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KRONECKER PRODUCTS OF PERRON SIMILARITIES*

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Abstract. An invertible matrix is called a Perron similarity if one of its columns and the corresponding row of its inverse are both nonnegative or both nonpositive. Such matrices are of relevance and import in the study of the nonnegative inverse eigenvalue problem. In this work, Kronecker products of Perron similarities are examined and used to construct ideal Perron similarities all of whose rows are extremal.

Key words. Kronecker product, Perron similarity, Ideal Perron similarity, Nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15B48, 52B12.

1. Introduction.

An invertible matrix is called a *Perron similarity* if one of its columns and the corresponding row of its inverse are both nonnegative or both nonpositive. Real Perron similarities were introduced by Johnson and Paparella [4, 5], and the case for complex matrices is forthcoming [3].

These matrices were introduced to examine the celebrated *nonnegative inverse eigenvalue problem* vis-á-vis the polyhedral cone:

$$\mathcal{C}(S) := \{x \in \mathbb{R}^n \mid SD_xS^{-1} \geq 0\},$$

called the *(Perron) spectracone of* S, and the set:

$$\mathcal{P}(S) := \left\{x \in \mathcal{C}(S) \,\middle|\, \left\|x\right\|_\infty = 1\right\},$$

called the *(Perron) spectratope of* S. The latter is not necessarily a polytope, but in some cases it is finitely generated (this is true for some complex matrices as well). Notice that the entries of of any element in $\mathcal{P}(S)$ form a *normalized spectrum* (i.e., $x_k = 1$ for some k and $\max_i\{|x_i|\} \leq 1$) of a nonnegative matrix.

In particular, Johnson and Paparella [4] showed that if

$$H_n := \begin{cases} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, & n = 2 \\ H_2 \otimes H_{n-1} = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, & n > 2 \end{cases},$$

then $\mathcal{C}(H_n)$ and $\mathcal{P}(H_n)$ coincide with the conical hull and the convex hull of the rows of H_n, respectively.

In this work, Kronecker products of Perron similarities are examined. It is shown that the Kronecker product of Perron similarities is a Perron similarity. An example is constructed to refute a result presented by Johnson and Paparella [4, Corollary 3.17] (see Example 11), and the error in their proof is identified (see Remark 12). It is also shown that $\mathcal{C}(S) \otimes \mathcal{C}(T) \subset \mathcal{C}(S \otimes T)$ and $\mathcal{P}(S) \otimes \mathcal{P}(T) \subset \mathcal{P}(S \otimes T)$. Kronecker products of *ideal* Perron similarities yield Perron similarities all of whose rows are *extremal* (see Section 5).

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2. Notation and background. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, a mod n is abbreviated to $a\%n$. For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $\langle n \rangle$.

The set of m-by-n matrices over a field $\mathbb{F}$ is denoted by $\mathsf{M}_{m\times n}(\mathbb{F})$; when $m = n$, the set $\mathsf{M}_{m\times n}(\mathbb{F})$ is abbreviated to $\mathsf{M}_n(\mathbb{F})$. If $A \in \mathsf{M}_{m\times n}(\mathbb{F})$, then the (i,j)-entry of A is denoted by $[A]_{ij}$, a_{ij}, or $a_{i,j}$.

In this work, $\mathbb{F}$ stands for $\mathbb{C}$ or $\mathbb{R}$. The set of m-by-n matrices with entries over $\mathbb{F}$ is denoted by $\mathsf{M}_{m\times n}(\mathbb{F}) = \mathsf{M}_{m\times n}$; when $m = n$, $\mathsf{M}_{n\times n}(\mathbb{F})$ is abbreviated to $\mathsf{M}_n(\mathbb{F}) = \mathsf{M}_n$. The set of all n-by-1 column vectors is identified with the set of all ordered n-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. The set of nonsingular matrices in M_n is denoted by $\mathsf{GL}_n(\mathbb{F}) = \mathsf{GL}_n$.

Given $x \in \mathbb{F}^n$, $[x]_i = x_i$ denotes the i^{th} entry of x and $\operatorname{diag}(x) = D_x = D_{x^\top} \in \mathsf{M}_n(\mathbb{F})$ denotes the diagonal matrix whose (i,i)-entry is x_i. Notice that for scalars α, $\beta \in \mathbb{F}$, and vectors x, $y \in \mathbb{F}^n$, $D_{\alpha x+\beta y} = \alpha D_x + \beta D_y$.

