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GROUP INVERSES FOR MATRICES OVER A BEZOUT DOMAIN*

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Abstract. Suppose $\mathbf{R}$ is a Bezout domain. In this paper, some necessary and sufficient conditions for the existence of the group inverse for square matrix over $\mathbf{R}$ are given, the conditions for the existence of the group inverse of products of matrices are studied, and the equivalent conditions for reverse order law of group inverse of product of matrices are obtained. Also the existence and the representation of the group inverse for a class 2×2 block matrices over $\mathbf{R}$ are studied, and some well known relative results are generalized.

Key words. Bezout domain, Group inverse, Right $\mathbf{R}$–module, Block matrix.

AMS subject classifications. 15A09.

1. Introduction. Research on generalized inverses of matrices over commutative rings, especially the condition for the regularity of a matrix, is abundant; see e.g., [1] and references therein. However, research on the generalized inverses of matrices over non–commutative rings is relatively sparse; see [2], [4], [14], [15]. The purpose of this paper is to study the group inverses of matrices over an important associative ring – a Bezout domain, and generalize some results which are well known and given at present.

If every finitely generated left (right) ideal in a non-zero ring $\mathbf{R}$ which has a unit element 1 and no zero divisors is principal, then $\mathbf{R}$ is called a Bezout domain. Integral rings, non-commutative principal ideal domains, division rings, polynomial rings in an indeterminate over field, valuation rings and so on are Bezout domains; see [9], [11].

Let $\mathbf{R}$ be a Bezout domain, we denote the right $\mathbf{R}$–module of n-dimensional column vector and left $\mathbf{R}$–module of n-dimensional row vector by $\mathbf{R}_r^n$ and $\mathbf{R}_l^n$, respectively. Let $\mathbf{R}^{m\times n}$ be the set of all $m \times n$ matrices over $\mathbf{R}$. For a matrix $A \in \mathbf{R}^{n\times n}$, we denote the right $\mathbf{R}$–module which is generated by the columns, left $\mathbf{R}$–module which is generated by the rows, and right nullspace of A by $R_r(A)$, $R_l(A)$, and $N_r(A)$, respectively. The dimension of $R_r(A)$ and $R_l(A)$ is called the column rank and row

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rank of A, respectively. It is well known that the row rank of A is equal to its column rank, then their common value is called the rank of A, which is denoted by $rankA$; see [9], [11]. A matrix is said to be nonsingular if it is neither 0 nor a left or right zero-divisor. We say that $A^{\#}$ is the group inverse of A if $A^{\#}$ is a common solution of the matrix equations: $AXA = A$, $XAX = X$, $AX = XA$. It is well known that if $A^{\#}$ exists then $A^{\#}$ is unique.

We have two important results of matrices over $\mathbf{R}$, presented in the following lemmas; see [11].

Lemma 1.1. *A ring* $\mathbf{R}$ *is a Bezout domain if and only if every non-zero matrix* A *over* $\mathbf{R}$ *has a factorization* $A = P\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}Q$*, where* P*,* Q *are invertible matrices over* $\mathbf{R}$*,* $\Delta \in \mathbf{R}^{k\times k}$ *and* $rank(A) = k$*.*

Lemma 1.2. *Let* $\mathbf{R}$ *be a Bezout domain,* $A \in \mathbf{R}^{n\times n}$*. Then* A *is an idempotent matrix if and only if there exists invertible matrix* $p \in \mathbf{R}^{n\times n}$ *such that* $A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$*.*

This paper is divided into four sections. In the second section, we study the existence of the group inverse for matrix A, obtaining several equivalent conditions, as well as the existence and representation of the group inverse for 2×2 block upper triangular matrix. This generalizes the relative results in papers [4] and [13]. In the third section, we study the group inverses of products of two matrices, obtain the condition for the reverse order law of group inverses, and generalize the relative results in paper [5]. In the fourth section, we study the existence and representation of the group inverse for a class 2×2 block matrices, generalizing the relative results in paper [2]. All results we obtain are new even for commutative principal ideal domains.

