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BANACH SPACES OF GLT SEQUENCES AND FUNCTION SPACES*

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Abstract. Generalized locally Toeplitz (GLT) sequences of matrices originated from the spectral study of certain partial differential equations. To be more precise, such matrix sequences arise when we numerically approximate either partial differential equations or fractional differential equations using any discretization by local methods (finite differences, finite elements, finite volumes, isogeometric analysis, etc.). The study of the asymptotic spectral behavior of GLT sequences is important in analyzing the solution of the corresponding partial differential equations and in finding fast and efficient methods for the corresponding large linear systems. Approximating classes of sequences (a.c.s.) and spectral symbols are important notions connected to GLT sequences. Recently, G. Barbarino obtained some results regarding the theoretical aspects of such notions. He obtained the completeness of the space of matrix sequences with respect to pseudometric a.c.s. Also, he identified the space of GLT sequences with the space of measurable functions. In this article, we follow the same research line and obtain various connections between the subalgebras of matrix-sequence spaces and the subalgebras of function spaces. In some cases, these are identifications as Banach spaces and some of them are Banach algebra identifications. We also prove that the convergence notions in the sense of eigenvalue/singular value clustering are equivalent to the convergence with respect to the metrics introduced here. These convergence notions are related to the study of preconditioners in the case of matrix/operator sequences. As an application of our main results, we establish a Korovkin-type result in the setting of GLT sequences.

Key words. Singular value and eigenvalue asymptotics, Generalized locally Toeplitz sequences, Korovkin-type approximation.

AMS subject classifications. 15A60, 15B05, 47A58, 65F08.

1. Introduction and preliminaries.

A matrix sequence is a sequence of the form $\{A_n\}_n$ where each A_n is an $n \times n$ matrix. The correspondence between matrix sequences and Lebesgue integrable functions is very natural in many important cases such as the case of Toeplitz matrices. Here the spectral information of the operator/matrix sequence is stored in the corresponding symbol function (recall the celebrated Szegö distribution theorem [18]). Also, such matrix sequences in the multilevel setting arise naturally in the study of partial differential equations with certain boundary conditions, using local methods like finite difference and finite elements.

For example, if we consider the Schrödinger operator that maps $f \mapsto -f^{''} + vf$, where v is a real-valued periodic potential function, then the corresponding finite difference approximation leads to a sequence of block Toeplitz matrices. If we consider more general differential operators like those arising from diffusion problems $(f \mapsto (-af^{'})^{'} + vf)$ or convection–diffusion–reaction problems $(f \mapsto (-af^{'})^{'} + bf^{'} + vf)$, we end up with locally Toeplitz (LT) or generalized locally Toeplitz (GLT) sequences [16]. In most cases, the resulting sequence of discretization matrices $\{A_n\}_n$ enjoys an asymptotic spectral distribution. This is somehow related to the spectrum of the differential operator associated with the considered Partial differential equations (PDE). The notion of asymptotic spectral distribution and asymptotic singular value distribution is closely related.

Definition 1.1. *Let $\{A_n\}_n$ be a matrix sequence and $f : D \subset \mathbb{R}^k \to \mathbb{C}$ be a measurable function.*

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- *We say that* $\{A_n\}_n$ *has an asymptotic singular value distribution with symbol* f *and write* $\{A_n\}_n \sim_\sigma f$*, if, for all* $F \in C_c(\mathbb{R})$*, the space of complex-valued continuous functions defined on* $\mathbb{R}$ *and with bounded support,*
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)}\int_D F(|f(x)|)dx,$$
where $\sigma_i(A_n)$*,* $i = 1, \cdots, n$*, are the singular values of* A_n*,* μ_k *is the Lebesgue measure in* $\mathbb{R}^k$*.*
- *We say that* $\{A_n\}_n$ *has an asymptotic eigenvalue distribution with symbol* f *and write* $\{A_n\}_n \sim_\lambda f$*, if, for all* $F \in C_c(\mathbb{C})$*, the space of complex-valued continuous functions defined on* $\mathbb{C}$ *and with bounded support,*
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)}\int_D F(f(x))dx,$$
where $\lambda_i(A_n)$*,* $i = 1, \cdots, n$*, are the eigenvalues of* A_n*.*

Because of this inherent connection between matrix sequences and the corresponding symbol functions, many researchers explored the possible generalizations of such results. To understand the spectral asymptotic, they tried to determine symbol functions or function spaces corresponding to some classes of matrix sequences. Recently, such studies have been initiated in the setting of GLT sequences by researchers like S. Serra-Capizzano, A. Böttcher, G. Barbarino, C. Garoni, etc. [1, 2, 3, 8, 16].

An equivalence between GLT sequences and measurable functions was obtained in [1]. In this article, we follow the same research line and obtain various connections between the subalgebras of the space of all matrix sequences and the subalgebras of space of measurable functions. In some cases, these connections are identifications as Banach spaces and some of them are Banach algebra identifications. These spaces of matrix sequences are defined using various pseudometric functions introduced in this article. These notions are motivated from the pseudometric induced by the notion of approximating class of sequences (a.c.s.) defined in [9] and used in [16]. As in [16], we also obtain characterizations of convergence notions in the sense of eigenvalue/singular value clustering (these notions were originated from the preconditioning problems in numerical linear algebra).

Definition 1.2. *Let* $\{A_n\}_n$ *be a matrix sequence and* $\{\{B_{n,m}\}_n\}_m$ *a sequence of matrix sequences. We say that* $\{\{B_{n,m}\}_n\}_m$ *is an approximating class of sequences (a.c.s.) for* $\{A_n\}_n$*, and we write* $\{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{A_n\}_n$*, if the following condition is met: for every* m *there exists an* n_m *such that, for* $n \geq n_m$*,*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad rank(R_{n,m}) \leq c(m)n, \quad \|N_{n,m}\| \leq \omega(m),$$

where $\| \,.\, \|$ *is the spectral norm,* $n_m, c(m)$ *and* $\omega(m)$ *depend only on* m *and*

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

The notion of a.c.s. is a powerful tool in the numerical linear algebra literature (see [3, 5, 17, 25] and references therein). In particular, the asymptotic distribution of singular values/eigenvalues of $\{\{B_{n,m}\}_n\}_m$ can be used to compute the asymptotic distribution of singular values/eigenvalues of $\{A_n\}_n$. For the definition of LT sequence, we need the notion of tensor product of functions and matrices and the direct sum of matrices.

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Definition 1.3. *Let $f_i : D_i \to \mathbb{C}, i = 1, 2$, be two functions and A, B be $m\times n$, $p\times q$ matrices, respectively. Then, $f_1 \otimes f_2$ is the function defined on $D_1 \times D_2$ by*

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2), \quad (x_1, x_2) \in D_1 \times D_2.$$

Also,

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}, \quad A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Now, following [16], we give the construction of the GLT matrix sequences, originally defined in [10, 25].

Definition 1.4. *Let $a : [0, 1] \to \mathbb{C}$ be a Riemann-integrable function and $f \in L^1([-\pi, \pi])$. We say that a matrix sequence $\{A_n\}_n$ is a LT sequence with symbol $a \otimes f$, and we write $\{A_n\}_n \sim_{\mathrm{LT}} a \otimes f$, if $\{\{LT^m_n(a, f)\}_n\}_{m\in\mathbb{N}}$ is an a.c.s. for $\{A_n\}_n$, where*

$$\begin{aligned} LT^m_n(a, f) &= [D_m(a) \otimes T_{\lfloor n/m\rfloor}(f)] \oplus O_{n(mod\, m)} \\ &= \operatorname*{diag}_{i=1,\dots,n} [a\left(\frac{i}{m}\right) T_{\lfloor n/m\rfloor}(f)] \oplus O_{n(mod\, m)}. \end{aligned}$$

Here $T_n(f)$ is the Toeplitz matrix generated by the function f and $D_m(a)$ is a $(m \times m)$ diagonal matrix associated with a given by

$$D_m(a) = \operatorname*{diag}_{i=1,\dots,n} a\left(\frac{i}{m}\right).$$

Definition 1.5. *Let $\kappa : [0, 1] \times [-\pi, \pi] \to \mathbb{C}$ be a measurable function. We say that a matrix sequence $\{A_n\}_n$ is a GLT sequence with symbol κ, and we write $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$, if the following condition is met.*

For every m varying in some infinite subset of $\mathbb{N}$, there exists a finite number of LT sequences $\{A^{(i,m)}_n\}_n \sim_{\mathrm{LT}} a_{i,m} \otimes f_{i,m}, i = 1, \dots, k_m$, such that:

- $\sum_{i=1}^{k_m} a_{i,m} \otimes f_{i,m} \to \kappa$ *in measure over* $[0, 1] \times [-\pi, \pi]$ *when* $m \to \infty$*;*
- $\left\{\left\{\sum_{i=1}^{k_m} A^{(i,m)}_n\right\}_n\right\}_m$ *is an a.c.s. for* $\{A_n\}_n$.

In this article, the results presented are for $D = [0, 1]\times[-\pi, \pi]$. All these results can be directly extended to the multilevel case, $D = [0, 1]^d \times [-\pi, \pi]^d$, by following the multidimensional GLT approach (see [5, 17]).

Let $\mathcal{E} = \{\{A_n\}_n : A_n \in M_n(\mathbb{C})\}$, where $M_n(\mathbb{C})$ is the space of $n\times n$ complex matrices. For $A_n \in M_n(\mathbb{C})$, let

$$P(A_n) = inf \left\{\frac{rank(R_n)}{n} + \|N_n\| \,:\, R_n + N_n = A_n, \quad R_n, N_n \in M_n(\mathbb{C})\right\},$$

where infimum is taken over all decompositions $A_n = R_n + N_n$.

Given $\{A_n\}_n \in \mathcal{E}$, we define

$$p(\{A_n\}_n) = \limsup_{n\to\infty} P(A_n).$$

For $\{A_n\}_n, \{B_n\}_n \in \mathcal{E}$, define

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = p(\{A_n - B_n\}_n).$$

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It was proved in [1, 15] that d_{acs} is a pseudometric on $\mathcal{E}$ which turns $\mathcal{E}$ into a complete pseudometric space $(\mathcal{E}, d_{acs})$.

LEMMA 1.6 (Theorem 4.1 [16]). *Let $\{A_n\}_n$ be a matrix sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix sequences. Then the following conditions are equivalent*

1. $\{\{B_{n,m}\}_n\}_m$ *is an a.c.s. for* $\{A_n\}_n$
2. $p(\{A_n - B_{n,m}\}_n) \to 0$ *as* $m \to \infty$

A class of matrix sequences that plays a central role in the framework of the theory of GLT sequences is the class of zero-distributed sequences.

DEFINITION 1.7. *A matrix sequence $\{Z_n\}_n$ is said to be a zero-distributed sequence if $p(\{Z_n\}_n) = 0$.*

Next theorem gives a sufficient condition for a matrix sequence $\{Z_n\}_n$ to be zero distributed.

THEOREM 1.8 (Theorem 2.10 of [16]). *Let $\{N_n\}_n$ be a matrix sequence and suppose that, for some $p \in [1, \infty)$, $\lim_{n\to\infty} \frac{\|N_n\|_p}{n^{1/p}} = 0$. Then $p(\{N_n\}_n) = 0$.*

Here $\|N_n\|_p$ denotes the Schatten p-norm, that is, the l^p norm of the vector of the singular values (see [7]).