Denote by I, e, and e_i the identity matrix, the all-ones vector, and the i^{th} canonical basis vector, respectively. The sizes of these objects are determined from the context in which they appear.

If $A \in \mathsf{M}_{m\times n}$ and $B \in \mathsf{M}_{p\times q}$, then the the *Kronecker product of A and B*, denoted by $A \otimes B$, is the mp-by-nq matrix defined blockwise by $A \otimes B = \begin{bmatrix} a_{ij}B \end{bmatrix}$. More precisely, but less intuitively,

$$[A \otimes B]_{ij} = a_{\lceil i/p\rceil,\lceil j/q\rceil} b_{[(i-1)\%p]+1,[(j-1)\%q]+1}. \tag{1}$$

If $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, then (1) simplifies to

$$[x \otimes y]_i = x_{\lceil i/n\rceil} y_{(i-1)\%n+1}. \tag{2}$$

If $U, V \subseteq \mathbb{F}^n$, then $U \otimes V := \{u \otimes v \mid u \in U,\ v \in V\}$.

If $S \in \mathsf{GL}_n$, then the *(Perron) spectracone of S*, denoted by $\mathcal{C}(S)$, is defined by $\mathcal{C}(S) = \{x \in \mathbb{F}^n \mid SD_xS^{-1} \geq 0\}$, and the *(Perron) spectratope of S*, denoted by $\mathcal{P}(S)$, is defined by:

$$\mathcal{P}(S) = \left\{ x \in \mathcal{C}(S) \,\middle|\, \left\|x\right\|_\infty = 1 \right\}.$$

If V is a vector space over $\mathbb{R}$ and U is a nonempty subset of V, then the *conical hull of U*, denoted by $\operatorname{coni} U$, is the set of all possible *conical combinations* of vectors in U, that is,

$$\operatorname{coni} U = \left\{ \sum_{i=1}^{k} \alpha_i u_i \in V \mid k \in \mathbb{N},\ u_i \in U,\ \alpha_i \geq 0 \right\},$$

and the *convex hull of U*, denoted by $\operatorname{conv} U$, is the set of all possible *convex combinations* of vectors in U, that is,

$$\operatorname{conv} U = \left\{ \sum_{i=1}^{k} \alpha_i u_i \in V \mid k \in \mathbb{N},\ u_i \in U,\ \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

The conical hull and convex hull of the rows of S are denoted by $\mathcal{C}_r(S)$ and $\mathcal{P}_r(S)$, respectively. If $S \in \mathsf{GL}_n$, then $SD_eS^{-1} = SIS^{-1} = I \geq 0$, that is, $e \in \mathcal{C}(S)$.

If there is an $i \in \langle n \rangle$ such that Se_i and $e_i^\top S^{-1}$ are both nonnegative or both nonpositive for $S \in \mathsf{GL}_n$, then S is called a *Perron similarity*.

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3. Preliminary results.

LEMMA 1. *If $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, then*

$$i = (\lceil i/n \rceil - 1)n + (i-1)\%n + 1.$$

Proof. By the division algorithm,

$$i - 1 = \left\lfloor \frac{i-1}{n} \right\rfloor n + (i-1)\%n,$$

and because

$$\left\lfloor \frac{i-1}{n} \right\rfloor = \left\lceil \frac{i}{n} \right\rceil - 1,$$

it follows that

$$i - 1 = (\lceil i/n \rceil - 1)n + (i-1)\%n,$$

that is,

$$i = (\lceil i/n \rceil - 1)n + (i-1)\%n + 1.$$

∎

LEMMA 2. *If $e_k \in \mathbb{F}^m$ and $e_\ell \in \mathbb{F}^n$, then $e_k \otimes e_\ell = e_{(k-1)n+\ell} \in \mathbb{F}^{mn}$.*

Proof. It suffices to show that $[e_k \otimes e_\ell]_i = 1$ if and only if $i = (k-1)n + \ell$; to this end, if $i = (k-1)n + \ell$, then

$$\left\lceil \frac{i}{n} \right\rceil = \left\lceil \frac{(k-1)n + \ell}{n} \right\rceil = \left\lceil k - 1 + \frac{\ell}{n} \right\rceil = k,$$

and

$$\begin{aligned} (i-1)\%n + 1 &= ((k-1)n + \ell - 1)\%n + 1 \\ &= (\ell - 1)\%n + 1 \\ &= \ell - 1 + 1 = \ell. \end{aligned}$$