2. The Existence of the Group Inverse.

Theorem 2.1. *Suppose* $A \in \mathbf{R}^{n\times n}$*. Then the following conditions are equivalent:*

(i) $A^{\#}$ *exists .*
(ii) $R_r(A) = R_r(A^2)$*.*
(iii) $R_l(A) = R_l(A^2)$*.*
(iv) There exist invertible matrices $D \in \mathbf{R}^{r\times r}$ *and* $N \in \mathbf{R}^{n\times n}$ *such that* $A = N\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}N^{-1}$*.*

Proof. $(i) \Rightarrow (ii)$ It follows from the definition of the group inverse and $A^{\#}$ exists that $A = A^2A^{\#}$. Hence $R_r(A) \subseteq R_r(A^2)$. On the other hand $R_r(A) \supseteq R_r(A^2)$ is obvious. Then $R_r(A) = R_r(A^2)$.

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$(i) \Rightarrow (iii)$ The proof is similar to $(i) \Rightarrow (ii)$.

$(iv) \Rightarrow (i)$ Let $X = N\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}N^{-1}$. It follows from the definition of the group inverse that $X = A^{\#}$.

$(ii) \Rightarrow (iv)$ By Lemma 1.1, we can find invertible matrices $Q,\ P \in \mathbf{R}^{n\times n}$ such that $A = P\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}Q$, where Δ is a $r \times r$ matrix with rank$\Delta = r$. Since $R_r(A) = R_r(A^2)$, we may write $A = A^2X$ for some matrix X. Let $QP = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, $X = P\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}P^{-1}$, where $P_1,\ X_1 \in \mathbf{R}^{r\times r}$. Then

$$A = P\begin{bmatrix} \Delta P_1 & \Delta P_2 \\ 0 & 0 \end{bmatrix}P^{-1}, \text{ and}$$

$$A^2X = P\begin{bmatrix} \Delta P_1\Delta P_1X_1 + \Delta P_1\Delta P_2X_3 & \Delta P_1\Delta P_1X_2 + \Delta P_1\Delta P_2X_4 \\ 0 & 0 \end{bmatrix}P^{-1}.$$

Hence

$$\Delta P_1\Delta P_1X_1 + \Delta P_1\Delta P_2X_3 = \Delta P_1,\ \Delta P_1\Delta P_1X_2 + \Delta P_1\Delta P_2X_4 = \Delta P_2,$$

$$P_1\Delta P_1X_1 + P_1\Delta P_2X_3 = P_1, \text{ and } P_1\Delta P_1X_2 + P_1\Delta P_2X_4 = P_2.$$

It follows that there exists matrix Z such that $\begin{bmatrix} P_1 & P_2 \end{bmatrix}Z = I$ from invertibility of the matrices QP, then

$$\begin{bmatrix} P_1\Delta P_1X_1 + P_1\Delta P_2X_3 & P_1\Delta P_1X_2 + P_1\Delta P_2X_4 \end{bmatrix}Z = I, \text{ and}$$

$$P_1\Delta\begin{bmatrix} P_1X_1 + P_2X_3 & P_1X_2 + P_2X_4 \end{bmatrix}Z = I.$$

Hence P_1 and Δ are invertible matrices. Namely ΔP_1 is an invertible matrix. Let $N = P\begin{bmatrix} I & -P_1^{-1}P_2 \\ 0 & I \end{bmatrix}$ and $D = \Delta P_1$, then $A = N\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}N^{-1}$.

$(iii) \Rightarrow (iv)$ The proof is similar to $(ii) \Rightarrow (iv)$. □

Remark 2.2. If $\mathbf{R}$ is a skew field, then $R_r(A) = R_r(A^2)$ if and only if $rank(A) = rank(A^2)$. However, if $\mathbf{R}$ is a general Bezout domain, then $R_r(A) = R_r(A^2)$ and $rank(A) = rank(A^2)$ are not equivalent, e.g. if $\mathbf{R}$ is a ring of integral number, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. It is obvious that $rank(A) = rank(A^2)$, but $R_r(A) \neq R_r(A^2)$.

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THEOREM 2.3. *Suppose* $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathbf{R}^{n\times n}$, $A \in \mathbf{R}^{m\times m}$. *Then:*

(1) $M^\#$ *exists if and only if* $A^\#$, *and* $C^\#$ *exist and* $(I - AA^\#)B(I - CC^\#) = 0$.

(2) If $M^\#$ *exists, then* $M^\# = \begin{bmatrix} A^\# & X \\ 0 & C^\# \end{bmatrix}$, *where* $X = (A^\#)^2B(I - CC^\#) + (I - AA^\#)B(C^\#)^2 - A^\#BC^\#$.

