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SPECTRAL PROPERTIES OF CERTAIN SEQUENCES OF PRODUCTS OF TWO REAL MATRICES*

MICHELA BRUNDU† AND MARINO ZENNARO†

Abstract. The aim of this paper is to analyze the asymptotic behavior of the eigenvalues and eigenvectors of particular sequences of products involving two square real matrices A and B, namely of the form $B^k A$, as $k \to \infty$. This analysis represents a detailed deepening of a particular case within a general theory on finite families $\mathcal{F} = \{A_1, \dots, A_m\}$ of real square matrices already available in the literature. The Bachmann–Landau symbols and related results are largely used and are presented in a systematic way in the final Appendix.

Key words. Eigenvalue, Eigenvector, Elementary symmetric function, Matrix power sequence, Asymptotic behavior, Bachmann–Landau symbol.

AMS subject classifications. 15A18, 26A12, 20B30.

1. Introduction. It is well known that, given a finite family $\mathcal{F} = \{A_1, \dots, A_m\}$ of $n \times n$ real matrices and the associated multiplicative semigroup $\Sigma(\mathcal{F})$ (i.e., the set of all the possible finite products $P = A_{i_1}^{k_1} \dots A_{i_s}^{k_s}$), generally the eigenvalues and the structure of the eigenspaces of a given $P \in \Sigma(\mathcal{F})$ are not easily correlatable to the eigenvalues and eigenspaces of its factors. Recently, in [4], we have considered the particular case in which all of the products P are *asymptotically rank-one* matrices (i.e., the eigenvalue of maximum modulus is unique and simple) and, under some additional technical assumptions, we have proved that the set of the leading eigenvectors of all the products $P \in \Sigma(\mathcal{F})$ determines a so-called *leading multicone*, which is a particular symmetric subset of $\mathbb{R}^n$ (see also [3]), invariant under the action of $\mathcal{F}$.

In this paper, we make a first study of the precise behavior of all the eigenvalues and eigenvectors of specific sequences of products $P_k \in \Sigma(\mathcal{F})$ as $k \to \infty$. We consider one of the simplest cases, in which the products involve only two elements of $\mathcal{F}$ and have the form $P_k = B^k A$. However, it is worth noting that the results on sequences of this particular form can be extended in a straightforward way to more general cases of product sequences such as, for example, $P_k = (A_{i_1}^{k_1} \dots A_{i_s}^{k_s})^k A_{i_{s+1}}^{k_{s+1}} \dots A_{i_r}^{k_r}$, where $i_1, \dots, i_r \in \{1, \dots, m\}$ and $k_1, \dots, k_r \geq 1$ are fixed.

An important field of applications of what above is, for example, the investigation of the asymptotic behavior of the solutions to *discrete-time linear switched systems* such as:

$$x(k+1) = A_{\sigma(k)}\, x(k), \quad \sigma : \mathbb{N} \longrightarrow \{1, \dots, m\},$$

where $x(0) \in \mathbb{R}^n$, $A_{\sigma(k)} \in \mathcal{F}$, and σ denotes the *switching law*. For an introduction to this subject, see, for example, the monograph by Blanchini and Miani [2] and the paper by De Iuliis et al. [5].

In our statements and proofs, we make large use of the Bachmann–Landau symbols and related results. Unfortunately, not all of them are easy to find in the literature. Therefore, we needed to give some proofs *ex novo* and, so, we decided to collect systematically the whole necessary theory in a final appendix (Section 6).

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Corresponding Author: Michela Brundu
†Dipartimento di Matematica e Geoscienze, Università di Trieste, I-34100 Trieste, Italy (brundu@units.it, zennaro@units.it).

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We think that it constitutes an interesting part of the paper in itself even if, at a first sight, it could seem to be not very consistent with the main scope of our work.

The paper is organized as follows.

In Section 2, we recall some basic definitions and notions regarding the *symmetric polynomials* and the *term-orderings* of monomials.