Thus, according to (2),

$$[e_k \otimes e_\ell]_i = [e_k]_{\lceil \frac{i}{n} \rceil} [e_\ell]_{(i-1)\%n+1} = [e_k]_k [e_\ell]_\ell = 1.$$

Conversely, if

$$1 = [e_k \otimes e_\ell]_i = [e_k]_{\lceil \frac{i}{n} \rceil} [e_\ell]_{(i-1)\%n+1},$$

then $k = \lceil i/n \rceil$ and $\ell = (i-1)\%n + 1$. Hence, by the division algorithm, there is a positive integer q such that $(i-1) = qn + \ell - 1$, that is, $i = qn + \ell$. Thus,

$$k = \left\lceil \frac{qn + \ell}{n} \right\rceil = \left\lceil q + \frac{\ell}{n} \right\rceil = q + 1,$$

that is, $q = k - 1$. Therefore, $i = qn + \ell = (k-1)n + \ell$. ∎

LEMMA 3. *If $e_i \in \mathbb{F}^{mn}$, then $e_i = e_{\lceil i/n \rceil} \otimes e_{(i-1)\%n+1}$, where $e_{\lceil i/n \rceil} \in \mathbb{F}^m$ and $e_{(i-1)\%n+1} \in \mathbb{F}^n$.*

Proof. If $e_{\lceil i/n \rceil} \in \mathbb{F}^m$ and $e_{(i-1)\%n+1} \in \mathbb{F}^n$, then

$$e_{\lceil i/n \rceil} \otimes e_{(i-1)\%n+1} = e_{(\lceil i/n \rceil - 1)n + (i-1)\%n+1} = e_i,$$

by Lemmas 1 and 2. ∎

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LEMMA 4. *If $x \in \mathbb{F}^m$ and $y \in \mathbb{F}^n$, then $D_x \otimes D_y = D_{x\otimes y}$.*

Proof. If $i, j \in \langle mn \rangle$, then

$$[D_x \otimes D_y]_{ij} = [D_x]_{\lceil \frac{i}{n} \rceil, \lceil \frac{j}{n} \rceil} [D_y]_{(i-1)\%n+1,(j-1)\%n+1},$$

in view of (1). Since

$$[D_x]_{ij} = \begin{cases} x_i, & i = j \\ 0, & i \neq j, \end{cases}$$

it follows that $[D_x \otimes D_y]_{ij} \neq 0$ if and only if $\lceil i/n \rceil = \lceil j/n \rceil$ and $(i-1)\%n + 1 = (j-1)\%n + 1$. These equations hold, in light of Lemma 1, if and only if $i = j$. Thus, $D_x \otimes D_y$ is a diagonal matrix and, when $i = j$, notice that

$$\begin{aligned} [D_x]_{\lceil \frac{i}{n} \rceil, \lceil \frac{i}{n} \rceil} [D_y]_{(i-1)\%n+1,(i-1)\%n+1} &= x_{\lceil \frac{i}{n} \rceil} y_{(i-1)\%n+1} \\ &= [x \otimes y]_i \\ &= [D_{x\otimes y}]_{ii}, \end{aligned}$$

as required. ∎

LEMMA 5. *If $x \in \mathbb{F}^m$ and $y \in \mathbb{F}^n$, then $\|x \otimes y\|_p = \|x\|_p \|y\|_p$, $\forall p \in [1, \infty]$.*

Proof. Notice that

$$\begin{aligned} \|x \otimes y\|_p &= \sqrt[p]{\sum_{k=1}^{mn} |[x \otimes y]_k|^p} \\ &= \sqrt[p]{\sum_{k=1}^{mn} |x_{\lceil \frac{k}{n} \rceil} y_{(k-1)\%n+1}|^p} \\ &= \sqrt[p]{\sum_{k=1}^{mn} |x_{\lceil \frac{k}{n} \rceil}|^p |y_{(k-1)\%n+1}|^p}. \end{aligned}$$