Proof. (1) ('if') Let $Y = \begin{bmatrix} A^\# & X \\ 0 & C^\# \end{bmatrix}$, where $X = (A^\#)^2B(I - CC^\#) + (I - AA^\#)B(C^\#)^2 - A^\#BC^\#$. It follows from the existence of $A^\#$ and $C^\#$, and $(I - AA^\#)B(I - CC^\#) = 0$ that $MYM = M$, $YMY = Y$, and $MY = YM$. Therefore $M^\#$ exists.

('only if') It follows from the existence of $M^\#$ and Theorem 2.1(ii) that $R_r(M) = R_r(M^2)$. Thus, for any $x_1 \in \mathbf{R}_r^m$, $x_2 \in \mathbf{R}_r^{n-m}$, there exists $y_1 \in \mathbf{R}_r^m$, $y_2 \in \mathbf{R}_r^{n-m}$, such that:

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A^2 & AB + BC \\ 0 & C^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence $Cx_2 = C^2y_2$. This means $R_r(C) \subseteq R_r(C^2)$. $R_r(C) \supseteq R_r(C^2)$ is obvious. Hence $R_r(C) = R_r(C^2)$. Again applying Theorem 2.1(ii), we know $C^\#$ exists. By Theorem 2.1(iii), for any $u_1 \in \mathbf{R}_l^m$, $u_2 \in \mathbf{R}_l^{n-m}$ there exists $z_1 \in \mathbf{R}_l^m$, $z_2 \in \mathbf{R}_l^{n-m}$ such that

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} A^2 & AB + BC \\ 0 & C^2 \end{bmatrix}.$$

Hence $u_1A = z_1A^2$. This means $R_l(A) = R_l(A^2)$, that is $A^\#$ exists. It follows from Theorem 2.1(iv) that $A = P\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$ and $C = Q^{-1}\begin{bmatrix} D_2 & 0 \\ 0 & 0 \end{bmatrix}Q$, where $D_1 \in \mathbf{R}^{s\times s}$ and $D_2 \in \mathbf{R}^{t\times t}$ are invertible matrices. Let

$$P^{-1}BQ^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \; B_4 \in \mathbf{R}^{(m-s)\times(n-m-t)}.$$

It is easy to see

$$(I - AA^\#)B(I - CC^\#) = P\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}P^{-1}BQ^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}Q = P\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}Q.$$

In order to prove the conclusion, we only need to prove $B_4 = 0$. In fact,

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} D_1 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix}.$$

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¿From it we know $M \sim \begin{bmatrix} D_1 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_4 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since $M^{\#}$ exists, we have $N^{\#} = \begin{bmatrix} 0 & B_4 \\ 0 & 0 \end{bmatrix}^{\#}$. Obviously, $N^2 = 0$. Then $N = N^2 N^{\#} = 0$, and $B_4 = 0$.

(2) It is obvious from the proof of (1). □

Corollary 2.4. *Suppose* $B_1 \in \mathbf{R}^{m\times m}$ *and* $M = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}$. *Then*

(1) $M^{\#}$ *exists if and only if* $B_1^{\#}$ *exists and* $R_r(B_2) \subset R_r(B_1)$.
(2) If $M^{\#}$ *exists, then* $M^{\#} = \begin{bmatrix} B_1^{\#} & (B_1^{\#})^2 B_2 \\ 0 & 0 \end{bmatrix}$.

Proof. By Theorem 2.3(2), let $C = 0$. The result follows easily. □

Corollary 2.5. *Suppose* $B_1 \in \mathbf{R}^{m\times m}$ *and* $M = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}$. *Then*

(1) $M^{\#}$ *exists if and only if* $B_1^{\#}$ *exists and* $R_l(B_2) \subset R_l(B_1)$.
(2) If $M^{\#}$ *exists, then* $M^{\#} = \begin{bmatrix} B_1^{\#} & 0 \\ B_2(B_1^{\#})^2 & 0 \end{bmatrix}$.

Proof. We can get the formula for the group inverse of $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ by the same proof of Theorem 2.3. Let $C = 0$. The result follows easily. □

3. Group Inverses of Products of Two Matrices.

Theorem 3.1. *Suppose* $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times m}$. *Then the following conditions are equivalent:*

(1) $(AB)^{\#}$ *and* $(BA)^{\#}$ *exist.*
(2) $R_r(AB) = R_r(ABA)$ *and* $R_r(BA) = R_r(BAB)$.
(3) $R_l(AB) = R_l(ABA)$ *and* $R_l(BA) = R_l(BAB)$.