Let $Z = \{\{A_n\}_n \in \mathcal{E} : p(\{A_n\}_n) = 0\}$, the set of all zero-distributed sequences. Then the quotient space $\tilde{\mathcal{E}} = \mathcal{E}/Z$ is a metric space with respect to the metric $\tilde{d}_{acs} : \tilde{\mathcal{E}} \times \tilde{\mathcal{E}} \to \mathbb{R}$ defined by

$$\tilde{d}_{acs}(\{A_n\}_n + Z, \{B_n\}_n + Z) = d_{acs}(\{A_n\}_n, \{B_n\}_n).$$

The following theorem gives an equivalent definition for $P(\cdot)$.

THEOREM 1.9 (Theorem 5 of [15]). *For any matrix $A_n \in M_n(\mathbb{C})$,*

$$P(A_n) = \min_{i=0,1,\ldots,n} \left\{ \frac{i}{n} + \sigma_{i+1}(A_n) \right\},$$

where $\sigma_i(A_n)$ is the i^{th} singular value of A_n arranged in non-increasing order and we assume by convention that $\sigma_{n+1}(A_n) = 0$.

REMARK 1.10. $\tilde{d}_{acs}$ in $\tilde{\mathcal{E}}$ is not induced by any norm, because if $\{A_n\}_n = \{I_n\}_n$ is the sequence of identity matrices and $\{B_n\}_n = \{O_n\}_n$ is the sequence of zero matrices, then

$$d_{acs}(\{A_n\}_n, \{B_n\}_n) = d_{acs}(\{I_n\}_n, \{O_n\}_n) = 1,$$
$$d_{acs}(\{2A_n\}_n, \{2B_n\}_n) = d_{acs}(\{2I_n\}_n, \{O_n\}_n) = 1,$$

and so $d_{acs}(\{2A_n\}_n, \{2B_n\}_n) \neq 2d_{acs}(\{A_n\}_n, \{B_n\}_n)$.

It is known that the class of GLT matrix sequences with respect to a.c.s. metric forms a complete *-algebra [16]. This space is isometrically isomorphic to the class of measurable functions on D [1].

The article is organized as follows. In Section 2, we introduce the seminorms $q_p, 1 \leq p \leq \infty$, and obtain their relation with Type 2 weak cluster (see Definition 2.2). Also, we identify the Banach spaces $\tilde{\mathcal{A}}_p, 1 \leq p \leq \infty$, of matrix sequences with respect to the norms induced by these seminorms. In Section 3, we introduce the spaces $\tilde{\mathcal{G}}_p$ for $1 \leq p \leq \infty$, the set of all equivalence classes of GLT matrix sequences which

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belongs to $\tilde{\mathcal{A}}_p$ and obtain that these are Banach spaces. Also we prove our main result which states that $\tilde{\mathcal{G}}_p$ and $L^p(D)$ are isometrically isomorphic. In Section 4, as an application of our main results, we obtain a Korovkin-type approximation theorem for GLT matrix sequences analogous to the result for Toeplitz sequences. The article ends with a concluding section, mentioning some further possibilities.

2. Banach spaces of matrix sequences. Motivated by the notion of a.c.s., we introduce certain seminorms on the space of all matrix sequences.

Definition 2.1. *Let $\{A_n\}_n$ be a matrix sequence and define the functions $q_p : \mathcal{E} \to \mathbb{R}, 1 \le p \le \infty$, as*

$$q_p(\{A_n\}_n) = \inf \left\{ \limsup_{n\to\infty} \frac{\|N_n\|_p}{n^{1/p}} :\ R_n + N_n = A_n,\ rank(R_n) = o(n) \right\},\ 1 \le p < \infty,$$

$$q_\infty(\{A_n\}_n) = \inf \left\{ \limsup_{n\to\infty} \|N_n\| :\ R_n + N_n = A_n,\ rank(R_n) = o(n) \right\}.$$

Here the infimum is taken over all such decompositions of A_n.

Define the subspaces $\mathcal{A}_p$ of $\mathcal{E}$ as follows:

$$\mathcal{A}_p = \{\{A_n\}_n \in \mathcal{E}\ :\ q_p(\{A_n\}_n) < \infty\}.$$

Now we recall the notion of weak cluster convergence, strong cluster convergence, and uniform cluster convergence used in [20, 23].

Definition 2.2. *Let $\{A_n\}_n$ and $\{B_n\}_n$ be two matrix sequences. We say that $\{A_n - B_n\}_n$ converges to the sequence of zero matrices $\{O_n\}_n$ in Type 2 weak cluster sense if for any $\epsilon > 0$, there exist integers $n_{1,\epsilon}, n_{2,\epsilon}$ such that, for $n > n_{2,\epsilon}$,*

$$A_n - B_n = R_n + N_n,\ rank(R_n) \le n_{1,\epsilon},\ \|N_n\| < \epsilon,$$

where $n_{1,\epsilon}$ depends on both n and ϵ and is $o(n)$.

The convergence is in the Type 2 uniform cluster sense if $n_{1,\epsilon}$ is independent of ϵ and in the Type 2 strong cluster sense if $n_{1,\epsilon}$ depends only on ϵ.

Remark 2.3 ([20]). $\{A_n - B_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense if and only if for any $\epsilon > 0$ there exist integers $n_{1,\epsilon}, n_{2,\epsilon}$ such that, for all $n > n_{2,\epsilon}$ except at most $n_{1,\epsilon}$ ($n_{1,\epsilon} = o(n)$) singular values, all singular values of $A_n - B_n$ lie in the interval $[0, \epsilon)$. The Type 2 convergence is equivalent to the singular value clustering. There is a notion of Type 1 convergence that is equivalent to eigenvalue clustering. Both originated from the study of preconditioners in numerical linear algebra problems (see, e.g., [23]).

The following lemma is a consequence of the results in [26] and provides a criterion to establish the convergence notions defined above.

Lemma 2.4. *Let $\{A_n\}_n$ and $\{B_n\}_n$ be two sequences of $n \times n$ matrices of growing order. If $\|A_n - B_n\|_2^2 = o(n)$, then we have the convergence in the Type 2 weak cluster sense. If $\|A_n - B_n\|_2^2 = O(1)$, then the convergence is in Type 2 strong cluster sense.*

In [16], C. Garoni and S. Serra-Capizzano proved that a.c.s. convergence and Type 2 weak convergence are equivalent. We state the result below.

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THEOREM 2.5 (Theorem 4.1 of [16]). *Let* $\{A_n\}_n$ *and* $\{B_n\}_n$ *be two matrix sequences. Then* $\{A_n - B_n\}_n$ *converges to* $\{O_n\}_n$ *in Type* 2 *weak cluster sense if and only if* $d_{acs}(\{A_n\}_n, \{B_n\}_n) = 0$.

The following theorem gives a characterization for q_∞ and leads to a relation between Type 2 weak convergence and q_∞, analogous to Theorem 2.5.

THEOREM 2.6. *Let* $\{A_n\}_n$ *be a matrix sequence and* $\sigma_i(A_n)$ *be the* i^{th} *singular value of the matrix* A_n *arranged in non-increasing order. Then,*

$$q_\infty(\{A_n\}_n) = \inf \{\alpha \in [0,\infty) : \#(\sigma(A_n) > \alpha) = o(n)\},$$

$$q_p(\{A_n\}_n) = \inf \left\{ \alpha \in [0,\infty) : \sum_{i=j_n+1}^{n} \frac{\sigma_i^p(A_n)}{n} < \alpha^p \text{ except for finitely many } n \text{ and } j_n = o(n) \right\}, 1 \le p < \infty,$$

where $\#(\sigma(A_n) > \alpha)$ *is the number of singular values of* A_n *greater than* α.

Proof. Let $\{R_n\}_n$ and $\{N_n\}_n$ be any matrix sequences such that $\{R_n\}_n + \{N_n\}_n = \{A_n\}_n$ and $\text{rank}(R_n) = o(n)$. Let $\sigma_1(A_n) \ge \sigma_2(A_n) \ge \ldots \ge \sigma_n(A_n)$ be the singular values of A_n arranged in non-increasing order. We know that from [7]

$$\sigma_i(A_n) \le \sigma_i(R_n) + \|N_n\|.$$

Setting $r_n = rank\, R_n$, we have $\sigma_i(A_n) \le \|N_n\|$ for all $i > r_n$. Let r_n' be the smallest integer such that $\sigma_i(A_n) \le \|N_n\|$ for all $i > r_n'$. r_n' may be zero, first we consider $r_n' \ne 0$.

$r_n \ge r_n'$ and $\sigma_{r_n'}(A_n) > \|N_n\| \ge \sigma_{r_n'+1}(A_n)$. Let $A_n = U_n \Sigma_n V_n^*$ be a singular value decomposition (SVD) of A_n and set

$$\tilde{R}_n = U_n\, diag(\sigma_1(A_n), \ldots, \sigma_{r_n'}(A_n), 0, \ldots, 0) V_n^*,$$
$$\tilde{N}_n = U_n\, diag(0, \ldots, 0, \sigma_{r_n'+1}(A_n), \ldots, \sigma_n(A_n)) V_n^*.$$

Then, $A_n = \tilde{R}_n + \tilde{N}_n$. If $r_n' = 0$, then $\tilde{R}_n = 0$ and $\tilde{N}_n = A_n$. We have

$$rank(R_n) = r_n \ge r_n' = rank(\tilde{R}_n), \quad \|N_n\| \ge \sigma_{r_n'+1}(A_n) = \|\tilde{N}_n\|.$$

Let $\limsup\limits_{n\to\infty} \|\tilde{N}_n\| = \alpha$. Then for every $\epsilon > 0$, there exists an n_0 such that for all $n > n_0$, $\|\tilde{N}_n\| < \alpha + \epsilon$ and $\#(\sigma(A_n) \ge \alpha + \epsilon) \le r_n'$. Now for every $\epsilon > 0$, we have $\limsup\limits_{n\to\infty} \dfrac{\#(\sigma(A_n) \ge \alpha+\epsilon)}{n} \le \limsup\limits_{n\to\infty} \dfrac{r_n'}{n}$. Now, $\limsup\limits_{n\to\infty} \dfrac{\#(\sigma(A_n) > \alpha+\epsilon)}{n} \le \limsup\limits_{n\to\infty} \dfrac{r_n'}{n} \le \limsup\limits_{n\to\infty} \dfrac{r_n}{n} = 0$. Therefore,

$$\begin{aligned} q_\infty(\{A_n\}_n) &\ge \inf \{\alpha \in [0,\infty) : \#(\sigma(A_n) > \alpha + \epsilon) = o(n)\} \\ &\ge \inf \{\alpha \in [0,\infty) : \#(\sigma(A_n) > \alpha) = o(n)\} - \epsilon \qquad \forall \epsilon > 0. \end{aligned}$$

To prove the other inequality, let $A_n = U_n \Sigma_n V_n^*$ be a SVD of A_n. Let $\alpha \in [0,\infty)$ such that $\#(\sigma(A_n) \ge \alpha) = o(n)$. Let $R_n = U_n \tilde{\Sigma}_n V_n^*$, $N_n = U_n \hat{\Sigma}_n V_n^*$, where $\tilde{\Sigma}_n$ is the diagonal matrix obtained from Σ_n by setting to 0 all the singular values of A_n that are less than or equal to α, and $\hat{\Sigma}_n = \Sigma - \tilde{\Sigma}_n$. Hence, $\text{rank}(\tilde{\Sigma}_n) = o(n)$. Then, $A_n = R_n + N_n$, $rank(R_n) = \#(\sigma(A_n) \ge \alpha)$ and $\limsup_{n\to\infty} \|N_n\| \le \alpha$. By taking infimum over all such α, we get

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$$\begin{aligned} q_\infty(\{A_n\}_n) &\leq \inf\left\{\alpha\in[0,\infty) : \limsup_{n\to\infty}\|N_n\|\leq\alpha,\ \ \mathrm{rank}(R_n)=o(n),\ \ A_n=R_n+N_n\right\} \\ &\leq \inf\left\{\alpha\in[0,\infty) : \#(\sigma(A_n)>\alpha)=o(n)\right\}. \end{aligned}$$

Hence, $q_\infty(\{A_n\}_n)=\inf\{\alpha\in[0,\infty):\#(\sigma(A_n)>\alpha)=o(n)\}$.