Since $\lceil k/n \rceil \in \langle m \rangle$ and $(k-1)\%n + 1 \in \langle n \rangle$, it follows that

$$\begin{aligned} \|x \otimes y\|_p &= \sqrt[p]{\sum_{i=1}^{m} \sum_{j=1}^{n} |x_i|^p |y_j|^p} \\ &= \sqrt[p]{\sum_{i=1}^{m} \left(|x_i|^p \left(\sum_{j=1}^{n} |y_j|^p \right) \right)} \\ &= \sqrt[p]{\left(\sum_{i=1}^{m} |x_i|^p \right) \left(\sum_{j=1}^{n} |y_j|^p \right)} \\ &= \sqrt[p]{\sum_{i=1}^{m} |x_i|^p} \cdot \sqrt[p]{\sum_{j=1}^{n} |y_j|^p} = \|x\|_p \|y\|_p. \end{aligned}$$

The case when $p = \infty$ follows from the fact that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$. ∎

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4. Main results. In this section, it is shown that the Kronecker product of spectracones (spectratopes) is contained in the spectracone (spectratope) of the Kronecker product (Theorem 6). Furthermore, this containment is shown to be strict (Theorem 10).

THEOREM 6. *If $S \in \mathsf{GL}_m$ and $T \in \mathsf{GL}_n$, then $\mathcal{C}(S) \otimes \mathcal{C}(T) \subseteq \mathcal{C}(S \otimes T)$ and $\mathcal{P}(S) \otimes \mathcal{P}(T) \subseteq \mathcal{P}(S \otimes T)$.*

Proof. If $z \in \mathcal{C}(S) \otimes \mathcal{C}(T)$, then $z = x \otimes y$, where $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$. Thus, $SD_xS^{-1} \geq 0$ and $TD_yT^{-1} \geq 0$. By Lemma 4 and properties of the Kronecker product,

$$\begin{aligned}(SD_xS^{-1}) \otimes (TD_yT^{-1}) &= (S \otimes T)(D_x \otimes D_y)(S^{-1} \otimes T^{-1}) \\ &= (S \otimes T)(D_{x\otimes y})(S \otimes T)^{-1} \geq 0,\end{aligned}$$

since the Kronecker product of nonnegative matrices is nonnegative. Therefore, $z \in C(S \otimes T)$ and $\mathcal{C}(S) \otimes \mathcal{C}(T) \subseteq C(S \otimes T)$.

If, in addition, $z \in \mathcal{P}(S) \otimes \mathcal{P}(T)$, then $x \in \mathcal{P}(S)$ and $y \in \mathcal{P}(T)$, that is, $\left\|x\right\|_\infty = \left\|y\right\|_\infty = 1$. By Lemma 5,

$$\left\|z\right\|_\infty = \left\|x \otimes y\right\|_\infty = \left\|x\right\|_\infty \left\|y\right\|_\infty = 1,$$

that is, $z \in P(S \otimes T)$ and $P(S) \otimes P(T) \subseteq P(S \otimes T)$. ◻

THEOREM 7. *If $S \in \mathsf{GL}_m$ and $T \in \mathsf{GL}_n$ are Perron similarities, then $S \otimes T$ is a Perron similarity.*

Proof. By definition, $\exists k \in \langle m \rangle$ and $\exists \ell \in \langle n \rangle$ such that the vectors Se_k and $e_k^\top S^{-1}$ are both nonnegative or both nonpositive, and the same holds for Te_ℓ and $e_\ell^\top T^{-1}$. By Lemma 2,

$$(S \otimes T)e_{(k-1)n+\ell} = (S \otimes T)(e_k \otimes e_\ell) = Se_k \otimes Te_\ell,$$

and, since

$$\begin{aligned}e_{(k-1)n+\ell}^\top (S \otimes T)^{-1} &= (e_k \otimes e_\ell)^\top (S \otimes T)^{-1} \\ &= (e_k^\top \otimes e_\ell^\top)(S^{-1} \otimes T^{-1}) \\ &= e_k^\top S^{-1} \otimes e_\ell^\top T^{-1},\end{aligned}$$

it follows that the vectors $(S\otimes T)e_{(k-1)n+\ell}$ and $e_{(k-1)n+\ell}^\top (S\otimes T)^{-1}$ are both nonnegative or both nonpositive. Thus, $S \otimes T$ is a Perron similarity. ◻

Remark 8. If $x, y \in \mathcal{C}(S)$ and $\alpha, \beta \geq 0$, then $\alpha x + \beta y \in \mathcal{C}(S)$, that is, $\mathcal{C}(S)$ is a convex cone.