Proof. (1) ⇒ (2) It follows from the existence of $(AB)^{\#}$ and Theorem2.1 that $R_r(AB) = R_r(ABAB)$. $R_r(ABAB) \subseteq R_r(ABA) \subseteq R_r(AB)$ is obvious. Thus $R_r(AB) = R_r(ABA)$. Similarly $R_r(BA) = R_r(BAB)$.

(2) ⇒ (1) From $R_r(AB) = R_r(ABA) = AR_r(BA)$ and $R_r(BA) = R_r(BAB)$, we have $R_r(AB) = R_r(ABA) = R_r(ABAB)$. Thus $(AB)^{\#}$exists. Similarly $(BA)^{\#}$ exists.

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(1) $\Leftrightarrow$ (3) The proof is similar to (1) $\Leftrightarrow$ (2). ☐

Corollary 3.2. *Suppose* $A \in \mathbf{R}^{m\times n}$*,* $B \in \mathbf{R}^{n\times m}$*, and* $R_r(A) = R_r(ABA)$*, then* $(AB)^\#$ *and* $(BA)^\#$ *exist.*

Proof. Since $R_r(ABA) \subseteq R_r(AB) \subseteq R_r(A)$, and $R_r(A) = R_r(ABA)$, we have $R_r(ABA) = R_r(AB) = R_r(A)$. Thus $R_r(AB) = R_r(ABA)$. By $R_r(ABA) = R_r(A)$, we have $R_r(BABA) = R_r(BA)$. Thus $R_r(BA) = R_r(BAB)$. From Theorem 3.1, we can prove immediately. ☐

Corollary 3.3. *Suppose* $A \in \mathbf{R}^{m\times n}$*,* $B \in \mathbf{R}^{n\times m}$*, and* $R_r(AB) = R_r(A)$*,* $R_r(BA) = R_r(B)$*, then* $(AB)^\#$ *and* $(BA)^\#$*exist.*

Proof. It follows from $R_r(AB) = R_r(A)$, and $R_r(BA) = R_r(B)$ that $R_r(AB) = AR_r(B) = AR_r(BA) = R_r(ABA)$. Similarly, $R_r(BA) = R_r(BAB)$. ¿From Theorem 3.1, we can prove immediately. ☐

Corollary 3.4. *Suppose* $A \in \mathbf{R}^{m\times n}$*,* $B \in \mathbf{R}^{n\times m}$*,* $R_r(AB) = R_r(A)$ *and* $(AB)^\#$ *exist, then* $(BA)^\#$ *exist.*

Proof. It follows from the existence of $(AB)^\#$ that$R_r(AB) = R_r(ABAB)$. Hence $R_r(AB) = R_r(ABA)$. From $R_r(AB) = R_r(A)$ it follows that $BR_r(AB) = BR_r(A)$. Therefore $R_r(BA) = R_r(BAB)$. Then we can obtain the existence of $(BA)^\#$ from Theorem 3.1 . ☐

Lemma 3.5. *Suppose* $A \in \mathbf{R}^{m\times n}$*,* $B \in \mathbf{R}^{n\times m}$*, then there exist invertible matrices* $P \in \mathbf{R}^{m\times m}$ *and* $Q \in \mathbf{R}^{n\times n}$*, such that*

$$A = P\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}Q, \text{ and } B = Q^{-1}\begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0_{r-s} & 0 \\ C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix}P^{-1},$$

where $A_1 \in \mathbf{R}^{r\times r}$*,* $rankA = rankA_1 = r$*,* $B_1 \in \mathbf{R}^{s\times s}$*,* $C_1 \in \mathbf{R}^{t\times s}$*.*

Proof. By Lemma 1.1, we can find invertible matrices $M \in \mathbf{R}^{m\times m}$, $N \in \mathbf{R}^{n\times n}$ such that $A = M\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}N$, where $\Delta \in \mathbf{R}^{r\times r}$ and $rankA = r$. We write $B = N^{-1}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}M^{-1}$, where $\Delta_1 \in \mathbf{R}^{r\times m}$, $\Delta_1 = Q_1^{-1}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}P_1^{-1}$, and E is a $s\times s$ non-singular matrix over $\mathbf{R}$. $\Delta_2 = Q_2^{-1}\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}P_2^{-1}$, where F is a $t\times t$ non-singular matrix over $\mathbf{R}$. Let