Using the inequality $\sigma_{i+j-1}(A_n)\leq\sigma_i(R_n)+\sigma_j(N_n), \forall i+j\leq n+1$, we can prove the case $1\leq p<\infty$ in same manner. ∎

Corollary 2.7. *Let $\{A_n\}_n$ and $\{B_n\}_n$ be two matrix sequences and let $1\leq p\leq\infty$. Then $\{A_n-B_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense if and only if $q_p(\{A_n-B_n\}_n)=0$.*

Proof. The case $p=\infty$ follows from Theorem 2.6 and Remark 2.3. For $1\leq p<\infty$, the result follows from $\dfrac{\|N_n\|_p}{n^{1/p}}\leq\|N_n\|$ and Theorem 1.8. ∎

Corollary 2.8. *A matrix sequence $\{Z_n\}_n$ is a zero-distributed sequence if and only if $q_p(\{Z_n\}_n)=0$.*

Proof. Result follows from Corollary 2.7 and Definition 1.7. ∎

Let $Z=\{\{A_n\}_n\in\mathcal{E}:p(\{A_n\}_n)=0\}$, the set of all zero-distributed sequences and $Z_p=\{\{A_n\}_n\in\mathcal{A}_p: q_p(\{A_n\}_n)=0\}$. In the view of Corollary 2.8, $Z=Z_p$ for every $1\leq p\leq\infty$. Then $\tilde{\mathcal{A}}_p=\mathcal{A}_p/Z$ is the quotient space of $\mathcal{A}_p$.

Now we prove that $\tilde{\mathcal{A}}_p$ are Banach spaces.

Theorem 2.9. *$\tilde{\mathcal{A}}_p, 1\leq p\leq\infty$, are Banach spaces with respect to the norms induced by the seminorms q_p. In particular, $\tilde{\mathcal{A}}_\infty$ forms a C^*-algebra and $\tilde{\mathcal{A}}_2$ is a Hilbert space.*

Proof. Here we prove only the case of $\tilde{\mathcal{A}}_p, 1\leq p<\infty$. For $p=\infty$, the proof is similar. First we fix some notations. Let $q_A=q_p(\{A_n\}_n), q_B=q_p(\{B_n\}_n), q_{A+B}=q_p(\{A_n\}_n+\{B_n\}_n)$ and $q_{AB}=q_p(\{A_nB_n\}_n)$. From the definition of q_p, for every $m\in\mathbb{N}$, there exist four matrix sequences $\{R^A_{n,m}\}$,$\{N^A_{n,m}\}$, $\{R^B_{n,m}\}$,$\{N^B_{n,m}\}$ such that $\{R^A_{n,m}\}+\{N^A_{n,m}\}=\{A_n\}_n$, $\{R^B_{n,m}\}+\{N^B_{n,m}\}=\{B_n\}_n$, and

$$\limsup_{n\to\infty}\frac{\|N^A_{n,m}\|_p}{n^{1/p}}\leq q_A+\frac{1}{m},\quad \limsup_{n\to\infty}\frac{\|N^B_{n,m}\|_p}{n^{1/p}}\leq q_B+\frac{1}{m}.$$

Also, rank $(R^A_{n,m})=$ rank $(R^B_{n,m})=o(n)$.

Now we verify the axioms of seminorm. The nonnegativity and $q_p(\{O_n\}_n)=0$ are trivial. For triangular inequality,

$$\begin{aligned} q_{A+B} &\leq \limsup_{n\to\infty}\frac{\|N^A_{n,m}+N^B_{n,m}\|_p}{n^{1/p}} \\ &\leq \limsup_{n\to\infty}\frac{\|N^A_{n,m}\|_p}{n^{1/p}}+\limsup_{n\to\infty}\frac{\|N^B_{n,m}\|_p}{n^{1/p}} \\ &\leq q_A+q_B+\frac{2}{m}. \end{aligned}$$

Thus, $q_{A+B}\leq q_A+q_B$.

The equation $q_{\alpha A}=|\alpha|q_A$ is obvious for every $\alpha\in\mathbb{C}$.

Hence, q_p is a seminorm in $\mathcal{A}_p$. Then the function $\tilde{q}_p:\tilde{\mathcal{A}}_p\times\tilde{\mathcal{A}}_p\to\mathbb{R}$ defined as

$$\tilde{q}_p(\{A_n\}_n+Z)=q_p(\{A_n\}_n),$$

becomes a norm on $\tilde{\mathcal{A}}_p$.

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Next, we prove the completeness of $\tilde{\mathcal{A}}_p$. Let $\{\{B_{n,m}\}_n + L_w\}_m$ be a Cauchy sequence in $\tilde{\mathcal{A}}_p$. It suffices to show the convergence of a subsequence. We can extract a subsequence and name it as $\{\{B_{n,m}\}_n + Z\}_m$ itself such that

$$\tilde{q}_p(\{B_{n,m+1} - B_{n,m}\}_n + Z) < 2^{-m}, \quad m = 1, 2, 3, \ldots$$

Then,

$$\tilde{q}_p(\{B_{n,m+i} - B_{n,m}\}_n + Z) = q_p(\{B_{n,m+i} - B_{n,m}\}_n) < 2^{(1-m)}, \quad i = 1, 2, 3, \ldots$$

Also we can construct two matrix sequences $\{R^m_{n,i}\}_n$ and $\{N^m_{n,i}\}_n$ such that $B_{n,m+i} - B_{n,m} = R^m_{n,i} + N^m_{n,i}$, where $\frac{\|N^m_{n,i}\|_p}{n^{1/p}} < 2^{-(m-1)}$ and $\text{rank}(R^m_{n,i}) = o(n)$.

We can find a strictly increasing sequence of positive integers $\{n_{i,m}\}_i$ such that for all $n \geq n_{i,m}$, $\frac{rank(R^m_{n,i})}{n} < \frac{1}{i}$. Also we choose $\{\{n_{i,m}\}_m\}_i$ such that

$$n_{i,m+1} > n_{i+1,m}. \tag{2.1}$$

This inequality helps us to obtain the required estimate. Since $n_{i,m+1} > n_{i+1,m} > n_{i,m}$, for a fixed i, $\{n_{i,m}\}_m$ is an increasing sequence.

Now consider $\{n_{2,m}\}_m$ and construct a matrix sequence $\{A_n\}_n$ in such a way that $A_n = B_{n,j+1}$, whenever $n_{2,j-1} \leq n < n_{2,j}$.

Consider $A_n - B_{n,m}$; for $n_{2,m+i-1} \leq n < n_{2,m+i}$,

$$A_n - B_{n,m} = B_{n,m+i+1} - B_{n,m} = R^m_{n,i+1} + N^m_{n,i+1},$$

where $\frac{\|N^m_{n,i+1}\|_p}{n^{1/p}} < 2^{(1-m)}$ and $\frac{rank(R^m_{n,i+1})}{n} < \frac{1}{i+1}$, for all $n \geq n_{i+1,m}$.

Here by inequality (2.1), $n \geq n_{2,m+i-1} > n_{i+1,m}$, then $q_p(\{A_n\}_n - \{B_{n,m}\}_n) < 2^{(1-m)}$.

Hence, $\lim_{m\to\infty} \tilde{q}_p(\{A_n\}_n - \{B_{n,m}\}_n + Z) = \lim_{m\to\infty} q_p(\{A_n\}_n - \{B_{n,m}\}_n) = 0$.

Thus, $\tilde{\mathcal{A}}_p$ are Banach spaces for every $1 \leq p \leq \infty$.

For the Banach algebra inequality, we consider

$$\{A_n B_n\}_n = \{R^A_{n,m} R^B_{n,m} + R^A_{n,m} N^B_{n,m} + N^A_{n,m} R^B_{n,m}\}_n + \{N^A_{n,m} N^B_{n,m}\}_n.$$

Here $\text{rank}(R^A_{n,m} R^B_{n,m} + R^A_{n,m} N^B_{n,m} + N^A_{n,m} R^B_{n,m}) = o(n)$. Then,

$$\begin{aligned} q_\infty(\{A_n B_n\}_n) &\leq \limsup_{n\to\infty} \|N^A_{n,m} N^B_{n,m}\| \\ &\leq \limsup_{n\to\infty} \|N^A_{n,m}\| \limsup_{n\to\infty} \|N^B_{n,m}\| \\ &\leq (q_\infty(\{A_n\}_n) + \frac{1}{m})(q_\infty(\{B_n\}_n) + \frac{1}{m}). \end{aligned}$$

Thus, $q_\infty(\{A_n B_n\}_n) \leq q_\infty(\{A_n\}_n) q_\infty(\{B_n\}_n)$ (note that this result is not true for $\tilde{\mathcal{A}}_p, p \neq \infty$).

Hence, $\tilde{\mathcal{A}}_\infty$ is a C^*-algebra with usual complex conjugate transpose of the matrix as the involution, that is, $\{A_n\}^*_n = \{A^*_n\}_n$.