Remark 9. If $S \in \mathsf{GL}_n$ is a Perron similarity, then

$$SD_{e_i}S^{-1} = (Se_i)(e_i^\top S^{-1}) \geq 0,$$

that is, $\exists x \in \mathcal{C}(S)$ such that $x \neq \alpha e$ for every nonnegative α.

THEOREM 10. *If $S \in \mathsf{GL}_m$ and $T \in \mathsf{GL}_n$ are Perron similarities such that $m > 1$ and $n > 1$, then $\mathcal{C}(S) \otimes \mathcal{C}(T) \subset \mathcal{C}(S \otimes T)$ and $\mathcal{P}(S) \otimes \mathcal{P}(T) \subset \mathcal{P}(S \otimes T)$.*

Proof. By Theorem 7, $S\otimes T$ is a Perron similarity. Thus, by Remark 9, $\exists i \in \langle mn \rangle$ such that $e_i \in \mathcal{C}(S\otimes T)$. If $z := \frac{1}{2}e_i + \frac{1}{2}e$, then $z \in \mathcal{C}(S \otimes T)$ (Remark 8), $\left\|z\right\|_\infty = 1$, and z is totally nonzero. In particular, notice that

$$z_k = \begin{cases} \frac{1}{2}, & k \neq i \\ 1, & k = i. \end{cases}$$

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For contradiction, assume that $z = x \otimes y$, where $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$. Notice that the vectors x and y must be totally nonzero (otherwise, z would not be totally nonzero).

By definition of the Kronecker product, $\exists \alpha \in \langle m \rangle$ and $\exists \beta \in \langle n \rangle$ such that $z_i = x_\alpha y_\beta$. Select $\hat{\beta} \in \langle n \rangle$ such that $\hat{\beta} \neq \beta$. Then, z contains the entry $z_j = x_\alpha y_{\hat{\beta}}$ and

$$\frac{z_i}{z_j} = \frac{y_\beta}{y_{\hat{\beta}}}.$$

Select $\hat{\alpha} \in \langle m \rangle$ such that $\hat{\alpha} \neq \alpha$. Then, z contains the entries $z_k = x_{\hat{\alpha}} y_\beta$ and $z_\ell = x_{\hat{\alpha}} y_{\hat{\beta}}$. Furthermore,

$$\frac{z_k}{z_\ell} = \frac{y_\beta}{y_{\hat{\beta}}} = \frac{z_i}{z_j},$$

however, $\frac{z_i}{z_j} = 2$ and $\frac{z_k}{z_\ell} = 1$, a contradiction. $\square$

Example 11. Johnson and Paparella [4, Corollary 3.17] stated that S is a Perron similarity if and only if $\operatorname{coni}(e)$ is properly contained in $\mathcal{C}(S)$. A contribution of this work is the refutation of this result with a counterexample constructed via the Kronecker product.

Indeed, the matrix:

$$S := \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

is the Kronecker product of

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and

$$T := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The inverse of S is

$$\begin{bmatrix} -0.5 & 1 & -0.5 & 1 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & 1 & 0.5 & -1 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}.$$

Notice that neither the first or second row of S^{-1} are nonnegative. Furthermore, if

$$D = \operatorname{diag}(\begin{bmatrix} 2 & 2 & -1 & -1 \end{bmatrix}),$$

then the matrix:

$$A := SDS^{-1} = \begin{bmatrix} 0.5 & 0 & 1.5 & 0 \\ 0 & 0.5 & 0 & 1.5 \\ 1.5 & 0 & 0.5 & 0 \\ 0 & 1.5 & 0 & 0.5 \end{bmatrix},$$

is nonnegative and nonscalar. Thus, $\operatorname{coni}(e)$ is properly contained in $\mathcal{C}(S)$, but S is not a Perron similarity.

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Remark 12. If S is a Perron similarity, then $\operatorname{coni}(e)$ is properly contained in $\mathcal{C}(S)$ (Remark 9). However, as demonstrated in Example 11, the converse fails.

The argument for the converse in the proof of Corollary 3.17 [4] is as follows: if $\mathcal{C}(S)\setminus\operatorname{coni}(e) \neq \emptyset$, then there is a vector $x \neq \alpha e$ such that $A := SD_xS^{-1} \geq 0$. By the Perron–Frobenius theorem, there are nonnegative vectors x and y such that $Ax = \rho x$ and $y^\top A = \rho y^\top$ (here, ρ denotes the spectral radius of A). However, these vectors are not unique and the situation is further complicated if the matrix is reducible (as illustrated by Example 11). The Perron–Frobenius theorem does not guarantee that a nonnegative left eigenvector appears in S^{-1} even when a nonnegative right eigenvector is selected from the eigenspace corresponding to the spectral radius.

