{1,r_1} , K_{1,r_2} , $\ldots, K_{1,r_n}$ denote directed star graphs. The digraph linked stars, by digraph D, denoted $gls(D, r_1, r_2, \ldots, r_n)$ are obtained by taking K_{1,r_1} , K_{1,r_2} , $\ldots, K_{1,r_n}$ and introducing an edge from the center vertex of the i-th star K_{1,r_i} to the center vertex of the j-th star K_{1,r_j} when (v_i, v_j) is an edge in D.*

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Example 2.2. Consider the digraph D given by

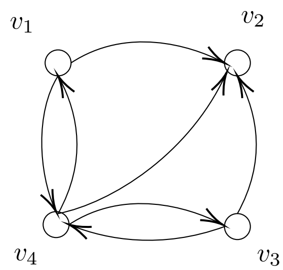

Figure 1. D.

Then the graph linked stars digraph $gls(D, 2, 1, 2, 3)$ is

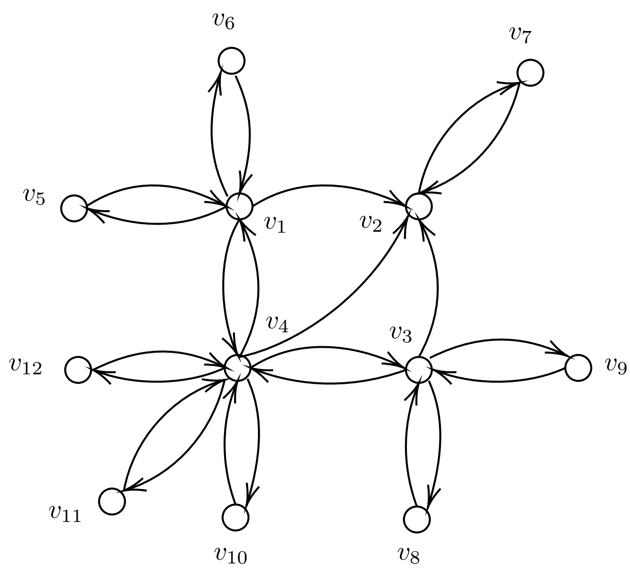

Figure 2. $gls(D, 2, 1, 2, 3)$.

It may be verified that the underlying graph of graph linked stars digraph is a particular case of *cluster networks*, see [21, 2]. In [21], authors are given Kirchhoff index formulae for composite graphs known as join, corona, and cluster of two graphs, in terms of the pieces. The Kirchhoff index formulae and the effective resistances of generalized composite networks, such as generalized cluster or corona network, are obtained, in terms of the pieces, in [2]. Also, in [2], an expression for the Green's function of the generalized composite graphs, viz., corona or cluster networks is given, in terms of the pieces.

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Let A be an $n \times n$ real matrix. For $i = 1, 2, \ldots, n$, let x_i and y_i be (column) vectors of length $r_i \in \mathbb{N}$ such that every coordinate is nonzero. We shall refer to such vectors as *strictly nonzero* vectors. Set $B = \begin{pmatrix} x_1^T & 0 & \ldots & 0 \\ 0 & x_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & x_n^T \end{pmatrix}$ and $C = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & y_n \end{pmatrix}$. Let

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}. \tag{2.1}$$

Then $D(M)$ is a digraph linked stars, by the digraph $D(A)$. On the other hand, any matrix associated with the digraph linked stars by digraph $D(A)$ is permutationally similar to a matrix of the form described in (2.1).

PROPOSITION 2.3. *Let A be an $n \times n$ real matrix. Let $D(M) = gls(D(A), r_1, r_2, \ldots, r_n)$, $r_i \geq 2$ for some i. Then the matrix M is singular.*

Proof. Since $r_i \geq 2$ for some i, the matrix C in the form (2.1) has y_i with at least two components. Thus, the rows in C corresponding to these components are linearly dependent, so that M is singular. $\square$

For a real matrix A, let $A^\Omega = I - AA^\dagger$ and $A^\pi = I - AA^\#$, when $A^\#$ exists.

THEOREM 2.4 ([20, Theorem 3.1]). *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$ with $A \in \mathbb{C}^{n\times n}$. Suppose that $\begin{pmatrix} B^\Omega A B^\Omega & 0 \\ C B^\Omega & 0 \end{pmatrix}^\#$ exists. Then $(B^\Omega A B^\Omega)^\#$ exists. Set $\Gamma = BC + A(B^\Omega A B^\Omega)^\pi B^\Omega A$. If $BCB^\Omega = 0$ and $rank(BC) = rank(B)$, then*

(i) *$M^\#$ exists if and only if $rank(\Gamma) = rank(B)$*
(ii) *If $M^\#$ exists, then $M^\# = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where*

$$\begin{aligned} &J = (B^\Omega A B^\Omega)^\pi B^\Omega A \Gamma^\dagger,\ H = \Gamma^\dagger A (B^\Omega A B^\Omega)^\pi B^\Omega,\ G = (B^\Omega A B^\Omega)^\#, \\ &X = JAGAH - JAH - GAH - JAG + G + H + J, \\ &Y = \Gamma^\dagger B + JAGA\Gamma^\dagger B - JA\Gamma^\dagger B - GA\Gamma^\dagger B, \\ &Z = (C - CGA)\Gamma^\dagger (I + AGAH - AH - AG) + CG^2(I - AH) \end{aligned}$$

and

$$W = (C - CGA)\Gamma^\dagger A (GA\Gamma^\dagger B - \Gamma^\dagger B) - CG^2 A \Gamma^\dagger B.$$

Here is our first main result.