In Section 3, we consider such polynomials evaluated on a set of complex functions $T_1, \ldots, T_n$, defined on a given domain $\mathcal{D} \subseteq \mathbb{R}$, to form the *elementary symmetric functions*. Then, making use of the results on the Bachman–Landau symbols reported in Section 6, we analyze the order of the elementary symmetric functions near an accumulation point of $\mathcal{D}$, under specific assumptions on the mutual ordering of the T_i's.

In Section 4, we specialize the general results obtained in Section 3 to the particular case of the domain $\mathcal{D} = \mathbb{N}$ with accumulation point $k_0 = +\infty$. In this framework, we consider the eigenvalues $\mu_i(k)$ of the matrix $B^k A$, proving that they grow as λ_i^k, where the λ_i's are the eigenvalues of B (see Proposition 4.7). Consequentely, if we initially label them in such a way that $|\mu_1(k)| \geq \cdots \geq |\mu_n(k)|$, it turns out that $|\mu_i| \in \omega(\mu_{i+1})$ (see Corollary 4.8).

In Section 5, we apply the previous results on the asymptotic behavior of the eigenvalues of the matrices $B^k A$ as $k \to \infty$, to the corresponding eigenvectors. In this way, we are able to prove that they converge to vectors – explicitly computed – depending on the eigenvectors of B and on suitable submatrices of A (see Theorem 5.5).

2. Elementary symmetric polynomials. We start by recalling some basic notions and results.

In the sequel, if $z \in \mathbb{C}$, then $|z|$ will denote its module. Moreover, if $v \in \mathbb{R}^n$, then we set $\| v \|$ to be its Euclidean norm and $\mathrm{vers}(v) := v / \| v \|$.

Definition 2.1. *Let K be a field and $K[X_1, \ldots, X_n]$ be the ring of polynomials in a set of variables $X_1, \ldots, X_n$. Each element can be written as:*

$$P(X_1, \ldots, X_n) = \sum_{\text{finite}} \alpha_{r_1 \cdots r_n} X_1^{r_1} \cdots X_n^{r_n}, \quad \textit{where} \quad \alpha_{r_1 \cdots r_n} \in K, \quad r_i \geq 0. \tag{2.1}$$

We say that each $\alpha_{r_1 \cdots r_n} X_1^{r_1} \cdots X_n^{r_n}$ is a monomial of P and that $X_1^{r_1} \cdots X_n^{r_n}$ is a term.

Note that a term is just a monomial with coefficient 1 and that a monomial can be zero, while a term cannot.

Let us recall the well known *lexicographic order* among all the terms:

$$X_1^{r_1} \cdots X_n^{r_n} > X_1^{s_1} \cdots X_n^{s_n} \iff r_1 > s_1 \textit{ or } \exists\, i \mid r_1 = s_1, \ldots, r_i = s_i, r_{i+1} > s_{i+1}. \tag{2.2}$$

As immediate consequence of (2.2), we have an ordering among the variables:

$$X_1 > X_2 > \cdots > X_n. \tag{2.3}$$

Obviously, (2.2) induces a term-ordering on the terms of a given polynomial. So its *maximum term* is uniquely defined.

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NOTATION 2.2. *We denote the monomials (respectively, terms) also by:*

$$\alpha_R \overline{X}_R := \alpha_{r_1 \cdots r_n} X_1^{r_1} \cdots X_n^{r_n} \quad \textit{(respectively by} \quad \overline{X}_R := X_1^{r_1} \cdots X_n^{r_n}),$$

where R stands for the multi-index $(r_1 \cdots r_n) \in \mathbb{N}^n$.

With this notation, (2.2) induces an order on the (nonzero) monomials *belonging to any given polynomial.* Namely, if we take $P(\overline{X}) = \sum_{R \in \mathcal{I}} \alpha_R \overline{X}_R$, where $\mathcal{I} \subset \mathbb{N}^n$ is a finite set, for all $I, J \in \mathcal{I}$ it is natural to set

$$\alpha_I \overline{X}_I > \alpha_J \overline{X}_J \iff \overline{X}_I > \overline{X}_J,$$

where the second inequality is given by (2.2). So, the *maximum monomial* of $P(\overline{X})$ is defined.