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}P_1^{-1} = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0_{r-s} & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}P_2^{-1} = \begin{bmatrix} C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix}$$

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where $B_1 \in \mathbf{R}^{s\times s}$, and $C_1 \in \mathbf{R}^{t\times s}$. Then

$$B = N^{-1}\begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix}\begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0_{r-s} & 0 \\ C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix} M^{-1}, \text{ and}$$

$$A = M\begin{bmatrix} \Delta Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} N.$$

Let $M = P$, $\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} N = Q$, and $\Delta Q_1^{-1} = A_1$, then $A_1 \in \mathbf{R}^{r\times r}$, $rankA = rankA_1 = r$, and

$$A = P\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } B = Q^{-1}\begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0_{r-s} & 0 \\ C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}. \square$$

THEOREM 3.6. *Suppose $A, B \in \mathbf{R}^{n\times n}$. Then from any two of the following conditions, we can obtain the other one.*

(1) $(AB)^{\#}$ exists.
(2) $(BA)^{\#}$ exists.
(3) $AB \sim BA$.

Proof. $(1),(2) \Rightarrow (3)$ Let A and B be the form as in Lemma 3.5, then

$$AB = P\begin{bmatrix} A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & A_1\begin{bmatrix} B_3 \\ 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} P^{-1}, \text{ and}$$

$$BA = Q^{-1}\begin{bmatrix} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1 & 0 \\ \begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix} A_1 & 0 \end{bmatrix} Q.$$

¿From Corollary 2.4 and Corollary 2.5, we obtain the following:

$$(AB)^{\#} \text{ exists} \Leftrightarrow (A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix})^{\#} \text{ exists, and}$$
$$R_r(A_1\begin{bmatrix} B_3 \\ 0 \end{bmatrix}) \subset R_r(A_1\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}),$$

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$$(BA)^{\#} \text{ exists} \Leftrightarrow (\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1)^{\#}\text{exists, and}$$
$$R_l(\begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix} A_1) \subset R_l(\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1).$$

Hence there exists $E \in \mathbf{R}^{r\times(n-r)}$, $F \in \mathbf{R}^{(n-r)\times r}$ such that

$$A_1 \begin{bmatrix} B_3 \\ 0 \end{bmatrix} = A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} E, \text{ and} \begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix} A_1 = F \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1.$$

Then

$$AB = P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} P^{-1}, \text{ and}$$

$$BA = Q^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} Q.$$

Let $\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} S$, for invertible matrices R, S, where $D \in \mathbf{R}^{r_1\times r_1}$ and $rankD = r_1$. Then

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \sim SA_1R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \text{ and} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1 \sim \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} SA_1R.$$

Let $C = SA_1R$, $C = \begin{bmatrix} C_4 & C_5 \\ C_6 & C_7 \end{bmatrix}$. Then $rankC = r$,

$$C \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} C = \begin{bmatrix} DC_4 & DC_5 \\ 0 & 0 \end{bmatrix}.$$

It follows from $\left(A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\right)^{\#}$ exists that $\left(C \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{\#}$ exist, then $(C_4D)^{\#}$ exists, and $R_l(C_6D) \subset R_l(C_4D)$, then $C_6D = GC_4D$ for some matrix G. Hence

$$C \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } rank(C_4D) = r_1.$$

Similarly, we can prove $(DC_4)^{\#}$ exists,

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} C \sim \begin{bmatrix} DC_4 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } rank(DC_4) = r_1.$$

It follows from the definition of the group inverse and the existence of $(C_4D)^{\#}$. Thus

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$$C_4DXC_4D = C_4D \Rightarrow C_4D(XC_4D - I) = 0 \Rightarrow XC_4D - I = 0 \Rightarrow XC_4D = I$$

for matrix a X. Hence C_4, and D are invertible matrices, and $C_4D \sim DC_4$. Therefore $AB \sim BA$.