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Finally, we prove that $\tilde{\mathcal{A}}_2$ is a Hilbert space. Let $\tilde{q}_2(\{A_n\}_n + Z) = \alpha$ and $\tilde{q}_2(\{B_n\}_n + Z) = \beta$. Then for every $\epsilon > 0$, there exist matrix sequences $\{R_n^A\}_n, \{R_n^B\}_n, \{N_n^A\}_n, \{N_n^B\}_n$ and a positive integer n_0 such that

$$\frac{\|N_n^A\|_2^2}{n} \leq \alpha^2 + \epsilon \ \ \forall n > n_0, \quad A_n = R_n^A + N_n^A, \quad \text{rank}(R_n^A) = o(n),$$

$$\frac{\|N_n^B\|_2^2}{n} \leq \beta^2 + \epsilon \ \ \forall n > n_0, \quad B_n = R_n^B + N_n^B, \quad \text{rank}(R_n^B) = o(n).$$

Now $A_n + B_n = R_n^A + R_n^B + N_n^A + N_n^B$ and $A_n - B_n = R_n^A - R_n^B + N_n^A - N_n^B$. Then

$$q_2(\{A_n + B_n\})^2 \leq \limsup_{n\to\infty} \frac{\|N_n^A + N_N^B\|_2^2}{n}, \quad q_2(\{A_n - B_n\})^2 \leq \limsup_{n\to\infty} \frac{\|N_n^A - N_N^B\|_2^2}{n}.$$

$$q_2(\{A_n + B_n\})^2 + q_2(\{A_n - B_n\})^2 \leq \limsup_{n\to\infty} \frac{\|N_n^A + N_n^B\|_2^2}{n} + \limsup_{n\to\infty} \frac{\|N_n^A - N_n^B\|_2^2}{n} \leq 2\alpha^2 + 2\beta^2 + 4\epsilon.$$

Thus,

$$\tilde{q}_2(\{A_n + B_n\} + Z)^2 + \tilde{q}_2(\{A_n - B_n\} + Z)^2 \leq 2\tilde{q}_2(\{A_n\}_n + Z)^2 + 2\tilde{q}_2(\{B_n\}_n + Z)^2. \tag{2.2}$$

Let $\tilde{q}_2(\{A_n + B_n\}_n + Z) = \alpha'$ and $\tilde{q}_2(\{A_n - B_n\}_n + Z) = \beta'$. Then for every $\epsilon > 0$, there exist matrix sequences $\{R_n\}_n, \{R'_n\}_n, \{N_n\}_n, \{N'_n\}_n$ and a positive integer n_1 such that

$$\frac{\|N_n\|_2^2}{n} \leq {\alpha'}^2 + \epsilon \ \ \forall n > n_1, \quad A_n + B_n = R_n + N_n, \quad \text{rank}(R_n) = o(n),$$

$$\frac{\|N'_n\|_2^2}{n} \leq {\beta'}^2 + \epsilon \ \ \forall n > n_1, \quad A_n - B_n = R'_n + N'_n, \quad \text{rank}(R'_n) = o(n).$$

Now $2A_n = R_n + R'_n + N_n + N'_n$ and $2B_n = R_n - R'_n + N_n - N'_n$. Then,

$$4q_2(\{A_n\}_n)^2 \leq \limsup_{n\to\infty} \frac{\|N_n + N'_n\|_2^2}{n}, \quad 4q_2(\{B_n\}_n)^2 \leq \limsup_{n\to\infty} \frac{\|N_n - N'_n\|_2^2}{n}.$$

$$4q_2(\{A_n\}_n)^2 + 4q_2(\{B_n\}_n)^2 \leq \limsup_{n\to\infty} \frac{\|N_n + N'_n\|_2^2}{n} + \limsup_{n\to\infty} \frac{\|N_n - N'_n\|_2^2}{n} \leq 2{\alpha'}^2 + 2{\beta'}^2 + 4\epsilon.$$

Thus,

$$2\tilde{q}_2(\{A_n\}_n + Z)^2 + 2q_2(\{B_n\}_n + Z)^2 \leq \tilde{q}_2(\{A_n + B_n\} + Z)^2 + \tilde{q}_2(\{A_n - B_n\} + Z)^2. \tag{2.3}$$

From (2.2) and (2.3),

$$\tilde{q}_2(\{A_n + B_n\} + Z)^2 + \tilde{q}_2(\{A_n - B_n\} + Z)^2 = 2\tilde{q}_2(\{A_n\}_n + Z)^2 + 2\tilde{q}_2(\{B_n\}_n + Z)^2.$$

Since the norm $\tilde{q}_2$ satisfies parallelogram equality, $\tilde{\mathcal{A}}_2$ forms a Hilbert space. ☐

The convergence of a sequence $\{\{B_{n,m}\}_m\}_n$ to $\{A_n\}_n$ in the topology induced by q_p is denoted by $\{\{B_{n,m}\}_n\}_m \xrightarrow{q_p} \{A_n\}_n$.

The following theorem gives the relation between a.c.s. convergence and the convergence with respect to q_p.

Theorem 2.10. *q_∞ convergence $\Longrightarrow$ q_p convergence $\Longrightarrow$ a.c.s. convergence, $1 \leq p < \infty$.*

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Proof. q_∞ convergence implies q_p convergence and a.c.s. convergence follows from Definition 2.1 and Lemma 1.6.

Now we prove that q_p convergence implies a.c.s. convergence.

Consider a sequence $\{\{A_{n,m}\}_n\}_m$ and $q_p(\{A_{n,m}\}_n - \{A_n\}) \to 0$ as $m \to \infty$.

Hence for $\epsilon > 0$, there exists a positive integer m_0 such that for all $m > m_0$,

$$q_p(\{A_{n,m}\}_n - \{A_n\}_n) < \epsilon/2.$$

Fix a $m > m_0$. Then there exists a decomposition $A_{n,m} - A_n = R_n^m + N_n^m$ such that

$$\limsup_{n\to\infty} \frac{\|N_n^m\|_p}{n^{1/p}} < \epsilon, \quad \operatorname{rank}(R_n^m) = o(n).$$

Therefore, there exists a positive integer n_0 such that, for all $n > n_0$, $\dfrac{\|N_n^m\|_p}{n^{1/p}} < \epsilon$.

Let $\sigma_i(N_n^m)$ be the i^{th} singular value of N_n^m when arranged in non-increasing order. So, for all $n > n_0$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} \sigma_i(N_n^m)^p\right)^{1/p} < \epsilon,$$

$$\frac{1}{n}\sum_{i=1}^{n} \sigma_i(N_n^m)^p < \epsilon^p.$$

Let k_n be the number of singular values of N_n^m which are greater than $\sqrt{\epsilon}$. Then for all $n > n_0$,

$$\frac{1}{n}\sum_{i=1}^{k_n} \sigma_i(N_n^m)^p \le \frac{1}{n}\sum_{i=1}^{n} \sigma_i(N_n^m)^p < \epsilon^p, \quad \frac{1}{n}\sum_{i=1}^{k_n} \sigma_i(N_n^m)^p > \frac{k_n}{n}\epsilon^{p/2}.$$

Thus, for all $n > n_0$, $\frac{k_n}{n} < \epsilon^{p/2}$. Now consider,

$$P(N_n^m) = \min_{i=0,1,\ldots,n} \left\{\frac{i}{n} + \sigma_{i+1}(N_n^m)\right\}.$$

For $i = k_n$,

$$\frac{i}{n} + \sigma_{i+1}(N_n^m) = \frac{k_n}{n} + \sigma_{k+1}(N_n^m) < \epsilon^{p/2} + \epsilon^{1/2},$$

$$p(\{N_n^m\}_n) = \limsup_{n\to\infty} P(N_n^m) \le \epsilon^{p/2} + \epsilon^{1/2}.$$

Since $p(\{R_n^m\}_n) = 0$,

$$p(\{A_{n,m}\}_n - \{A_n\}_n) \le p(\{R_n^m\}_n) + p(\{N_n^m\}_n) \le \epsilon^{p/2} + \epsilon^{1/2}.$$

Thus, for all $m > m_0$,

$$p(\{A_{n,m}\}_n - \{A_n\}_n) \le \epsilon^{p/2} + \epsilon^{1/2}.$$

Hence, q_p convergence implies a.c.s. convergence. ☐

Remark 2.11. Reverse implication is false, as we shall see in the following example. Also, q_p convergence implies q_r convergence if $1 \le r < p < \infty$.

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EXAMPLE 2.12. Let $B_{n,m}$ be the diagonal matrix with its first $\lfloor \frac{n}{m} \rfloor$ diagonal entries 1 and others 0.

$$\begin{aligned} p(\{B_{n,m}\}_n) &= \inf\left\{\limsup_{n\to\infty}\left\{\frac{rank(R)}{n} + \|N\| :\ R+N = B_{n,m}\right\}\right\} \\ &\leq \limsup_{n\to\infty}\frac{rank(B_{n,m})}{n} \\ &= \frac{\lfloor \frac{n}{m} \rfloor}{n} \\ &\leq \frac{1}{m} \end{aligned}$$

Then, by Lemma 1.6, $\{\{B_{n,m}\}_n\}_m \xrightarrow{a.c.s.} \{O_n\}_n$. But $q_\infty(\{B_{n,m}\}_n)$ is 1 for all m. Hence, $\{\{B_{n,m}\}_n\}_m$ does not converge to $\{O_n\}_n$ in $\tilde{\mathcal{A}}_\infty$.

For a.c.s. convergence does not imply q_p convergence, take the same example with m instead of 1 in $B_{n,m}$.

3. Main results: GLT matrix sequences and L^p spaces. In this section, we prove our main result. Recall the definition of GLT matrix sequences (Definition 1.5). Let $\mathcal{G}^p$ be the space defined as follows.

$$\mathcal{G}^p = \{\{A_n\}_n \in \mathcal{A}_p : \{A_n\}_n \sim_{GLT} f\}.$$

Let

$$Z^p = \{\{A_n\}_n \in \mathcal{G}^p : q_p(\{A_n\}_n) = 0\}.$$

From Corollary 2.8, it follows that $Z = Z^p$ for every $1 \leq p \leq \infty$. Let $\tilde{\mathcal{G}}^p = \mathcal{G}^p/Z$ be the quotient space of $\mathcal{G}^p$.

We recall two theorems of GLT matrix sequences from [1, 16];

THEOREM 3.1. *Let $\{A_n\}_n \sim_{GLT} f$ and $\{B_n\}_n \sim_{GLT} g$. Then,*

1. $\{A_n^*\}_n \sim_{GLT} \bar{f}$.
2. $\{\alpha A_n + \beta B_n\}_n \sim_{GLT} \alpha f + \beta g$, *for all* $\alpha, \beta \in \mathbb{C}$.
3. $\{A_n B_n\}_n \sim_{GLT} fg$.
4. *if* $\{A_n\}_n \sim_{GLT} h$ *then* $f = h$ *a.e.*
5. $\{A_n\}_n$ *is zero distributed iff* $f = 0$ *a.e.*

THEOREM 3.2. *For all measurable functions f defined on $D = [0,1] \times [-\pi,\pi]$, there exists a matrix sequence $\{A_n\}_n$ such that $\{A_n\}_n \sim_{GLT} f$.*

Let $D = [0,1]\times[-\pi,\pi]$ and define a function $\phi_p : \tilde{\mathcal{G}}^p \to L^p(D), 1 \leq p \leq \infty$, such that whenever $\{A_n\}_n \sim_{GLT} f$,

$$\phi_p(\{A_n\}_n) = \begin{cases} f & \text{if } p = \infty, \\ \dfrac{1}{(2\pi)^{\frac{1}{p}}} f & \text{if } p \neq \infty. \end{cases}$$

ϕ_p is well defined by Theorem 3.1, item 4. Now we are in a position to prove our main result (Theorem 3.6). In fact, we prove that ϕ_p is an isometric isomorphism between $\tilde{\mathcal{G}}^p$ and $L^p(D), 1 \leq p \leq \infty$. This is a consequence of Theorems 3.1, 3.2, and the following two lemmas.