DEFINITION 2.3. *We say that the integer d is the degree of a nonzero monomial (respectively, term) $\alpha_{r_1 \cdots r_n} X_1^{r_1} \cdots X_n^{r_n}$ if $d = r_1 + \cdots + r_n$. Moreover, the degree of a polynomial (2.1) is the maximum degree of its monomials. Finally, a polynomial is said homogeneous if all its monomials have the same degree.*

In the sequel, we will often consider square-free terms: for them we can use a simpler notation and denote them by:

$$X_{i_1} \cdots X_{i_d}, \quad \text{where} \quad 1 \leq i_1 < \cdots < i_d \leq n.$$

It is clear that the set of square-free terms is finite, and here the lexicographic order can be expressed also as follows. If $X_{i_1} \cdots X_{i_d}$ and $X_{j_1} \cdots X_{j_c}$ are two square-free terms, then

$$X_{i_1} \cdots X_{i_d} > X_{j_1} \cdots X_{j_c} \iff i_1 < j_1 \textit{ or } \exists\, k \mid i_1 = j_1, \ldots, i_k = j_k \text{ and } i_{k+1} < j_{k+1}. \tag{2.4}$$

DEFINITION 2.4. *The elementary symmetric polynomials σ_d in $X_1, \ldots, X_n$ are defined, for each $d = 1, \ldots, n$, as the sums of all the possible square-free terms of degree d in $X_1, \ldots, X_n$, that is,*

$$\begin{aligned}
\sigma_0 &:= 1 \\
\sigma_1 &:= X_1 + X_2 + \cdots + X_n \\
\sigma_2 &:= X_1 X_2 + X_1 X_3 + \cdots + X_{n-1} X_n \\
&\cdots \\
\sigma_d &:= \sum_{1 \leq i_1 < \cdots < i_d \leq n} X_{i_1} \cdots X_{i_d} \\
&\cdots \\
\sigma_n &:= X_1 \cdots X_n.
\end{aligned}$$

Clearly, each σ_d is homogeneous of degree d. Moreover, as seen before, the *maximum term* of σ_d is uniquely defined and turns out to be $X_1 \cdots X_d$. It is also clear that the maximum term of $\sigma_d - X_1 \cdots X_d$ is given by $X_1 \cdots X_{d-1} X_{d+1}$.

The elementary symmetric polynomials are particular *symmetric polynomials* (i.e., the elements of $K[X_1, \ldots, X_n]$ invariant under all the permutations of the variables). It is known that they generate (as K–algebra) all the symmetric polynomials.

In the sequel, we are going to use some slightly more general polynomials: substantially, we consider finite sums of square-free monomials, not necessarily terms. These are still homogeneous but no more symmetric and, so, it is worthwhile to give them a specific name.

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Definition 2.5. *A quasi-elementary symmetric polynomial of degree d in $X_1,\dots,X_n$ is*

$$q_d(X_1,\dots,X_n)=\sum_{1\le i_1<\dots<i_d\le n}\alpha_{i_1\cdots i_d}\,X_{i_1}\cdots X_{i_d},\quad \textit{where}\quad \alpha_{i_1\cdots i_d}\in K,\ \alpha_{1\cdots d}\neq 0.$$

Clearly, these polynomials specialize to the elementary symmetric polynomial σ_d as far as all the coefficients are equal to 1.

Remark 2.6. Accordingly to the previous definitions and remarks, the maximum monomial of the polynomial $q_d(X_1,\dots,X_n)$ is $\alpha_{1\cdots d}X_1\cdots X_d$.

3. Elementary symmetric functions and asymptotic properties. From now on, the role of the field K as the field of coefficients of $K[X_1,\dots,X_n]$ will be played by the field $\mathbb{R}$ of real numbers.

Throughout this section, $\mathcal{D}\subseteq\mathbb{R}$ denotes a domain and $x_0\in\mathbb{R}\cup\{\pm\infty\}$ an accumulation point of $\mathcal{D}$.