$(1),(3) \Rightarrow (2)$ and $(2),(3) \Rightarrow (1)$ are obvious. ◻

THEOREM 3.7. *Suppose $A, B \in \mathbf{R}^{n\times n}$, and $A^\#$ and $B^\#$ exist. Then the following conditions are equivalent:*

(i) $(AB)^\#$ *exists and* $(AB)^\# = B^\# A^\#$.
(ii) $R_r(AB) = R_r(BA)$ *and* $N_r(AB) = N_r(BA)$.
(iii) There exists a invertible $P \in \mathbf{R}^{n\times n}$ *such that*

$$\begin{cases} A = P(A_1 \oplus \cdots \oplus A_t)P^{-1}, \\ B = P(B_1 \oplus \cdots \oplus B_t)P^{-1}, \end{cases}$$

where $A_i = 0$ or A_i is invertible for every i, $B_i = 0$ or B_i is invertible for every i, the orders of A_i and B_i are equal for every i, and $A_jB_j = 0$ for $j \geq 2$.

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ can be obtained by a direct computation.

$(i) \Rightarrow (iii)$ We proceed by induction on n. If $n = 1$, the proof is obvious. Suppose the lemma is true when $k < n$, where $k \geq 2$; we will prove that it is true when $k = n$. Without loss of generality, we assume $0 < rankA < n$. It follows from Theorem 2.1(iv) and the existence of $A^\#$ that

$$A = N\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}N^{-1}, \text{ and } A^\# = N\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}N^{-1} \tag{3.1}$$

for some invertible matrices$N \in \mathbf{R}^{n\times n}$, $D \in \mathbf{R}^{r\times r}$, where $r = rankA$. Let

$$B = N\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}N^{-1}, \text{ and } B^\# = N\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}N^{-1} \tag{3.2}$$

where $B_1, C_1 \in \mathbf{R}^{r\times r}$. Then

$$AB = N\begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix}N^{-1}, \text{ and } B^\# A^\# = N\begin{bmatrix} C_1D^{-1} & 0 \\ C_3D^{-1} & 0 \end{bmatrix}N^{-1}.$$

By Corollary 2.4 and $(AB)^\# = B^\# A^\#$, we have

$$[(DB_1)^\#]^2 DB_2 = 0, \text{ and } C_3 = 0. \tag{3.3}$$

Again applying Corollary 2.4 and the existence of $(AB)^\#$, we have $R_r(DB_2) \subset R_r(DB_1)$. Thus, $DB_2 = DB_1X$ for some $r \times (n - r)$ matrix X, i.e., $B_2 = B_1X$. Replacing B_2 by B_1X in (3.3), we have $DB_1X = 0$. Hence

$$B_2 = D^{-1}(DB_2) = D^{-1}(DB_1X) = 0 \tag{3.4}$$

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Combining (3.2), (3.3), and (3.4), we have $\begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ 0 & C_4 \end{bmatrix}^{\#}$. It follows from Theorem 2.3 that $B_3 = 0$. In summary,

$$A = N(D \oplus 0)N^{-1}, \text{ and } B = N(B_1 \oplus B_4)N^{-1}. \tag{3.5}$$

Again applying $(AB)^{\#} = B^{\#}A^{\#}$, we obtain $(DB_1)^{\#} = B_1^{\#}D^{\#}$, and $(0B_4)^{\#} = B_4^{\#}0$. By the induction hypothesis, it is easy to see that (iii) holds.

$(ii) \Rightarrow (iii)$ We proceed by induction on n. If $n = 1$, the proof is obvious. Suppose the lemma is true when $k < n$, where $k \geq 2$; we will prove that it is true when $k = n$. We can assume that (3.1) and (3.2) hold. It follows from $R_r(AB) = R_r(BA)$ that $B_3D = 0$, i.e., $B_3 = 0$. Again applying $N_r(AB) = N_r(BA)$, we have $N_r(\begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix}) = N_r(\begin{bmatrix} B_1D & 0 \\ 0 & 0 \end{bmatrix})$. Noting $\begin{bmatrix} 0 \\ y \end{bmatrix} \in N_r(\begin{bmatrix} B_1D & 0 \\ 0 & 0 \end{bmatrix})$ for any $y \in \mathbf{C}^{n-r}$, we obtain $\begin{bmatrix} 0 \\ y \end{bmatrix} \in N_r(\begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix})$, and hence $DB_2y = 0$. That is $B_2 = 0$. In summary Equation (3.5) holds. Again applying (ii), we obtain that $R_r(DB_1) = R_r(B_1D)$, $N_r(DB_1) = N_r(B_1D)$, $R_r(0B_4) = R_r(B_40)$, and $N_r(0B_4) = N_r(B_40)$. By the induction hypothesis, it is easy to see that (iii) holds. □