5. Ideal Perron similarities. If $S \in \mathsf{GL}_n$ is a Perron similarity, then S is called *ideal* if $\mathcal{C}(S) = \mathcal{C}_r(S)$. For real matrices, it is known that S is ideal if and only if $\exists k \in \langle n\rangle$ such that $e_k^\top S = e^\top$ and $e_i^\top S \in \mathcal{C}(S)$ for all $i \in \langle n\rangle$ [5, Theorem 3.8]. A careful examination of the arguments also applies to complex matrices.

Theorem 13. *If $S \in \mathsf{GL}_m$ and $T \in \mathsf{GL}_n$ are ideal, then $S \otimes T$ is ideal.*

Proof. By hypothesis, there are integers $k \in \langle m\rangle$ and $\ell \in \langle n\rangle$ such that $e_k^\top S = e^\top$ and $e_\ell^\top T = e^\top$. Notice that $(k-1)n + \ell \in \langle mn\rangle$ and by Lemma 2,

$$\begin{aligned} e_{(k-1)n+\ell}^\top(S\otimes T) &= (e_k \otimes e_\ell)^\top(S\otimes T) \\ &= (e_k^\top \otimes e_\ell^\top)(S\otimes T) \\ &= (e_k^\top S)\otimes(e_\ell^\top T) = e^\top \otimes e^\top = e^\top. \end{aligned}$$

If $i \in \langle mn\rangle$, then, following Lemma 3,

$$\begin{aligned} e_i^\top(S\otimes T) &= (e_{\lceil i/n\rceil} \otimes e_{(i-1)\%n+1})^\top(S\otimes T) \\ &= (e_{\lceil i/n\rceil}^\top \otimes e_{(i-1)\%n+1}^\top)(S\otimes T) \\ &= (e_{\lceil i/n\rceil}^\top S)\otimes(e_{(i-1)\%n+1}^\top T) \in \mathcal{C}(S\otimes T), \end{aligned}$$

since $e_{\lceil i/n\rceil}^\top S \in \mathcal{C}(S)$, $e_{(i-1)\%n+1}^\top T \in \mathcal{C}(T)$, and $\mathcal{C}(S)\otimes\mathcal{C}(T) \subseteq \mathcal{C}(S\otimes T)$ (Theorem 6). ∎

Theorem 14. *If $U = \{u_1,\ldots,u_p\} \subseteq \mathbb{F}^m$ and $V = \{v_1,\ldots,v_q\} \subseteq \mathbb{F}^n$, then* $\operatorname{coni}(U)\otimes\operatorname{coni}(V) \subseteq \operatorname{coni}(U\otimes V)$ *and* $\operatorname{conv}(U)\otimes\operatorname{conv}(V) \subseteq \operatorname{conv}(U\otimes V)$.

Proof. If $x \in \operatorname{coni}(U)\otimes\operatorname{coni}(V)$, then $x = u\otimes v$, where $u \in \operatorname{coni}(U)$ and $v \in \operatorname{coni}(V)$. By definition,

$$u = \sum_{i=1}^{p} \lambda_i u_i,\ \lambda_i \geq 0,\ \forall i \in \langle p\rangle,$$

and

$$v = \sum_{j=1}^{q} \mu_j v_j,\ \mu_j \geq 0,\ \forall j \in \langle q\rangle.$$

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By properties of the Kronecker product,

$$\begin{aligned} x = u \otimes v &= \left(\sum_{i=1}^{p} \lambda_i u_i \right) \otimes \left(\sum_{j=1}^{q} \mu_j v_j \right) \\ &= \sum_{i=1}^{p} \left(\lambda_i u_i \otimes \sum_{j=1}^{q} (\mu_j v_j) \right) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j (u_i \otimes v_j) \in \operatorname{coni}(U) \otimes \operatorname{coni}(V), \end{aligned}$$

since $\lambda_i \mu_j \geq 0$, $\forall (i,j) \in \langle p \rangle \times \langle q \rangle$.