THEOREM 2.5. *Let $A = (a_{ij})$ be an $n \times n$ real matrix. Let x_i, y_i be positive vectors of length r_i. Set $\alpha_i := x_i^T y_i$ for all $i \in \{1, 2, \ldots, n\}$. Create the matrix M as described in Equation (2.1). Then $D(M) = gls(D(A), r_1, r_2, \ldots, r_n)$, is a digraph linked stars digraph. Let $W = (W_{ij})$ be the $n \times n$ block matrix, where $W_{ij} = (\frac{a_{ij}}{\alpha_i \alpha_j}) y_i x_j^T$, an $r_i \times r_j$ matrix. Then*

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$$M^{\#} = \begin{pmatrix} 0 & Y \\ Z & -W \end{pmatrix},$$

where $Y = \frac{x_1^T}{\alpha_1} \oplus \frac{x_2^T}{\alpha_2} \oplus \ldots \oplus \frac{x_n^T}{\alpha_n}$ *and* $Z = \frac{y_1}{\alpha_1} \oplus \frac{y_2}{\alpha_2} \oplus \ldots \oplus \frac{y_n}{\alpha_n}$.

Proof. First, we observe that, $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. Then

$$BC = \begin{pmatrix} \alpha_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha_n \end{pmatrix} \text{ so that } (BC)^{\dagger} = \begin{pmatrix} \frac{1}{\alpha_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \frac{1}{\alpha_n} \end{pmatrix}.$$

Note that $\alpha_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. So, $rank(BC) = rank(B) = n$. Also,

$$B^{\dagger} = \begin{pmatrix} \frac{x_1}{x_1^T x_1} & 0 & \ldots & 0 \\ 0 & \frac{x_2}{x_2^T x_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \frac{x_n}{x_n^T x_n} \end{pmatrix}$$

and $B^{\Omega} = I - BB^{\dagger} = 0$. By Theorem 2.4,

$$M^{\#} = \begin{pmatrix} 0 & (BC)^{\dagger}B \\ C(BC)^{\dagger} & -C(BC)^{\dagger}A(BC)^{\dagger}B \end{pmatrix}.$$

Now, $(BC)^{\dagger}B = \frac{x_1^T}{\alpha_1} \oplus \frac{x_2^T}{\alpha_2} \oplus \ldots \oplus \frac{x_n^T}{\alpha_n}$, $C(BC)^{\dagger} = \frac{y_1}{\alpha_1} \oplus \frac{y_2}{\alpha_2} \oplus \ldots \oplus \frac{y_n}{\alpha_n}$ and so

$$C(BC)^{\dagger}A(BC)^{\dagger}B = \begin{pmatrix} \frac{y_1}{\alpha_1} & 0 & \ldots & 0 \\ 0 & \frac{y_2}{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \frac{y_n}{\alpha_n} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1n} \\ a_{n1} & \ldots & a_{nn-1} & a_{nn} \end{pmatrix} \begin{pmatrix} \frac{x_1^T}{\alpha_1} & 0 & \ldots & 0 \\ 0 & \frac{x_2^T}{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \frac{x_n^T}{\alpha_n} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a_{11}}{\alpha_1\alpha_1} y_1 x_1^T & \frac{a_{12}}{\alpha_1\alpha_2} y_1 x_2^T & \ldots & \frac{a_{1n}}{\alpha_1\alpha_n} y_1 x_n^T \\ \frac{a_{21}}{\alpha_2\alpha_1} y_2 x_1^T & \frac{a_{22}}{\alpha_2\alpha_2} y_2 x_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{a_{n-1n}}{\alpha_{n-1}\alpha_n} y_{n-1} x_n^T \\ \frac{a_{n1}}{\alpha_n\alpha_1} y_n x_1^T & \ldots & \frac{a_{nn-1}}{\alpha_n\alpha_{n-1}} y_n x_{n-1}^T & \frac{a_{nn}}{\alpha_n\alpha_n} y_n x_n^T \end{pmatrix}.$$

This completes the proof. ∎

Corollary 2.6. *Let* $A = (a_{ij}) \leq 0$ *be an* $n \times n$ *real matrix and* $D(M)$ *be a digraph linked stars graph* $gls(D(A), r_1, r_2, \ldots, r_n)$. *Let* M *have the form* (2.1) *with* $x_i,\ y_i > 0$ *for all* $i \in \{1, 2, \ldots, n\}$. *Then* $M^{\#} \geq 0$.

Proof. By Theorem 2.5, $M^{\#} = \begin{pmatrix} 0 & Y \\ Z & -W \end{pmatrix}$, where Y, Z, W are as given above. Since x_i and y_i are positive, Y and Z both are nonnegative matrices. Also, since $A \leq 0$, $W \leq 0$. Thus, $M^{\#}$ is nonnegative. ∎

Remark 2.7. Let us recall that an A is said to be an irreducible matrix if $D(A)$ is strongly connected. Let A be an $n \times n$ real matrix such that $A^{\#}$ exists. Then A is irreducible if, and only if, $A^{\#}$ is irreducible.