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Lemma 3.3. *If* $\{A_n\}_n \sim_\sigma f$*, then* $q_\infty(\{A_n\}_n) = \|f\|_\infty$ *and* $q_p(\{A_n\}_n) = \frac{1}{(2\pi)^{(1/p)}}\|f\|_p$*, for* $1 \leq p < \infty$*.*

Proof. Suppose $\|f\|_\infty = \operatorname*{ess\,sup}_{x\in D} |f(x)| = l < \infty$.

By definition of essential supremum, for any $\epsilon > 0$,

$$\mu\{x : |f(x)| \geq l + \epsilon\} = 0,$$

where μ is the Lebesgue measure. Since $\{A_n\}_n \sim_\sigma f$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{2\pi}\int_D F(|f(x)|)dx,$$

for every $F \in C_c(\mathbb{R})$. Suppose that the singular values $\sigma_i(A_n)$ are arranged in non-increasing order: $\sigma_1(A_n) \geq \sigma_2(A_n) \geq \cdots \geq \sigma_n(A_n)$. Consider a real-valued continuous function F with compact support, such that $\chi_{[-\epsilon,l+2\epsilon]} \geq F \geq \chi_{[0,l+\epsilon]}$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)) \leq \liminf_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) \leq l + 2\epsilon),$$

and

$$\int_D F(|f(x)|)dx \geq \mu\{x : |f(x)| \leq l + \epsilon\}.$$

Now,

$$\liminf_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) \leq l + 2\epsilon) \geq \frac{1}{2\pi}\mu\{x : |f(x)| \leq l + \epsilon\},$$

$$\liminf_{n\to\infty} \frac{1}{n}\{n - \#(\sigma(A_n) > l + 2\epsilon)\} \geq \frac{1}{2\pi}(2\pi - \mu\{x : |f(x)| > l + \epsilon\}),$$

$$1 - \limsup_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) > l + 2\epsilon) \geq 1 - \frac{1}{2\pi}\mu\{x : |f(x)| > l + \epsilon\},$$

$$\limsup_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) > l + 2\epsilon) \leq \frac{1}{2\pi}\mu\{x : |f(x)| > l + \epsilon\} = 0.$$

By Theorem 2.6, $q_\infty(\{A_n\}_n) \leq l + 2\epsilon$. Thus, $q_\infty(\{A_n\}_n) \leq \|f\|_\infty$.

To prove the other inequality, suppose $q_\infty(\{A_n\}) = k < \infty$. By Theorem 2.6, for $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\#(\sigma(A_n) > k + \epsilon)}{n} = 0.$$

Since $\{A_n\}_n \sim_\sigma f$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{2\pi}\int_D F(|f(x)|)dx,$$

for every $F \in C_c(\mathbb{R})$. Consider a function $F \in C_c(\mathbb{R})$ such that $\chi_{[-\epsilon,k+2\epsilon]} \geq F \geq \chi_{[0,k+\epsilon]}$. Then,

$$\liminf_{n\to\infty} \frac{\#(\sigma(A_n) \leq k + \epsilon)}{n} \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(A_n)),$$

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$$\int_D F(|f(x)|)dx \leq \mu\{x : |f(x)| \leq k + 2\epsilon\},$$

$$\liminf_{n\to\infty} \frac{1}{n}\#(\sigma_i(A_n) \leq k + \epsilon) \leq \frac{1}{2\pi}\mu\{x : |f(x)| \leq k + 2\epsilon\},$$

$$\frac{1}{2\pi}\mu\{x : |f(x)| > k + 2\epsilon\} \leq \limsup_{n\to\infty} \frac{1}{n}\#(\sigma_i(A_n) > k + \epsilon) = 0.$$

Thus, $\|f\|_\infty \leq k + 2\epsilon$ and $\|f\|_\infty \leq q_\infty(\{A_n\}_n)$.

Now, the case $q_\infty(\{A_n\}_n) = \|f\|_\infty = \infty$ is straight forward.

The proof of $q_p(\{A_n\}_n) = \frac{1}{(2\pi)^{(1/p)}}\|f\|_p$ is almost similar.

Let $f \in L^p(D)$ and $\int_D |f(x)|^p dx = l^p$. Consider a function $F_1 \in C_c(\mathbb{R})$ such that $x^p\chi_{[0,m]} \leq F_1(x) \leq x^p\chi_{[0,m+1]}$. Then,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{E_m} \sigma_i^p(A_n) \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F_1(\sigma_i(A_n)) = \frac{1}{2\pi}\int_D F_1(|f(x)|)dx \leq \frac{1}{2\pi}l^p, \tag{3.4}$$

where $E_m = \{i : \sigma_i(A_n) \leq m\}$. Now take $F_2 \in C_c(\mathbb{R})$ such that $\chi_{[0,m-1]} \leq F_2 \leq \chi_{[0,m]}$. Then,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F_2(\sigma_i(A_n)) \leq \liminf_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) \leq m),$$

and

$$\int_D F_2(|f(x)|)dx \geq \mu\{x : |f(x)| \leq m - 1\}.$$

Now,

$$\liminf_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) \leq m) \geq \frac{1}{2\pi}\mu\{x : |f(x)| \leq m - 1\},$$

$$\limsup_{n\to\infty} \frac{1}{n}\#(\sigma(A_n) > m) \leq \frac{1}{2\pi}\mu\{x : |f(x)| > m - 1\}. \tag{3.5}$$

We can choose m such that $\frac{1}{2\pi}\mu\{x : |f(x)| > m - 1\} < \frac{1}{t}$. From (3.4) and (3.5), for each $t \in \mathbb{N}$ we got a decomposition $A_n = R_n^t + N_n^t$ such that

$$R_n^t = U_n\, diag(\sigma_1(A_n), \ldots, \sigma_{j_t}(A_n), 0, \ldots, 0)V_n^*,$$

$$N_n^t = U_n\, diag(0, \ldots, 0, \sigma_{j_t+1}(A_n), \ldots, \sigma_n(A_n))V_n^*,$$

where $\sigma_{j_t}(A_n) > m$ and $\sigma_{j_t+1}(A_n) \leq m$. Also

$$\limsup_{n\to\infty} \frac{rank(R_n^t)}{n} \leq \frac{1}{t}, \quad \limsup_{n\to\infty} \frac{\|N_n^t\|}{n^{1/p}} \leq \frac{1}{(2\pi)^{1/p}}l.$$

Fix $\epsilon > 0$, for each $t \in \mathbb{N}$, there exists $n_t \in \mathbb{N}$ such that for $n > n_t$,

$$\frac{rank(R_n^t)}{n} < \frac{2}{t}, \quad \frac{\|N_n^t\|_p}{n^{1/p}} < \frac{1}{(2\pi)^{1/p}}l + \epsilon.$$

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Also choose $n_{t+1} > n_t$. Now for $n_t + 1 \leq n \leq n_{t+1}$, define $R_n = R_n^t$ and $N_n = N_n^t$. So this implies

$$rank(R_n) = o(n), \quad \limsup_{n\to\infty} \frac{\|N_n\|}{n^{1/p}} \leq \frac{1}{(2\pi)^{1/p}} l + \epsilon.$$

Since ϵ is arbitrary, $q_p(\{A_n\}_n) \leq \frac{1}{(2\pi)^{1/p}} l$. To prove the other inequality, suppose $q_p(\{A_n\}) = k < \infty$. By Theorem 2.6, for $\epsilon > 0$,

$$\frac{1}{n} \sum_{i=j_n+1}^{n} \sigma_i^p(A_n) \leq k^p + \epsilon \text{ except for finitely many } n \text{ and } j_n = o(n). \tag{3.6}$$

Consider a function $F \in C_c(\mathbb{R})$ such that $x^p \chi_{[0,m-1]} \leq F(x) \leq x^p \chi_{[0,m]}$. Then,

$$\frac{1}{2\pi} \int_{D_m} |f(x)|^p dx \leq \frac{1}{2\pi} \int_{D} F(|f(x)|) dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{E_m} \sigma_i^p(A_n),$$

where $D_m = \{x : |f(x)| \leq m-1\}$ and $E_m = \{i : \sigma_i(A_n) \leq m\}$. If for all $m \in \mathbb{N}$, $\liminf_{n\to\infty} \frac{1}{n} \sum_{E_m} \sigma_i^p(A_n) \leq k^p + \epsilon$, we are done. Suppose this is not true, then there exists $m \in \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{E_m} \sigma_i^p(A_n) > k^p + \epsilon.$$

Therefore, there exists n_0 such that for all $n > n_0$,

$$\frac{1}{n} \sum_{i=r_n+1}^{n} \sigma_i^p(A_n) > k^p + \epsilon, \quad \sigma_{r_n}(A_n) > m, \quad \sigma_{r_n+1}(A_n) \leq m. \tag{3.7}$$

From (3.6) and (3.7),

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=r_n+1}^{j_n} \sigma_i^p(A_n) \leq \limsup_{n\to\infty} \frac{j_n}{n} m^p = 0.$$

Thus,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \sum_{i=r_n+1}^{n} \sigma_i^p(A_n) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=r_n+1}^{j_n} \sigma_i^p(A_n) + \liminf_{n\to\infty} \frac{1}{n} \sum_{i=j_n+1}^{n} \sigma_i^p(A_n) \\ &\leq k^p + \epsilon.\end{aligned}$$

This is contrary to our assumption. Hence, $\frac{1}{2\pi^{1/p}} \|f\|_p \leq q_p(\{A_n\}_n)$. ◻

Corollary 3.4. *Let $\{A_n\}_n \sim_\sigma f$. Then $\{A_n\}_n \in \mathcal{A}_p$ if and only if $f \in L^p(D), 1 \leq p \leq \infty$.*

We give an example of a matrix sequence that belongs to $\tilde{\mathcal{G}}^1$ but not to $\tilde{\mathcal{G}}^2$.

Example 3.5. Let $a : [0,1] \to \mathbb{R}$,

$$a(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

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let $g : [-\pi, \pi] \to \mathbb{C}$ be the constant function 1, and let $\{A_n\}_n$ be the matrix sequence given by

$$A_n = \begin{pmatrix} \sqrt{n} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{n}{2}} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\frac{n}{3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The matrix sequence $\{A_n\}_n$ belongs to $\tilde{\mathcal{G}}^1$ but not to $\tilde{\mathcal{G}}^2$ and its symbol is the function $f : [0,1] \times [-\pi, \pi] \to \mathbb{R}$ defined by $f = a \otimes g$.

THEOREM 3.6. *The Banach spaces $\tilde{\mathcal{G}}^p$ and $L^p(D)$ are isometrically isomorphic for every $1 \leq p \leq \infty$. In particular, $\tilde{\mathcal{G}}^\infty$ and $L^\infty(D)$ are isomorphic as C^*-algebras.*

Proof. ϕ_p is an injective *-homomorphism of Banach spaces, which can be readily inferred from Theorem 3.1. The surjectivity follows from the definition of $\tilde{\mathcal{G}}^p$ and Theorem 3.2. Hence it is a *-isomorphism. From Lemma 3.3, it follows that ϕ_p is an isometry. In particular, ϕ_∞ is a C^*-isomorphism. ∎

Theorem 3.6 yields a natural isometry between the spaces $\tilde{\mathcal{G}}^p$ and $L^p(D)$, for $1 \leq p \leq \infty$, which is analogous to the isometry identified in [1] between space of GLT matrix sequences and space of measurable functions. Notice that in [1], the author derived a metric space isometry. Here we achieved a Banach space isometric isomorphism. A general setting in a Toeplitz and multilevel Toeplitz setting is studied in [14, 23], while for general matrix sequences, the readers are referred to [11, 12].