If, in addition,

$$\sum_{i=1}^{p} \lambda_i = \sum_{j=1}^{q} \mu_j = 1,$$

then

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j = \sum_{i=1}^{p} \left(\lambda_i \sum_{j=1}^{q} \mu_j \right) = \sum_{i=1}^{p} \lambda_i = 1,$$

that is, $\operatorname{conv}(U) \otimes \operatorname{conv}(V) \subseteq \operatorname{conv}(U \otimes V)$. □

Recall that a matrix is *irreducible* if and only if its digraph is strongly connected (see, e.g., Brualdi and Ryser [1, Theorem 3.2.1]). The *index of imprimitivity* of an irreducible matrix is the greatest common divisor of the lengths of the closed directed walks in its digraph [1, p. 68].

An invertible matrix S is called *strong* if there is an irreducible nonnegative matrix A such that $A = SDS^{-1}$ (in such a case, S must be a Perron similarity since the eigenspace corresponding to the Perron root is one-dimensional). If S is strong, then S is ideal if and only if $\mathcal{P}(S) = \mathcal{P}_r(S)$ [3].

The following result is a consequence of a result stated by Harary and Trauth [2, p. 251] and follows from a result due to McAndrew [6, Theorem 2].

THEOREM 15. *If A and B are irreducible and k and ℓ are the indices of imprimitivity of A and B, respectively, then $A \otimes B$ is irreducible if and only if* $\gcd(k, \ell) = 1$.

COROLLARY 16. *Suppose that S and T are ideal and strong. Let A and B be irreducible nonnegative matrices with relatively prime indices of imprimitivity k and ℓ, respectively, and such that $A = SDS^{-1}$ and $B = T\hat{D}T^{-1}$. Then, $S \otimes T$ is ideal and strong.*

Proof. The matrix $S \otimes T$ is ideal by Theorem 13 and strong by Theorem 15. Thus, $\mathcal{P}_r(S) = \mathcal{P}(S)$, $\mathcal{P}_r(T) = \mathcal{P}(T)$, and $\mathcal{P}_r(S \otimes T) = \mathcal{P}(S \otimes T)$. The weak containment $\mathcal{P}(S) \otimes \mathcal{P}(T) \subseteq \mathcal{P}(S \otimes T)$ follows from Theorem 14. □

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Example 17. For $n \in \mathbb{N}$, let $F = F_n$ be the *discrete Fourier transform* matrix of order n, that is, F is the n-by-n matrix with (i, j)-entry equal to $\omega^{(i-1)(j-1)}$, where $\omega := \exp(2\pi i/n)$. Notice that

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^k & \cdots & \omega^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \omega^k & \cdots & \omega^{k^2} & \cdots & \omega^{k(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{k(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix},$$

and F is ideal as it is a Vandermonde matrix corresponding to the polynomial $p(t) := t^n - 1$. The companion matrix C corresponding to p is nonnegative and the spectrum of the nonnegative matrix C^{k-1} corresponds to the k^{th}-row of F, $k \in \langle n \rangle$. Furthermore, F is strong given that C is the adjacency matrix of the directed cycle of length n and, hence, is irreducible (it also admits positive circulant matrices).

A normalized, realizable spectrum x is called *extremal* if αx is not realizable whenever $\alpha > 1$. Notice that every row of F is extremal and every point in every row is extremal in the Karpelevič region.

At the 2019 Meeting of the International Linear Algebra Society in Rio de Janeiro, the second author asked whether other such matrices exist. Notice that $F_n \otimes F_m$, $F_m \otimes H_n$, and $H_n \otimes F_m$ are matrices all of whose rows and entries are extremal.

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REFERENCES

[1] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*, Vol. 39. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge, 1991.
[2] F. Harary and C.A. Trauth, Jr. Connectedness of products of two directed graphs. *SIAM J. Appl. Math.*, 14:250–254, 1966.
[3] C.R. Johnson and P. Paparella. Perron similarities and the nonnegative inverse eigenvalue problem. In preparation.
[4] C.R. Johnson and P. Paparella. Perron spectratopes and the real nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 493:281–300, 2016.
[5] C.R. Johnson and P. Paparella. Row cones, Perron similarities, and nonnegative spectra. *Linear Multilinear Algebra*, 65(10):2124–2130, 2017.
[6] M.H. McAndrew. On the product of directed graphs. *Proc. Amer. Math. Soc.*, 14:600–606, 1963.