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COROLLARY 2.8. *Let $A = (a_{ij}) \leq 0$ be an $n \times n$ real matrix and $D(M)$ be a digraph linked stars $gls(D(A), r_1, r_2, \dots, r_n)$. Let M have the form (2.1) with $x_i,\ y_i > 0$ for all $i \in \{1, 2, \dots, n\}$. Then the smallest positive eigenvalue λ of M is simple and there exists a positive eigenvector corresponding to it.*

Proof. By Corollary 2.6, $M^{\#} \geq 0$. Since A is irreducible, M is irreducible and so by Remark 2.7, $M^{\#}$ is irreducible, as well. By the Perron–Frobenius theorem, $\rho(M^{\#})$ is a simple eigenvalue. Since $\lambda = \frac{1}{\rho(M^{\#})}$, λ is simple. Let $x > 0$ be the Perron vector corresponding to $\rho(M^{\#})$ of $M^{\#}$ so that $M^{\#}x = \rho(M^{\#})x$. So, $x \in R(M^{\#}) = R(M)$ and so, $x = MM^{\#}x = \rho(M^{\#})Mx$. This means that $\lambda x = \frac{1}{\rho(M^{\#})}x = Mx$, showing that x is a positive eigenvector corresponding to λ. ◻

A symmetric matrix with zero diagonal elements is called a *hollow symmetric matrix*. We refer the reader to [13] for a study of Moore–Penrose inverse of hollow symmetric matrices, with certain specific applications to the distance matrix of weighted trees. Note that, for symmetric matrices, the Moore–Penrose inverse is equal to the group inverse.

In general, it is not true that group inverse of a hollow symmetric matrix is again a hollow symmetric matrix which is shown by the following example:

EXAMPLE 2.9. Consider a hollow symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Then its group inverse

$$A^{\#} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

is not a hollow symmetric matrix.

Let A be a symmetric matrix such that its associated digraph $D(A)$ is bipartite. Then the digraph $D(A^{\#})$ corresponding to its group inverse is also bipartite. This is an easy consequence of [10, Theorem 2.2]. For a given hollow symmetric matrix A, the matrix M having the form (2.1) with $x_i = y_i > 0$ for all $i \in \{1, 2, \dots, n\}$, is a hollow symmetric matrix. Interestingly, as a consequence of Theorem 2.5, the group inverse of matrix M with digraph linked stars graph $D(M) = gls(D(A), r_1, r_2, \dots, r_n)$ is again a hollow symmetric matrix.

3. Group inverses of matrices with double star digraph.

DEFINITION 3.1. *The digraph denoted by $S_{m,n}$, obtained by introducing an edge from the center vertex of star $K_{1,m-1}$ to the center vertex of star $K_{1,n-1}$ as well as an edge from the center vertex of star $K_{1,n-1}$ to the center vertex of star $K_{1,m-1}$, is called the double star digraph.*

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EXAMPLE 3.2. The digraph corresponding to the double star $S_{3,4}$ is given by

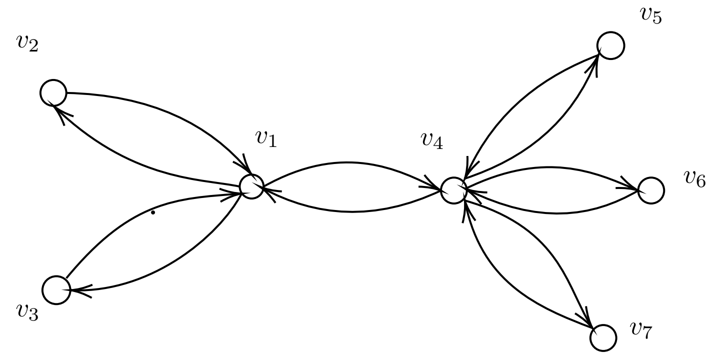

FIGURE 3. $S_{3,4}$.

Let x and y be strictly nonzero vectors of length m, while z and w are strictly nonzero vectors of length n. Let

$$A = \begin{pmatrix} 0 & x^T & a & 0 \\ y & 0 & 0 & 0 \\ b & 0 & 0 & z^T \\ 0 & 0 & w & 0 \end{pmatrix}, \tag{3.2}$$

where a and b are nonzero real numbers. Then $D(A)$ is a double star digraph $S_{(m+1),(n+1)}$ and any matrix A with the double star digraph $D(A) = S_{(m+1),(n+1)}$ is permutation similar to a matrix of the form described in (3.2).