REMARK 3.7. All the relations between the convergences of matrix sequences reflect the same properties between functions, where the convergence q_p is the convergence in L^p and the a.c.s. convergence is the convergence in measure.

4. Korovkin-Type theorem. P. P. Korovkin proved a classical approximation theorem in 1953, which unified several approximation processes. Korovkin-type theorems in the setting of Toeplitz operators acting on Hardy spaces and Fock spaces were obtained in [6, 20, 22]. Type 2 strong/weak cluster sense convergence was considered there. Here we obtain an analogous result for GLT matrix sequences.

Consider $M_n(\mathbb{C})$, with Frobenius norm induced from the inner product $\langle A, B\rangle = \mathrm{trace}(B^*A)$. Let $\{U_n\}_n$ be a sequence of unitary matrices such that each U_n is of order n. For each n, define the subalgebra M_{U_n} of $M_n(\mathbb{C})$ as

$$M_{U_n} = \{A \in M_n(\mathbb{C}) : U_n^* A U_n \text{ is diagonal}\}.$$

M_{U_n} is a closed subspace of $M_n(\mathbb{C})$. We denote the orthogonal projection of $M_n(\mathbb{C})$ onto M_{U_n} by $P_{U_n}(\cdot)$ and $P_{U_n}(A) = U_n^* \mathrm{diag}(U_n A U_n^*) U_n$. It is known that the operator norm $\|P_{U_n}(\cdot)\| = 1$ and the Frobenius norm $\|P_{U_n}(\cdot)\|_F = 1$ (see [22] for details). For $A \in M_n(\mathbb{C})$, $P_{U_n}(A)$ is called a preconditioner for A.

Preconditioners play a crucial role in solving linear systems by iterative techniques. They help to increase the convergence rate of iterations. For instance, consider the linear system with Toeplitz structure,

$$T_n(f)x = b_n.$$

For a fixed f, we can consider a sequence of Toeplitz matrices $\{T_n(f)\}_n$. If we can find a sequence of matrices $\{C_n(f)\}_n$ such that $\{C_n(f) - T_n(f)\}_n$ converges to $\{O_n\}_n$ in Type 2 strong/weak cluster sense, $\{C_n(f)\}_n$ can be considered as an efficient preconditioner [21]. In this case, the eigenvalues of $C_n(f)^{-1}T_n(f)$ will be

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clustered at 1. This will help to improve the stability of the corresponding linear system. In [13], R. H. Chan and M. C. Yeung proved that if $U_n = F_n$ (the Fourier matrix of order n) and f is a continuous periodic function, then $\{P_{U_n}(T_n(f)) - T_n(f)\}_n$ converges to $\{O_n\}_n$ in Type 2 strong cluster sense, the corresponding preconditioners are known as circulant preconditioners. Depending on the choice of U_n, we can obtain other efficient preconditioners such as Hartley [19], Tau [24], etc., for Toeplitz matrices.

Since linear systems involving GLT matrix sequences appear in various situations, finding efficient preconditioners for GLT matrix sequences is also an important problem.

In what follows, we propose an example of an efficient preconditioner for GLT matrix sequences. The method we use here is quite similar to the methods used by G. Barbarino in [4].

Consider a LT sequence $\{A_n\}_n \sim_{LT} a \otimes f$, where a is a Riemann integrable function on $[0,1]$ and f is a continuous periodic function on $[-\pi, \pi]$. We give an example of preconditioner for this LT sequence and also obtain a preconditioner for a GLT sequence. A general setting in a Toeplitz and multilevel Toeplitz setting is studied in [14, 23], while for general matrix sequences the reader is referred to [11, 12]. Let

$$U_n = \begin{pmatrix} F_{\lfloor \frac{n}{m} \rfloor} & 0 & 0 & \cdots & 0 \\ 0 & F_{\lfloor \frac{n}{m} \rfloor} & 0 & \cdots & 0 \\ 0 & 0 & F_{\lfloor \frac{n}{m} \rfloor} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F_{n(mod\, m)} \end{pmatrix},$$

where $F_n = \left(\frac{1}{\sqrt{n}} e^{\frac{2\pi i j l}{n}} \right)_{j,l=0}^{n-1}$ is the Fourier matrix of order n. Consider

$$LT_n^m(a,f) = D_m(a) \otimes T_{\lfloor \frac{n}{m} \rfloor}(f) \oplus O_{n(mod\, m)}.$$

We can construct a matrix sequence $\{\tilde{A}_n\}_n$ which is an a.c.s. limit for the sequence $\{\{LT_n^m(a,f)\}_n\}_m$ such that $\tilde{A}_n = LT_n^m(a,f)$ for some m and $n \geq m^2$. Now consider

$$P_{U_n}(\tilde{A}_n) - \tilde{A}_n = \begin{pmatrix} a(\frac{1}{m})P_{F_k}(T_k(f)) & 0 & 0 & \cdots & 0 \\ 0 & a(\frac{2}{m})P_{F_k}(T_k(f)) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & a(1)P_{F_k}(T_k(f)) & 0 \\ 0 & 0 & \cdots & 0 & O_{n(mod\, m)} \end{pmatrix}$$
$$- \begin{pmatrix} a(\frac{1}{m})T_k(f) & 0 & 0 & \cdots & 0 \\ 0 & a(\frac{2}{m})T_k(f) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & a(1)T_k(f) & 0 \\ 0 & 0 & \cdots & 0 & O_{n(mod\, m)} \end{pmatrix},$$

where $k = \lfloor \frac{n}{m} \rfloor$. Since $\{P_{F_k}(T_k(f)) - T_k(f)\}_n$ converges to $\{O_n\}_n$ in Type 2 strong cluster sense, we can show that $\{P_{U_n}(\tilde{A}_n) - \tilde{A}_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Since $q_\infty(\{A_n - \tilde{A}_n\}_n) = 0, \{P_{U_n}(\tilde{A}_n) - A_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense.

Definition 4.1. *A matrix sequence* $\{A_n\}_n$ *is said to be a norm bounded matrix sequence if there exists a real number* $M < \infty$ *such that for all* n, $\|A_n\| \leq M$.

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REMARK 4.2. Note that $\{A_n\}_n$ in the above example needs not be norm bounded. If $\{A_n\}_n$ is norm bounded, then $\{P_{U_n}(A_n) - A_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense (see Lemma 4.4).

Now consider a GLT sequence $\{B_n\}_n$ belongs to $\tilde{\mathcal{G}}^\infty$ with symbol κ, such that $\sum_{i=1}^{k_m} a_{i,m} \otimes f_{i,m}$ converges to κ in essential supremum norm (or in measure), where each $a_{i,m}$ is a Riemann integrable function on $[0,1]$ and $f_{i,m}$ is a continuous periodic function on $[-\pi,\pi]$. Then $\{\{\sum_{i=1}^{k_m} P_{U_n}(D_n(a_{i,m})T_n(f_{i,m}))\}_n\}_m$ converges to $\{B_n\}_n$ in $\tilde{\mathcal{G}}^\infty$ (or in $\tilde{G}$, the space of all GLT sequences). Then we can construct a sequence $\{\hat{A}_n\}_n$ as in the proof of Theorem 2.9 such that $\hat{A}_n = P_{U_n}(\sum_{i=1}^{k_m} D_n(a_{i,m})T_n(f_{i,m}))$ for some m (depending on n) and $q_\infty(\{\hat{A}_n\}_n - \{B_n\}_n) = 0$. Thus ,$\{\hat{A}_n\}_n$ is a good preconditioner for $\{B_n\}_n$.

Since the convergence of $\{P_{U_n}(T_n(f)) - T_n(f)\}_n$ to $\{O_n\}_n$ in Type 2 strong/weak cluster sense leads to efficient preconditioners, it is important to know when does this convergence hold. The Korovkin-type theorems obtained in [6, 20, 21, 22] reduce this task into a finite subset of the class of symbols. With the assumptions of convergence on a finite subset of symbols (test set), we get convergence in a class of operators generated by this test set. Here we obtain a similar result in the setting of GLT matrix sequences. First, we prove three lemmas.

LEMMA 4.3. *Suppose $\{A_n\}_n$ is a norm bounded matrix sequence with $\|A_n\| \le M < \infty$. Then $q_\infty(\{A_n\}_n) = 0$ if and only if $\|A_n\|_F^2 = o(n)$.*

Proof. If $\|A_n\|_F^2 = o(n)$, then by Lemma 2.4, $\{A_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. By Corollary 2.7, $q_\infty(\{A_n\}_n) = 0$.

Conversely assume that $q_\infty(\{A_n\}_n) = 0$. Then $\{A_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Using Theorem 2.5, we get $\lim_{n\to\infty} P(A_n) = 0$. Also, by Theorem 1.9, we have

$$P(A_n) = \min_{i=0,1,\ldots,n} \left\{ \frac{i}{n} + \sigma_{i+1}(A_n) \right\}.$$

Since $\lim_{n\to\infty} P(A_n) = 0$, for $\epsilon > 0$, there exists a positive integer n_ϵ such that for all $n \ge n_\epsilon$, $P(A_n) < \epsilon$. Hence, we have for $n \ge n_\epsilon$,

$$\min_{i=0,1,\ldots,n} \left\{ \frac{i}{n} + \sigma_{i+1}(A_n) \right\} < \epsilon.$$

Then there exists a j, such that $\frac{j}{n} + \sigma_{j+1}(A_n) < \epsilon$. Now

$$\frac{1}{n}\left(\sum_{i=1}^{j} {\sigma_i}^2\right) < M^2\epsilon, \qquad \sum_{i=j+1}^{n} {\sigma_i}^2 < \epsilon^2(n-j).$$

Then, for every $n \ge n_\epsilon$,

$$\|A_n\|_F^2 = \sum_{i=1}^{n} {\sigma_i}^2(A) < M^2\epsilon + (n-j)\epsilon^2, \quad \frac{\|A_n\|_F^2}{n} < \frac{M^2\epsilon}{n} + \epsilon^2.$$

Hence ,$\|A_n\|_F^2 = o(n)$. ∎

LEMMA 4.4. *Let $\{A_n\}_n$ be a matrix sequence such that for each $n, \|A_n\| \le M < \infty$. If $q_\infty(\{A_n\}_n) = 0$, then $q_\infty(\{P_{U_n}(A_n)\}_n) = 0$.*

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Proof. Suppose that $q_\infty(\{A_n\}_n) = 0$. Then by Lemma 4.3, $\|A_n\|_F^2 = o(n)$. Since $\|P_{U_n}\|_F = 1$, $\|P_{U_n}(A_n)\|_F^2 \leq \|A_n\|_F^2 = o(n)$. Thus, $q_\infty(\{P_{U_n}(A_n)\}_n) = 0$. ∎

Lemma 4.5. *For every $f \in L^\infty(D)$, there exists a norm bounded GLT matrix sequence $\{A_n\}_n$ such that $\{A_n\}_n \sim_{GLT} f$.*

Proof. Let $f \in L^\infty(D)$. From Theorem 3.6, there exists a GLT matrix sequence $\{B_n\}_n$ such that $\{B_n\}_n \sim_{GLT} f$ and by Lemma 3.3, $q_\infty(\{B_n\}_n) = \|f\|_\infty$.