Corollary 3.8. *Suppose $A, B \in \mathbf{R}^{n\times n}$, $A^{\#}$ and $B^{\#}$ exist, and $AB = BA$. Then $(AB)^{\#}$ exists and $(AB)^{\#} = B^{\#}A^{\#}$.*

Corollary 3.9. *Suppose $A, B \in \mathbf{R}^{n\times n}$, $A^{\#}$ and $B^{\#}$ exist. Then $(AB)^{\#}$ exists and $(AB)^{\#} = B^{\#}A^{\#}$ if and only if $(BA)^{\#}$ exists and $(BA)^{\#} = A^{\#}B^{\#}$.*

4. Group Inverse for a Class 2×2 Block Matrices. The group inverses of block matrices have various applications in singular differential and difference equations, Markov chains, iterative methods; see [1], [3], [6], [7], [8], [10], [12], [13], [16], [17]. We generalized the results of [2] to the Bezout domain in this section of the paper.

Lemma 4.1. *Let $A, B \in \mathbf{R}^{n\times n}$, if $A^2 = A$, $rank A = r$, $R_r(B) = R_r(BAB)$, then there is an invertible matrix $P \in \mathbf{R}^{n\times n}$, such that $B = P\begin{bmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{bmatrix}P^{-1}$ and $B_1^{\#}$ exists, where $B_1 \in \mathbf{R}^{r\times r}$, $X \in \mathbf{R}^{r\times(n-r)}$, $Y \in \mathbf{R}^{(n-r)\times r}$.*

Proof. Since $A^2 = A$, by Lemma 1.2 there exists invertible matrix $P \in \mathbf{R}^{n\times n}$ such that $A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, $B = P\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}P^{-1}$, where $B_1 \in \mathbf{R}^{r\times r}, B_4 \in \mathbf{R}^{(n-r)\times(n-r)}$. It follows from $R_r(B) = R_r(BAB)$ that $R_r(B) = R_r(BA)$. Then there exists $Z = P\begin{bmatrix} Z_1 & X \\ Z_3 & Z_4 \end{bmatrix}P^{-1} \in \mathbf{R}^{n\times n}$, where $X \in \mathbf{R}^{r\times(n-r)}$, such that $B = BAZ$,

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i.e.,

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} \begin{bmatrix} Z_1 & X \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} B_1Z_1 & B_1X \\ B_3Z_1 & B_3X \end{bmatrix},$$

then $B_2 = B_1X$, and $B_4 = B_3X$. It follows from $R_r(B) = R_r(BAB)$ and Corollary 3.2 that $(BA)^\#$ exists. By Corollary 2.5 and $BA = P \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} P^{-1}$, we get that $B_1^\#$ exists and $R_l(B_3) \subset R_l(B_1)$. So there is a matrix $Y \in \mathbf{R}^{(n-r)\times r}$, such that $B_3 = YB_1$. That is $B = P \begin{bmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{bmatrix} P^{-1}$. ∎

Lemma 4.2. *Let $A, B \in \mathbf{R}^{n\times n}$. If $A^2 = A$, $rankA = r$, $R_r(B) = R_r(BAB)$, then the following conclusions hold:*

(i) $(BA)^\# BAB = B$;
(ii) $A(AB)^\# = (AB)^\#$, $(BA)^\# A = (BA)^\#$, $(AB)^\# A = A(BA)^\#$, $(BA)^\# B = B(AB)^\#$;
(iii) $(AB)^\# ABA(AB)^\# = (AB)^\#$, $A(BA)^\#(AB)^\# AB = (AB)^\#$;
(iv) $(BA)^\# BA(AB)^\# A = (BA)^\#$, $B(AB)^\# ABA = BA$.