THEOREM 3.3. *Let A be a real matrix with double star digraph $S_{(m+1),(n+1)}$. Let A have the form (3.2) with $x^Ty \neq 0$ and $z^Tw \neq 0$. Then*

$$A^{\#} = \begin{pmatrix} 0 & \frac{1}{x^Ty}x^T & 0 & 0 \\ \frac{1}{x^Ty}y & 0 & 0 & -\frac{a}{(x^Ty)(z^Tw)}yz^T \\ 0 & 0 & 0 & \frac{1}{z^Tw}z^T \\ 0 & -\frac{b}{(x^Ty)(z^Tw)}wx^T & \frac{1}{z^Tw}w & 0 \end{pmatrix}.$$

Proof. The proof is skipped, as it is just a routine verification. $\square$

REMARK 3.4. Observe that the matrix A as in (3.2) is permutationally similar to the matrix

$$T = \begin{pmatrix} 0 & a & x^T & 0 \\ b & 0 & 0 & z^T \\ y & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix}.$$

Let $N = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Thus, $D(T)$ is a double star digraph $S_{(n+1),(n+1)}$ as well as a graph linked star digraph $gls(D(N), m, n)$. So, for $x, y, z, w > 0$, $A^{\#}$ can be determined from Theorem 2.5.

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4. Inverse and group inverse of the adjacency matrix of Dutch windmill graph.

DEFINITION 4.1. *The Dutch windmill graph D_n^m is a graph constructed by joining m copies of the cycle graph C_n with a common vertex.*

EXAMPLE 4.2. The Dutch windmill graph D_3^4 is given by

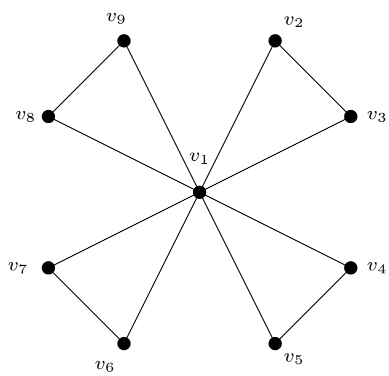

FIGURE 4. D_3^4.

REMARK 4.3. Note that D_{2n}^m is a bipartite graph while D_{2n+1}^m is non-bipartite. Now, by [3, Theorem 3.6, Chapter 3], we can conclude that D_{2n}^m is always a singular graph while D_{2n+1}^m is always non-singular.

THEOREM 4.4 ([3, Theorem 3.33., Chapter 3]). *Let T be a non-singular tree with $V(T) = \{v_1, \ldots, v_n\}$, A be the adjacency matrix of T, and $\mathcal{M}$ be the perfect matching in T. Let $B = (b_{ij})$ be the $n \times n$ matrix defined as follows: $b_{ij} = 0$ if $i = j$, or if the path between v_i and v_j is not alternating; $b_{ij} = (-1)^{\frac{d(v_i,v_j)-1}{2}}$, if the path between v_i and v_j is alternating. Then, $B = A^{-1}$.*

REMARK 4.5. Let P_{2n} be the path on $2n$ vertices and let its edge set be $\{v_iv_{i+1} \mid i \in \{1, 2, \ldots, 2n-1\}\}$. Let A_{2n} be the adjacency matrix of P_{2n}. Then, by Theorem 4.4,

$$A_{2n}^{-1} = \begin{pmatrix} B & -C^T & C^T & \cdots & (-1)^{n-1}C^T \\ -C & B & -C^T & \cdots & (-1)^{n-2}C^T \\ C & -C & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -C^T \\ (-1)^{n-1}C & (-1)^{n-2}C & \cdots & -C & B \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 4.6. *Let A_{2n} be the adjacency matrix of the path P_{2n} and $x = e_1 + e_{2n}$ be of length $2n$. Let $y = A_{2n}^{-1}x = (y_1, y_2, \ldots, y_{2n})^T$. Then, for $k = 1, 2, \ldots, n$,*

$$y_{2k-1} = (-1)^{n-k} \textit{ while } y_{2k} = (-1)^{k-1}.$$

Also, $x^Ty = 2(-1)^{n-1}$.

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Proof. Let $A_{2n}^{-1} = (\beta_{ij})$. Since A_{2n}^{-1} is a symmetric matrix, the coordinates of the vector $y = A_{2n}^{-1}x = (y_1, y_2, \ldots, y_{2n})^T$, are given by $y_i = \beta_{1i} + \beta_{i2n}, 1 \leq i \leq 2n$. By Theorem 4.4, for $k = 1, 2, \ldots, n$,

$$y_{2k-1} = 0 + \beta_{(2k-1)(2n)} = (-1)^{\frac{2n-(2k-1)-1}{2}} = (-1)^{n-k},$$

and

$$y_{2k} = \beta_{1(2k)} + 0 = (-1)^{\frac{(2k-1)-1}{2}} = (-1)^{k-1}.$$

Clearly, $x^T y = y_1 + y_{2n} = 2(-1)^{n-1}$. ∎

THEOREM 4.7. *Let A and A_{2n} be the adjacency matrices of D_{2n+1}^m and P_{2n}, respectively. Let y be the vector of length $2n$ defined by*

$$y_{2k-1} = (-1)^{n-k} \textit{ while } y_{2k} = (-1)^{k-1}, k = 1, 2, \ldots, n,$$

and $C = (-1)^n yy^T$. Then A^{-1} is the $(m+1) \times (m+1)$ block matrix given by

$$\frac{1}{2m}\begin{pmatrix} (-1)^n & (-1)^{n+1}y^T & (-1)^{n+1}y^T & \cdots & (-1)^{n+1}y^T \\ (-1)^{n+1}y & 2mA_{2n}^{-1}+C & C & \cdots & C \\ (-1)^{n+1}y & C & 2mA_{2n}^{-1}+C & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ (-1)^{n+1}y & C & \cdots & C & 2mA_{2n}^{-1}+C \end{pmatrix}.$$