By the definition of q_∞, for $\epsilon > 0$, there exist two matrix sequences $\{R_n\}_n$ and $\{N_n\}_n$ such that

$$\|N_n\| < \|f\|_\infty + \epsilon, \quad \text{rank}(R_n) = o(n), \quad B_n = R_n + N_n.$$

Consider $A_n = N_n$. Rank$(R_n) = o(n)$ implies $\{R_n\}_n \sim_{GLT} 0$. Hence, by Theorem 3.1, item 2, $\{A_n\}_n \sim_{GLT} f$. Also $\|A_n\| < \|f\|_\infty + \epsilon < \infty$. ∎

Theorem 4.6 (Uchiyama's inequality [27]). *Let Φ be a contractive positive map on a C^*-algebra $\mathbb{A}$ and let $\circ$ denote the Jordan product. For $f, g \in \mathbb{A}$, let*

$$X = \Phi(f^* \circ f) - \Phi(f^*) \circ \Phi(f) \geq 0,$$

$$Y = \Phi(g^* \circ g) - \Phi(g^*) \circ \Phi(g) \geq 0,$$

$$Z = \Phi(f^* \circ g) - \Phi(f^*) \circ \Phi(g).$$

Then $|\phi(Z)| \leq |\phi(X)|^{1/2}|\phi(Y)|^{1/2}$ for all states ϕ on $\mathbb{A}$.

Remark 4.7. Note that the above inequality holds for completely positive maps with norm less than or equal to 1 with respect to the usual C^* product. We will use Uchiyama's inequality for a particular state ϕ_x on $M_n(\mathbb{C})$, where ϕ_x is defined by $\phi_x(A_n) = \langle A_n x, x\rangle, x \in \mathbb{C}^n, \|x\| = 1$.

The next lemma is analogous to Lemma 2.6 of [20] and the proof is quite similar, so we omit the proof.

Lemma 4.8. *Let $\{A_n\}_n$ and $\{B_n\}_n$ be two positive matrix sequences (sequences of positive semi-definite matrices) such that $\{A_n + B_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Then $\{A_n\}_n$ and $\{B_n\}_n$ converge to $\{O_n\}_n$ in Type 2 weak cluster sense.*

Now we present the Korovkin-type theorem in the setting of GLT matrix sequences. Here we obtain preconditioners for the norm bounded GLT matrix sequences. For arbitrary GLT matrix sequences, see the Corollary 4.10.

Theorem 4.9. *Let $\{f_1, f_2, \ldots, f_k\} \subseteq L^\infty(D)$ and $\{A_n(f)\}_n \sim_{GLT} f$ is norm bounded for each $f \in \{f_1, f_2, \ldots, f_k, \sum_{i=1}^k f_i f_i^*\}$. Suppose that $\{P_{U_n}(A_n(g)) - A_n(g)\}$ converges to $\{O_n\}$ in Type 2 weak cluster sense for $g \in \{f_1, f_2, \ldots, f_k, \sum_{i=1}^k f_i f_i^*\}$. Then for every f in the C^*-algebra generated by $\{f_1, f_2, \ldots, f_k\}$, if $\{A_n(f)\}_n \sim_{GLT} f$ is norm bounded, then $\{P_{U_n}(A_n(f)) - A_n(f)\}$ converges to $\{O_n\}$ in Type 2 weak cluster sense.*

Proof. Let $f_i, f_j \in \{f_1, f_2, \ldots, f_k\} \subseteq L^\infty(D)$. Lemma 4.5 shows that for each $f \in L^\infty(D)$, there exists a GLT sequence $\{A_n(f)\}_n$ such that $\|A_n(f)\| \leq M < \infty, \forall n \in \mathbb{N}$.

Let $\{A_n(f_i)\}_n$ and $\{A_n(f_j)\}_n$ be sequences such that $\|A_n(f_i)\| \leq M_i < \infty$ and $\|A_n(f_j)\| \leq M_j < \infty$.

$P_{U_n}(\cdot) : M_n(\mathbb{C}) \to M_{U_n}$ is a completely positive map and $\|P_{U_n}(\cdot)\| = 1$. Let,

$$X_n = P_{U_n}(A_n(f_i^*)A_n(f_i)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i)) \geq 0,$$

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$$Y_n = P_{U_n}(A_n(f_j^*)A_n(f_j)) - P_{U_n}(A_n(f_j^*))P_{U_n}(A_n(f_j)) \geq 0,$$

$$Z_n = P_{U_n}(A_n(f_i^*)A_n(f_j)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_j)).$$

We know $\{P_{U_n}(A_n(\sum_{i=1}^k f_i^* f_i)) - A_n(\sum_{i=1}^k f_i^* f_i)\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. By Theorem 3.1, item 1,2 and 5, $\{A_n(f_i)^* - A_n(f_i^*)\}_n$ is zero distributed. Now,

$$q_\infty\left(A_n\left(\sum_{i=1}^k f_i^* f_i\right) - \sum_{i=1}^k A_n(f_i^*)A_n(f_i)\right) = 0. \tag{4.8}$$

Also $\left\|A_n\left(\sum_{i=1}^k f_i^* f_i\right) - \sum_{i=1}^k A_n(f_i^*)A_n(f_j)\right\| \leq K < \infty$. Therefore By Lemma 4.4,

$$q_\infty\left(P_{U_n}\left[A_n\left(\sum_{i=1}^k f_i^* f_i\right) - \sum_{i=1}^k A_n(f_i^*)A_n(f_i)\right]\right) = 0. \tag{4.9}$$

Now,

$$\begin{aligned}\sum_{i=1}^k & P_{U_n}(A_n(f_i^*)A_n(f_i)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i)) \\ &= \sum_{i=1}^k \{P_{U_n}(A_n(f_i^*)A_n(f_i)) - A_n(f_i^*)A_n(f_i)\} + \sum_{i=1}^k \{A_n(f_i^*)A_n(f_i) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i))\}.\end{aligned} \tag{4.10}$$

Consider the first term,

$$\begin{aligned}\sum_{i=1}^k P_{U_n}(A_n(f_i^*)A_n(f_i)) - P_{U_n}\left(A_n\left(\sum_{i=1}^k f_i^* f_i\right)\right) + P_{U_n}\left(A_n\left(\sum_{i=1}^k f_i^* f_i\right)\right) - A_n\left(\sum_{i=1}^k f_i^* f_i\right) \\ + A_n\left(\sum_{i=1}^k f_i^* f_i\right) - \sum_{i=1}^k A_n(f_i^*)A_n(f_i).\end{aligned}$$

By (4.8) and (4.9), we get that first sum in the right-hand side of (4.10) converges to $\{O_n\}_n$ in Type 2 weak cluster sense.

Since $\{P_{U_n}(A_n(f_i^*)) - A_n(f_i^*)\}_n$ converges to $\{O_n\}_n$ and $\{P_{U_n}(A_n(f_i)) - A_n(f_i)\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense, then it is easy to show that $\{P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i)) - A_n(f_i^*)A_n(f_i)\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense (see Lemma 2.5 of [20]). Hence, second sum in the right-hand side of (4.10) converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Hence, $\left\{\sum_{i=1}^k P_{U_n}(A_n(f_i^*)A_n(f_i)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i))\right\}$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. By Lemma 4.8, for each $i = 1, 2, \ldots, k$, $\{P_{U_n}(A_n(f_i^*)A_n(f_i)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_i))\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense.

Thus, $\{X_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Similarly, $\{Y_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Now consider the state ϕ_x on $M_n(\mathbb{C})$ defined by $\phi_x(A_n) = \langle A_n x, x\rangle$ where $x \in \mathbb{C}^n$ and $\|x\| = 1$. By Uchiyama's inequality

$$|\phi_x(Z_n)| = |\langle Z_n x, x\rangle| \leq \langle X_n x, x\rangle^{1/2}\langle Y_n x, x\rangle^{1/2}.$$

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Then as in the proof of Theorem 3.4 of [20], we obtain $\{Z_n\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Now

$$P_{U_n}(A_n(f_i^*)A_n(f_j)) - A_n(f_i^*)A_n(f_j) = (P_{U_n}(A_n(f_i^*)A_n(f_j)) - P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_j))) \\ + (P_{U_n}(A_n(f_i^*))P_{U_n}(A_n(f_j)) - A_n(f_i^*)A_n(f_j)).$$

The first term is Z_n and hence converges to $\{O_n\}_n$ in Type 2 weak cluster sense. The second term converges to $\{O_n\}_n$ in Type 2 weak cluster sense (see Lemma 2.5 of [20]). Hence, $\{P_{U_n}(A_n(f_i^*)A_n(f_j)) - A_n(f_i^*)A_n(f_j)\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. Since $q_\infty(\{A_n(f_i^*)A_n(f_j) - A_n(f_i^* f_j)\}_n) = 0$, $\{P_{U_n}(A_n(f_i^*)A_n(f_j) - A_n(f_i^* f_j))\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. *-algebra generated by the test set contains all linear combinations of products of elements in the test set and their adjoints. Hence by the above calculation, we obtain convergence on the whole *-algebra generated by the test set.

From the *-algebra to reach C^*-algebra (i.e., the closure in the C^*-algebra norm), we proceed as follows.

Let g belong to the C^*-algebra generated by $\{f_1, f_2, \ldots, f_k\}$ and $\{g_m\}$ be a sequence which converges to g, where each g_m belongs to *-algebra generated by $\{f_1, f_2, \ldots, f_k\}$. Then by Lemma 4.5, there exist norm bounded GLT matrix sequences $\{A_n(g_m)\}_n$ and $\{A_n(g)\}_n$ corresponding to each g_m and g, respectively, such that $\{\{A_n(g_m)\}_n\}_m$ converges to $\{A_n(g)\}_n$ in $\tilde{\mathcal{G}}^\infty$.