Proof. By Lemma 4.1. The proof is similar to Lemma 2.6 of [2]. ∎

Theorem 4.3. *Suppose $M = \begin{bmatrix} A & A \\ B & O \end{bmatrix}$, where $A, B \in \mathbf{R}^{n\times n}$, $A^2 = A$, $rankA = r$, then*

(i) $M^\#$ *exists if and only if* $R_r(B) = R_r(BAB)$;
(ii) *If* $M^\#$ *exists, then* $M^\# =$

$$\begin{bmatrix} A - (AB)^\# + (AB)^\# A - (AB)^\# ABA & A + (AB)^\# A - (AB)^\# ABA \\ (BA)^\# B + (BA)^\#(AB)^\# AB - (BA)^\# & -(BA)^\# \end{bmatrix}$$

Proof. (i) It is easy to see that

$$R_r(M) = \left\{ \begin{bmatrix} A & A \\ B & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \forall x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}, \text{ and}$$

$$R_r(M^2) = \left\{ \begin{bmatrix} A + AB & A \\ BA & BA \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \middle| \forall y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}.$$

('only if ') Let $x_1 \in \mathbf{R}^n$. It follows from the existence of $M^\#$ that $R_r(M) = R_r(M^2)$. So there exists $y_1, y_2 \in \mathbf{R}^n$ such that

$$\begin{bmatrix} A & A \\ B & O \end{bmatrix} \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} A & A \\ B & O \end{bmatrix}^2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A + AB & A \\ BA & BA \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

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Then $0 = Ay_1 + Ay_2 + ABy_1$, and $Bx_1 = BA(y_1 + y_2)$, so $Bx_1 = -BABy_1$. That is $R_r(B) \subseteq R_r(BAB)$. $R_r(B) \supseteq R_r(BAB)$ is obvious. Hence $R_r(B) = R_r(BAB)$.

(' if ') Given any $x_1, x_2 \in \mathbf{R}^n$. It follows from $R_r(B) = R_r(BAB)$ that there exists $a \in \mathbf{R}^n$ such that $B(x_1 - A(x_1 + x_2)) = BABa$; that is $Bx_1 = BA(x_1 + x_2) + BABa$. Hence

$$\begin{bmatrix} A & A \\ B & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A(x_1 + x_2) \\ Bx_1 \end{bmatrix} = \begin{bmatrix} Ax_1 + Ax_2 \\ BA(x_1 + x_2) + BABa \end{bmatrix} = \begin{bmatrix} 0 & A \\ -BAB & BA \end{bmatrix} \begin{bmatrix} -a \\ x_1 + x_2 \end{bmatrix}.$$

It follows from

$$\begin{bmatrix} 0 & A \\ -BAB & BA \end{bmatrix} = M^2 \begin{bmatrix} I & 0 \\ -I - B & I \end{bmatrix}$$

that $R_r(\begin{bmatrix} 0 & A \\ -BAB & BA \end{bmatrix}) = R_r(M^2)$. That is $R_r(M) \subseteq R_r(M^2)$. $R_r(M^2) \subseteq R_r(M)$ is obvious. Hence $R_r(M) = R_r(M^2)$, implying $M^\#$ exists.

(ii) By Lemma 4.1 and Lemma 4.2. The proof is similar to Theorem 3.1 of [2] ◻

Corollary 4.4. *Suppose* $M = \begin{bmatrix} A & B \\ A & O \end{bmatrix}$, *where* $A, B \in \mathbf{R}^{n \times n}$, $A^2 = A$, $rankA = r$, *then*

(i) $M^\#$ *exists if and only if* $R_l(B) = R_l(BAB)$;
(ii) If $M^\#$ *exists, then* $M^\# =$

$$\begin{bmatrix} A - (BA)^\# + (AB)^\# A - (AB)^\# ABA & (BA)^\# B + (BA)^\# (AB)^\# AB - (AB)^\# \\ A + (AB)^\# A - (AB)^\# ABA & -(AB)^\# \end{bmatrix}.$$

Proof. It is similar to the proof of Theorem 4.3. ◻

Theorem 4.5. *If* $\begin{bmatrix} A & A \\ B & O \end{bmatrix}^\#$ *exists, where* $A, B \in \mathbf{R}^{n \times n}$, $A^2 = A$, $rankA = r$, *then* $AB \sim BA$.

Proof. The conclusion is obvious by Corollary 3.2, Theorem 3.6, and Theorem 4.3. ◻

Remark 4.6. Theorem 4.3, Corollary 4.4, and Theorem 4.5 generalize Theorem 3.1, Corollary 1, and Corollary 2 of [2].

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