Proof. If we let $x = e_1 + e_{2n}$, then $y = A_{2n}^{-1}x$. The vertices of D_{2n+1}^m may be relabeled (if necessary), so that A can be written as the $(m+1) \times (m+1)$ block matrix

$$\begin{pmatrix} 0 & x^T & x^T & \cdots & x^T \\ x & A_{2n} & 0 & \cdots & 0 \\ x & 0 & A_{2n} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ x & 0 & \cdots & 0 & A_{2n} \end{pmatrix}.$$

Let

$$X = \frac{1}{2m}\begin{pmatrix} (-1)^n & (-1)^{n+1}y^T & (-1)^{n+1}y^T & \cdots & (-1)^{n+1}y^T \\ (-1)^{n+1}y & 2mA_{2n}^{-1}+C & C & \cdots & C \\ (-1)^{n+1}y & C & 2mA_{2n}^{-1}+C & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ (-1)^{n+1}y & C & \cdots & C & 2mA_{2n}^{-1}+C \end{pmatrix}.$$

Then

$$AX = \frac{1}{2m}\begin{pmatrix} m(-1)^{n+1}x^Ty & u & u & \cdots & u \\ v & 2mI+W & W & \cdots & W \\ v & W & 2mI+W & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & W \\ v & W & \cdots & W & 2mI+W \end{pmatrix},$$

where

$$u = 2mx^TA_{2n}^{-1} + mx^TC,$$

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$$v = (-1)^n x + (-1)^{n+1} A_{2n} y,$$

and

$$W = (-1)^{n+1} x y^T + A_{2n} C.$$

Now, by Lemma 4.6, $m(-1)^{n+1} x^T y = 2m$. Also,

$$u = 2mx^T A_{2n}^{-1} + mx^T C = 2my^T + 2m(-1)^{2n-1} y^T = 0.$$

Further, since $y = A_{2n}^{-1} x$, $v = 0$ as well as $W = 0$. So, $AX = I$, proving that $X = A^{-1}$. ◻

In what follows, we obtain a blockwise representation for the group inverse of the adjacency matrix of D_{2n}^m. We need two intermediate results, which we prove first.

Lemma 4.8. *Let $E \in \mathbb{R}^{n\times(n-1)}$ be a real matrix such that $E^T = \begin{pmatrix} G, & e_{n-1} \end{pmatrix}$, where $G \in \mathbb{R}^{(n-1)\times(n-1)}$ is the upper triangular matrix which has 1 along its main diagonal and super diagonal, and all of whose other entries are zero. Then $E^\dagger = \begin{pmatrix} (G^{-1})^T - qp^T, & q \end{pmatrix}$, where $p = (p_i) = ((-1)^{n-1-i}) \in \mathbb{R}^{n-1}$ and $q = (q_i) = (\frac{1}{n}(-1)^{n-1-i} \cdot i) \in \mathbb{R}^{n-1}$.*

Proof. Since G is a non-singular tree diagonal matrix, by [14, Theorem 1],

$$G^{-1} = \begin{pmatrix} 1 & -1 & 1 & \cdots & (-1)^{n-2} \\ 0 & 1 & -1 & \cdots & (-1)^{n-3} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let p be the last column of G^{-1} so that

$$p = G^{-1} e_{n-1} = \begin{pmatrix} (-1)^{n-2}, & (-1)^{n-3}, & \ldots, & -1, & 1 \end{pmatrix}^T.$$

Since $e_{n-1} - GG^{-1} e_{n-1} = 0$, by [6, Theorem 7, Chapter 5], $(E^T)^\dagger = \begin{pmatrix} G^{-1} - pq^T \\ q^T \end{pmatrix}$, where

$$q^T = (1 + p^T p)^{-1} p^T G^{-1} = \frac{1}{n} \begin{pmatrix} (-1)^{n-2}, & (-1)^{n-3} 2, & \ldots, & -1(n-2), & (n-1) \end{pmatrix}.$$

Now, since $(E^T)^\dagger = (E^\dagger)^T$, we have $E^\dagger = \begin{pmatrix} (G^{-1})^T - qp^T, & q \end{pmatrix}$. ◻