Therefore, for $\epsilon > 0$, there exists a positive integer t such that $q_\infty(\{A_n(g_t) - A_n(g)\}_n) < \epsilon/2$. Then there exist two norm bounded sequences $\{R_n\}_{n,t}$ and $\{N_n\}_{n,t}$ such that

$$A_n(g_t) - A_n(g) = R_{n,t} + N_{n,t}, \quad \lim_{n\to\infty} \frac{rank(R_{n,t})}{n} = 0, \quad \|N_{n,t}\| < \epsilon/2.$$

Now consider

$$\begin{aligned} q_\infty(\{P_{U_n}(A_n(g)) - A_n(g)\}_n) &= q_\infty(\{P_{U_n}(A_n(g)) - P_{U_n}(A_n(g_t)) \\ &\quad + P_{U_n}(A_n(g_t)) - A_n(g_t) + A_n(g_t) - A_n(g)\}_n) \\ &\le q_\infty(\{P_{U_n}(A_n(g) - A_n(g_t))\}_n) \\ &\quad + q_\infty(\{P_{U_n}(A_n(g_t)) - A_n(g_t)\}_n) \\ &\quad + q_\infty(\{A_n(g_t) - A_n(g)\})_n). \end{aligned}$$

The second term on the right-hand side is zero and $q_\infty(\{A_n(g_t) - A_n(g)\})_n) < \epsilon/2$. Now consider the first term on the right-hand side:

$$\begin{aligned} q_\infty(\{P_{U_n}(A_n(g) - A_n(g_t))\}_n) &= q_\infty(\{P_{U_n}(R_{n,t} + N_{n,t})\}_n) \\ &\le q_\infty(\{P_{U_n}(R_{n,t})\}_n) + q_\infty(\{P_{U_n}(N_{n,t})\}). \end{aligned}$$

Since $q_\infty(\{R_{n,t}\}_n) = 0$, by Lemma 4.4, $q_\infty(\{P_{U_n}(R_{n,t})\}_n) = 0$. Also we know that $\|P_{U_n}\| = 1$, then $\|P_{U_n}(N_{n,t})\| \le \|N_{n,t}\|$ and hence $q_\infty(\{P_{U_n}(N_{n,t})\}_n) < \epsilon/2$. Thus, $q_\infty(\{P_{U_n}(A_n(g)) - A_n(g)\}_n) < \epsilon$. Hence $\{P_{U_n}(A_n(g)) - A_n(g)\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense. ∎

COROLLARY 4.10. *Under the conditions of Theorem 4.9, if* $\sum_{i=1}^{k_m} g_{i,m}$ *converges to* g *in measure, where each* $g_{i,m}$ *belongs to the* C^**- algebra generated by* $\{f_1, f_2, \ldots, f_k\}$*, then we can extract a preconditioner sequence* $\{P_{U_n}(A_n(h_i))\}_n$ *for* $\{A_n(g)\}_n$*.*

Proof. $\sum_{i=1}^{k_m} g_{i,m}$ belongs to $C^*\{f_1, f_2, \ldots, f_m\}$, the C^*-algebra generated by $\{f_1, f_2, \ldots, f_m\}$. Hence, $\{P_{U_n}(A_n(\sum_{i=1}^{k_m} g_{i,m})) - A_n(\sum_{i=1}^{k_m} g_{i,m})\}_n$ converges to $\{O_n\}_n$ in Type 2 weak cluster sense if $A_n(\sum_{i=1}^{k_m} g_{i,m})$

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is of bounded norm and $A_n(\sum_{i=1}^{k_m} g_{i,m}) \sim_{GLT} \sum_{i=1}^{k_m} g_{i,m}$. Since $\sum_{i=1}^{k_m} g_{i,m}$ converges to g in measure, $A_n(\sum_{i=1}^{k_m} g_{i,m})$ and $\{P_{U_n}(A_n(\sum_{i=1}^{k_m} g_{i,m}))\}_n$ are a.c.s. for $A_n(g)$. We can extract a sequence $\{P_{U_n}(A_n(h_j))\}_n$ from $\{\{P_{U_n}(A_n(\sum_{i=1}^{k_m} g_{i,m}))\}_n\}_m$ such that $\{P_{U_n}(A_n(\sum_{i=1}^{k_m} g_{i,m}))\}_n \xrightarrow{a.c.s.} \{P_{U_n}(A_n(h_j))\}_n$, where $h_j = \sum_{i=1}^{k_m} g_{i,m}$ for some m (the construction is in the proof of Theorem 2.1 of [1]). Hence the result follows. □

Remark 4.11. Theorem 4.9 does not hold in general if $\{A_n(f)\}$ is not norm bounded. Let $f \in L^\infty(D)$ and consider a GLT sequence $\{A_n(f)\}_n$ such that $\|A_n(f)\| \leq M < \infty$, for all n. Let $q_\infty(\{P_{U_n}(A_n(f) - A_n(f))\})_n = 0$ and $\{B_n\}_n$ be another sequence such that $q_\infty(\{B_n - A_n(f)\}_n) = 0$. If $\|B_n\|$ is unbounded, then $q_\infty(\{P_{U_n}(B_n) - B_n\}_n)$ needs not be zero. Indeed, consider $B_n = A_n(f) + Z_n$, where $U_n^* Z_n U_n = (a_{ij})_{i,j=1}^n$ and $a_{ij} = 1$ for all $1 \leq i, j \leq n$. Clearly, $q_\infty(\{P_{U_n}(B_n) - B_n\}_n) \neq 0$. But $q_\infty(\{P_{U_n}(A_n(f)) - B_n\}) = 0$. So we can treat $P_{U_n}(A_n(f))$ as a preconditioner for $\{B_n\}_n$. Also note that the function g in Corollary 4.10 needs not be essentially bounded.

Notice that in Remark 4.2, we have already proved that for any $g = a \otimes f$ with Riemann Integrable $a(x)$ and continuous periodic $f(\theta)$, since they are bounded. As a consequence, $\{P_{U_n}(A_n(g)) - A_n(g)\}_n$ is zero distributed also for any linear combination $g(x, \theta)$ of functions in

$$\mathcal{F} = \{a(x) \otimes f(\theta)\} : a(x) R.I., f(\theta) \text{continuous, periodic}\},$$

but since the product of two elements of $\mathcal{F}$ is still an element of $\mathcal{F}$, then $\{P_{U_n}(A_n(g)) - A_n(g)\}_n$ is zero distributed for any g in the *-algebra generated by $\mathcal{F}$. By repeating the second part of the proof of Theorem 4.9,$\{P_{U_n}(A_n(g)) - A_n(g)\}_n$ is zero distributed for all g in the C^*-algebra generated by $\mathcal{F}$.

5. Concluding remarks. As we know, the theory of Toeplitz matrix sequences has a rich operator theoretic analogue on the Hardy space via the symbol function. There are variations of it into Bergman space, Fock space, etc. We expect such versions in the case of GLT matrix sequences also. The development must be through the identification of corresponding symbols. The major achievement of this article is that we are able to identify the connection between the space of symbols and the subspaces of GLT matrix sequences. We hope that these identifications will be helpful in establishing the operator theoretic analogue of the spectral distributional results of such matrix sequences. The Korovkin-type result we obtained in this article makes use of a topology on the space of GLT matrix sequences. The connection with the topologies on $B(\mathbb{H})$, the space of all bounded linear operators on Hilbert space $\mathbb{H}$, and the topologies introduced in this article would be another interesting point. Obtaining the convergence in eigenvalue clustering as convergence with respect to some topology on $B(\mathbb{H})$ is the main target of our future research.

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REFERENCES

[1] G. Barbarino. Equivalence between GLT Sequences and measurable functions. *Linear Algebra Appl.*, 529:397–412, 2017.
[2] G. Barbarino and C. Garoni. From convergence in measure to convergence of matrix-sequences through concave functions and singular values. *Electron. J. Linear Algebra*, 32:500–513, 2017.
[3] G. Barbarino, C. Garoni and S. Serra-Capizzano. Block generalized locally Toeplitz sequences: Theory and applications in the unidimensional case. *Electron. Trans. Numer. Anal.*, 53:28–112, 2020.

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Volume 38, pp. 295-316, April 2022.

[4] G. Barbarino. Normal form for GLT sequences, arXiv preprint arXiv:1805.08708, 2018.

[5] G. Barbarino, C. Garoni and S. Serra-Capizzano, Block generalized locally Toeplitz sequences: Theory and applications in the multidimensional case. *Electron. Trans. Numer. Anal.*, 53:113–216, 2020.

[6] W. Bauer, V.B. Kiran Kumar, and R. Rahul. *Korovkin-type Theorems on B(H) and their Applications to Function Spaces*. Monatsh. Math. Springer, 2021.

[7] R. Bhatia. *Matrix Analysis*, vol. 169. Springer Science & Business Media, 2013.

[8] A. Böttcher, C. Garoni, and S. Serra-Capizzano. Exploration of Toeplitz-like matrices with unbounded symbols is not a purely academic journey. *Sb. Math.*, 208(11):1602, 2017.

[9] S. Serra-Capizzano. Distribution results on the algebra generated by Toeplitz sequences: A finite-dimensional approach. *Linear Algebra Appl.*, 328(1–3):121–130, 2001.

[10] S. Serra-Capizzano. Generalized locally Toeplitz sequences: Spectral analysis and applications to discretized partial differential equations. *Linear Algebra Appl.*, 366:371–402, 2003.

[11] S. Serra-Capizzano. Korovkin tests, approximation, and ergodic theory. *Math. Comp.*, 69(232):1533–1558, 2000.

[12] S. Serra-Capizzano. Locally X matrices, spectral distributions, preconditioning, and applications. *SIAM J. Matrix Anal. Appl.*, 21(4):1354–1388, 2000.

[13] R.H. Chan and M.C. Yeung. Circulant preconditioners for Toeplitz matrices with positive continuous generating functions. *Math. Comp.*, 58(197):233–240, 1992.

[14] F. Di Benedetto and S. Serra Capizzano. A unifying approach to abstract matrix algebra preconditioning. *Numer. Math*, 82(1):57–90, 1999.

[15] C. Garoni. Topological foundations of an asymptotic approximation theory for sequences of matrices with increasing size. *Linear Algebra Appl.*, 513:324–341, 2017.

[16] C. Garoni and S. Serra-Capizzano. *Generalized Locally Toeplitz Sequences: Theory and Applications*, vol. 1. Springer, 2017.

[17] C. Garoni and S. Serra-Capizzano. *Generalized Locally Toeplitz Sequences: Theory and Applications*, vol. 2. Springer, 2018.

[18] U. Grenander and G. Szegö. *Toeplitz Forms and Their Applications,* 2nd edn. Chelsea, New York. 1984.

[19] X.-Q. Jin. Hartley preconditioners for Toeplitz systems generated by positive continuous functions. *BIT*, *34*(3):367–371, 1994.

[20] V.B. Kiran Kumar, M.N.N. Namboodiri, and R. Rajan. A Korovkin-type theory for non self-adjoint Toeplitz operators. *Linear Algebra Appl.*, 543:140–161, 2018.

[21] V.B. Kiran Kumar, M.N.N. Namboodiri, and R. Rajan. A short survey on preconditioners and Korovkin-type theorems. *J. Anal.*, 29(2):425–447, 2021.

[22] V.B. Kiran Kumar, M.N.N. Namboodiri, and S. Serra-Capizzano. Preconditioners and Korovkin-type theorems for infinite-dimensional bounded linear operators via completely positive maps. *Studia Math.*, 218(2):95–118, 2013.

[23] S. Serra-Capizzano. A Korovkin-type theory for finite Toeplitz operators via matrix algebras. *Numer. Math.*, 82(1):117–142, 1999.

[24] S. Serra-Capizzano. Superlinear PCG methods for symmetric Toeplitz systems. *Math. Comp.*, 68(226):793–803, 1999.

[25] S. Serra-Capizzano. The GLT class as a generalized Fourier analysis and applications. *Linear Algebra Appl.*, 419(1):180–233, 2006.

[26] E.E. Tyrtyshnikov. A unifying approach to some old and new theorems on distribution and clustering. *Linear Algebra Appl.*, 232:1–43, 1996.

[27] M. Uchiyama. Korovkin-type theorems for Schwarz maps and operator monotone functions in C*-algebras. *Math. Z.*, 230(4):785–797, 1999.