We shall use the following notation for the next result. Let p, q be defined as in Lemma 4.8. Define

$$s = (z^T, z^T, \ldots, z^T)^T \in \mathbb{R}^{mn},$$

where

$$z = (z_i) = (\frac{2}{n}(-1)^{i+1}) \in \mathbb{R}^n,$$
$$r = ((2q-p)^T, (2q-p)^T, \ldots, (2q-p)^T)^T \in \mathbb{R}^{m(n-1)},$$
$$w = ((v^T, \frac{n-1}{2}), (v^T, \frac{n-1}{2}), \ldots, (v^T, \frac{n-1}{2}))^T \in \mathbb{R}^{mn},$$

with

$$v = (v_i) = ((-1)^{n-1-i} \ \frac{n-2i+1}{2}) \in \mathbb{R}^{n-1},$$

and

$$u = (p^T, p^T, \ldots, p^T)^T \in \mathbb{R}^{m(n-1)}.$$

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Lemma 4.9. *Let $B^T \in \mathbb{R}^{mn\times(m(n-1)+1)}$ be the matrix given by*

$$B^T = \begin{pmatrix} E & 0 & 0 & \cdots & 0 & y \\ 0 & E & 0 & \cdots & 0 & y \\ 0 & 0 & E & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & y \\ 0 & 0 & \cdots & 0 & E & y \end{pmatrix},$$

with

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\times(n-1)},$$

and $y = e_1 + e_n \in \mathbb{R}^n$. Set $R = E^\dagger \oplus E^\dagger \oplus \cdots \oplus E^\dagger$ (m-times). Then, for odd n,

$$B^\dagger = \left(R^T - \tfrac{n}{4m} sr^T, \quad \tfrac{n}{4m} s\right),$$

while, for even n,

$$B^\dagger = \left(R^T - wu^T, \quad w\right),$$

where the vectors r, s, u, w are defined as earlier.

Proof. Let $D = E \oplus E \oplus \cdots \oplus E$ (m-times) so that $D^\dagger = E^\dagger \oplus E^\dagger \oplus \cdots \oplus E^\dagger$ (m-times) $= R$. If $x := \left(y^T, \quad y^T, \quad \ldots, \quad y^T\right)^T$, then we have $B^T = \left(D, \quad x\right)$. Also, by Lemma 4.8,

$$\begin{aligned} E^\dagger y &= \left((G^{-1})^T - qp^T, \quad q\right)\begin{pmatrix} e_1 \\ 1 \end{pmatrix} \\ &= (G^{-1})^T e_1 - qp^T e_1 + q \\ &= (-1)^{n-2} p - (-1)^{n-2} q + q. \end{aligned}$$

Now,

$$D^\dagger x = \begin{pmatrix} E^\dagger y \\ E^\dagger y \\ \vdots \\ E^\dagger y \end{pmatrix} = \begin{pmatrix} (-1)^n p - (-1)^n q + q \\ (-1)^n p - (-1)^n q + q \\ \vdots \\ (-1)^n p - (-1)^n q + q \end{pmatrix}.$$

Case (i): n is odd. Here, $D^\dagger x = \left((2q-p)^T, \quad (2q-p)^T, \quad \ldots, \quad (2q-p)^T\right)^T = r$ (say) and

$$y - EE^\dagger y = y - E(2q-p) = \frac{2}{n}\begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ -1 \\ 1 \end{pmatrix}.$$

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Now, let $s = x - DD^{\dagger}x = \begin{pmatrix} z^T, & z^T, & \ldots, & z^T \end{pmatrix}^T$, where $z = y - EE^{\dagger}y$. Since $s \neq 0$, once again by [6, Theorem 7, Chapter 5], $(B^T)^{\dagger} = \begin{pmatrix} D^{\dagger} - rs^{\dagger} \\ s^{\dagger} \end{pmatrix}$. Note that, $s^Ts = mz^Tz = \frac{4m}{n}$ and $s^{\dagger} = \frac{1}{s^Ts}s^T$. Thus, $(B^T)^{\dagger} = \begin{pmatrix} D^{\dagger} - \frac{n}{4m}rs^T \\ \frac{n}{4m}s^T \end{pmatrix}$ and so $B^{\dagger} = \begin{pmatrix} (D^{\dagger})^T - \frac{n}{4m}sr^T, & \frac{n}{4m}s \end{pmatrix}$.

Case (ii): n is even. Then $p = (1, -1, 1, -1, \ldots, 1)^T$. In this case $D^{\dagger}x = (p^T, p^T, \cdots, p^T)^T = u$ (say). Then

$$Ep = \begin{pmatrix} p_{11} \\ p_{11} + p_{21} \\ \vdots \\ p_{(n-2)1} + p_{(n-1)1} \\ p_{(n-1)1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = y.$$

Observe that $q^Tp = \frac{n-1}{2}$. Now, by Lemma 4.8,

$$\begin{aligned} (E^{\dagger})^Tp &= \begin{pmatrix} G^{-1} - pq^T \\ q^T \end{pmatrix} p \\ &= \begin{pmatrix} G^{-1}p - pq^Tp \\ q^Tp \end{pmatrix} \\ &= \begin{pmatrix} G^{-1}p - (\frac{n-1}{2})p \\ \frac{n-1}{2} \end{pmatrix} \\ &= \begin{pmatrix} v \\ \frac{n-1}{2} \end{pmatrix}, \end{aligned}$$

where v is defined as earlier. So, $p^TE^{\dagger} = \begin{pmatrix} v^T, & \frac{n-1}{2} \end{pmatrix}$.

Since $x - DD^{\dagger}x = 0$, again by [6, Theorem 7, Chapter 5], $(B^T)^{\dagger} = \begin{pmatrix} D^{\dagger} - uw^T \\ w^T \end{pmatrix}$, where

$$\begin{aligned} w^T &= (1 + u^Tu)^{-1}u^TD^{\dagger} \\ &= (1 + mp^Tp)^{-1}\left(p^TE^{\dagger}, p^TE^{\dagger}, \cdots, p^TE^{\dagger}\right) \\ &= \{1 + m(n-1)\}^{-1}\left(p^TE^{\dagger}, p^TE^{\dagger}, \cdots, p^TE^{\dagger}\right). \end{aligned}$$

Thus, $B^{\dagger} = \begin{pmatrix} (D^{\dagger})^T - wu^T, & w \end{pmatrix}$. ◻

We are now in a position to prove our result for D_{2n}^m. From the 2×2 block representation of the group inverse of the adjacency matrix given in [10, Theorem 2.2], we provide an explicit calculation of the group inverse, using Lemma 4.9.

Theorem 4.10. *Let A be the adjacency matrix of D_{2n}^m and $B^{\dagger}$ be as given in Lemma 4.9. Then*

$$A^{\#} = \begin{pmatrix} 0 & (B^{\dagger})^T \\ B^{\dagger} & 0 \end{pmatrix}.$$

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Proof. Since D_{2n}^m is a connected bipartite graph, there is a unique partition in D_{2n}^m containing $m(n-1)+1$ vertices in one partition and mn vertices in the other partition. Thus, A can be written as $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$, where B^T is as given in Lemma 4.9. Now, by [10, Theorem 2.2], the proof follows by observing that

$$\begin{aligned}(BB^T)^{\#}B &= (BB^T)^{\dagger}B \\ &= (B^T)^{\dagger}B^{\dagger}B \\ &= (B^{\dagger})^T B^{\dagger}B \\ &= (B^{\dagger})^T (B^{\dagger}B)^T \\ &= (B^{\dagger}BB^{\dagger})^T \\ &= (B^{\dagger})^T,\end{aligned}$$

as well as the identity $B^T(BB^T)^{\#} = ((BB^T)^{\#}B)^T = B^{\dagger}$. ∎

5. Group inverse of the adjacency matrix of multi-star graph.

Definition 5.1 ([16]). *Consider the star graph $K_{1,m}$. Introduce an edge to each of the pendant vertices and denote the resulting graph by $S^2(K_{1,m})$. Starting from $S^2(K_{1,m})$, introduce an edge to each of the pendant vertices to obtain the graph $S^3(K_{1,m})$. Repeating this process $(n-1)$ times, we get a graph containing $(mn+1)$ vertices, which we denote by $S^n(K_{1,m})$. This graph is called a multi-star graph. Here, $S^1(K_{1,m}) = K_{1,m}$*

Note that the multi-star graph is the graph obtained by iteratively subdividing the edges starting from the star graph $K_{1,m}$. For more details on the subdivision operation of a graph, we refer the reader to [7, Section 8.3] and [15, Section 1.2].

Example 5.2. The multi-star graph $S^4(K_{1,3})$ is given as

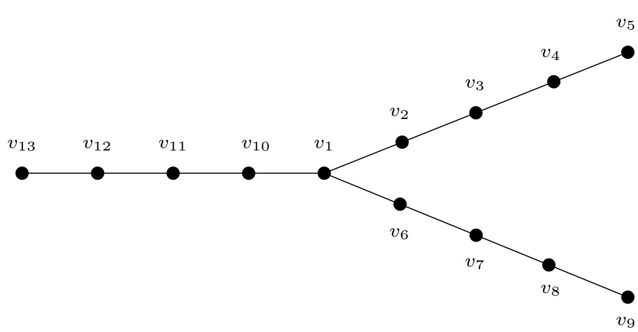

Figure 5. $S^4(K_{1,3})$.

Lemma 5.3. *Let A_{2n} be the adjacency matrix of P_{2n}. Let y be the vector of length $2n$ such that for $k = 1, 2, \ldots, n$,*

$$y_{2k-1} = 0 \text{ while } y_{2k} = (-1)^{k-1}.$$

Then

$$(A_{2n}^{-1}y)_{2k-1} = (-1)^{k-1}(n-k+1) \text{ while } (A_{2n}^{-1}y)_{2k} = 0.$$

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Proof. Note that y can be written as an $n \times 1$ block matrix

$$\begin{pmatrix} e_2 \\ -e_2 \\ e_2 \\ \vdots \\ (-1)^{n-1}e_2 \end{pmatrix}.$$

By Remark 4.5, for any $k = 1, 2, \ldots, n$, the k-th block of $A_{2n}^{-1}y$ is

$$\begin{aligned}(-1)^{k-1}\{(k-1)Ce_2 + Be_2 + (n-k)C^Te_2\} &= (-1)^{k-1}\{Be_2 + (n-k)C^Te_2\} \\ &= (-1)^{k-1}(n-k+1)e_1.\end{aligned}$$

It now follows that the $(2k-1)$-th entry of $A_{2n}^{-1}y$ is $(-1)^{k-1}(n-k+1)$ and the $2k$-th entry of $A_{2n}^{-1}y$ is 0. ∎

Let A be the adjacency matrix of $S^n(K_{1,m})$. Then the entries of $A^\#$ can be obtained from [19, Theorem 1] in terms of maximum matchings and alternating paths. Also, a 2×2 block representation of $A^\#$ is given in [10, Theorem 2.2]. In what follows, we give a new block representation of the group inverse of the adjacency matrix of $S^{2n}(K_{1,m})$ adopting a purely linear algebraic approach.

Theorem 5.4. *Let A and A_{2n} be the adjacency matrices of $S^{2n}(K_{1,m})$ and P_{2n}, respectively. Let y and z be vectors of length $2n$ such that for $k = 1, 2, \ldots, n$,*

$$y_{2k-1} = 0, \ \textit{with}\ y_{2k} = (-1)^{k-1},$$

and

$$z_{2k-1} = (-1)^{k-1}(n-k+1), \ \textit{while}\ z_{2k} = 0.$$

Then $A^\#$ is the $(m+1) \times (m+1)$ block matrix given by

$$\frac{1}{mn+1}\begin{pmatrix} 0 & z^T & z^T & \ldots & z^T \\ z & (mn+1)A_{2n}^{-1} + C & C & \ldots & C \\ z & C & (mn+1)A_{2n}^{-1} + C & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ z & C & \ldots & C & (mn+1)A_{2n}^{-1} + C \end{pmatrix},$$

where $C = -yz^T - zy^T$.

Proof. By Lemmas 4.6 and 5.3, it is clear that $y = A_{2n}^{-1}e_1$ and $z = A_{2n}^{-1}y$. Also, note that C is a symmetric matrix. The vertices of $S^{2n}(K_{1,m})$ can be relabeled (if necessary,) so that A has the following $(m+1) \times (m+1)$ block matrix representation:

$$\begin{pmatrix} 0 & e_1^T & e_1^T & \cdots & e_1^T \\ e_1 & A_{2n} & 0 & \cdots & 0 \\ e_1 & 0 & A_{2n} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ e_1 & 0 & \cdots & 0 & A_{2n} \end{pmatrix}.$$

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Let

$$X = \frac{1}{mn+1}\begin{pmatrix} 0 & z^T & z^T & \dots & z^T \\ z & (mn+1)A_{2n}^{-1}+C & C & \dots & C \\ z & C & (mn+1)A_{2n}^{-1}+C & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ z & C & \dots & C & (mn+1)A_{2n}^{-1}+C \end{pmatrix}.$$

Then

$$AX = \frac{1}{mn+1}\begin{pmatrix} me_1^T z & w & w & \cdots & w \\ y & (mn+1)I+W & W & \cdots & W \\ y & W & (mn+1)I+W & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & W \\ y & W & \cdots & W & (mn+1)I+W \end{pmatrix},$$

where

$$w = (mn+1)y^T + m(Ce_1)^T,$$

and

$$W = e_1 z^T + A_{2n}C.$$

Note that $y^T e_1 = 0$ and $y^T y = n$. So,

$$\begin{aligned}(Ce_1)^T &= -(yz^Te_1 + zy^Te_1)^T \\ &= -(y(A_{2n}^{-1}y)^T e_1)^T \\ &= -(yy^Ty)^T \\ &= -ny^T.\end{aligned}$$

Thus,

$$w = (mn+1)y^T + m(Ce_1)^T = (mn+1)y^T - mny^T = y^T.$$

Also,

$$A_{2n}C = A_{2n}(-yz^T - zy^T) = -e_1z^T - yy^T,$$

and so,

$$W = e_1z^T + A_{2n}C = -yy^T.$$

So, A, X are symmetric matrices such that AX is symmetric. To show that $X = A^{\#}$, it suffices to prove that $AXA = A$. Now,

$$AXA = \frac{1}{mn+1}\begin{pmatrix} 0 & u & u & \cdots & u \\ v & (mn+1)A_{2n}+Q & Q & \cdots & Q \\ v & Q & (mn+1)A_{2n}+Q & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Q \\ v & Q & \cdots & Q & (mn+1)A_{2n}+Q \end{pmatrix},$$

where

$$u = me_1^T z e_1^T + y^T A_{2n},$$

$$v = (mn+1)e_1 + mWe_1,$$

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$$Q = ye_1^T + WA_{2n},$$

and have made use of $y^Te_1 = 0$. Now,

$$\begin{aligned} u &= me_1^T(A_{2n}^{-1}y)e_1^T + (A_{2n}^{-1}e_1)^T A_{2n} \\ &= my^Tye_1^T + e_1^T A_{2n}^{-1} A_{2n} \\ &= mne_1^T + e_1^T \\ &= (mn+1)e_1^T. \end{aligned}$$

Further, as $We_1 = -yy^Te_1 = 0$, we have $v = (mn+1)e_1$. Finally,

$$\begin{aligned} Q &= ye_1^T + WA_{2n} \\ &= ye_1^T - yy^T A_{2n} \\ &= ye_1^T - y(A_{2n}^{-1}e_1)^T A_{2n} \\ &= ye_1^T - ye_1^T \\ &= 0. \end{aligned}$$

Thus, $AXA = A$, completing the proof that $X = A^\#$. □

REMARK 5.5. A 2×2 block representation of the group inverse of the adjacency matrix of $S^{2n+1}(K_{1,m})$ may be obtained in a manner similar to Theorem 4.10.

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