We shall make large use of the *Bachmann–Landau notation* in order to appropriately handle the orders of real and complex functions defined on $\mathcal{D}$ for $x\longrightarrow x_0$. For instance, $\mathcal{F}_{x_0}(\mathcal{D})$ will denote the set of the complex functions defined on $\mathcal{D}\backslash\{x_0\}$ (see Notation 6.36). For a detailed treatment of this subject in relation to the use we make in this paper, the reader is referred to the Appendix in Section 6.

Let

$$T_i:\quad \mathcal{D}\longrightarrow\mathbb{C},\qquad i=1,\dots,n,$$

be an ordered set of n complex functions of one real variable x satisfying, eventually (near x_0),

$$|T_1(x)|\ge|T_2(x)|\ge\cdots\ge|T_n(x)|. \tag{3.5}$$

We can consider the evaluation of any polynomial $P(X_1,\dots,X_n)\in\mathbb{R}[X_1,\dots,X_n]$ at the n-tuple of functions $(T_1(x),\dots,T_n(x))$ obtaining a complex function of the real variable x defined on the domain $\mathcal{D}$ as well. Therefore, we can consider the order (near x_0) of such a function and compare it with the order of any of its *evaluated monomials*, that is, of a function of the type $T_1^{r_1}(x)\cdots T_n^{r_n}(x)$.

In this section, we are going to consider two particular cases of polynomials evaluated on a set of given functions: the elementary symmetric ones and the quasi-elementary ones.

Definition 3.1. *The elementary symmetric functions in $T_1(x),\dots,T_n(x)$ are the elementary symmetric polynomials $\sigma_d(X_1,\dots,X_n)$ evaluated in such set of functions, that is, for any degree $d=0,\dots,n$, we set*

$$\mathrm{Sym}_d(x):=\sigma_d(T_1(x),\dots,T_n(x))=\sum_{1\le i_1<\dots<i_d\le n}T_{i_1}(x)\cdots T_{i_d}(x).$$

It is obvious that the orders of $\mathrm{Sym}_d(x)$ and of its maximum (evaluated) monomial are equal to each other for $d=n$ (see Definition 2.4). But, for a general d, this is no more true: it is necessary to make further assumptions. This will be done in the forthcoming Theorem 3.6. Its proof is quite tricky: essentially, we are going to apply Proposition 6.39 to a particular pair of functions f and g, where g is an elementary symmetric function and f is its maximum monomial and repeat the argument on the remaining parts.

We need first some technical results, which will be stated in a general form (i.e., concerning quasi-elementary polynomials), and in this version they will be useful also in the next section.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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NOTATION 3.2. *Consider, for each $d \in \{1, \ldots, n\}$, a quasi-elementary polynomial of degree d with real coefficients (see Definition 2.5):*

$$q_d(X_1, \ldots, X_n) = \sum_{1 \le i_1 < \cdots < i_d \le n} \alpha_{i_1 \cdots i_d} X_{i_1} \cdots X_{i_d}, \quad \textit{where} \quad \alpha_{i_1 \cdots i_d} \in \mathbb{R},\ \alpha_{1 \cdots d} \neq 0,$$

and denote the difference between q_d and its maximum monomial by:

$$\tilde{q}_d(X_1, \ldots, X_n) := q_d(X_1, \ldots, X_n) - \alpha_{1 \cdots d} X_1 \cdots X_d.$$

The evaluations of the above polynomials on $(T_1(x), \ldots, T_n(x))$ will be denoted, respectively, by:

$$Q_d(x) := q_d(T_1(x), \ldots, T_n(x)), \quad \tilde{Q}_d(x) := \tilde{q}_d(T_1(x), \ldots, T_n(x)),$$

which are both elements of $\mathcal{F}_{x_0}(\mathcal{D})$, as well.

LEMMA 3.3. *Keeping the notation above and assuming that $(T_1(x), \ldots, T_n(x))$ satisfies (3.5), for any $d \in \{1, \ldots, n\}$, the following property is verified:*

$$|T_1 \cdots T_d| \in \Omega(|Q_d|), \quad \textit{i.e.,} \quad |Q_d| \in \mathcal{O}(|T_1 \cdots T_d|). \tag{3.6}$$

Moreover, for any $d \in \{2, \ldots, n-1\}$, if $\alpha_{1 \cdots d-1, d+1} \neq 0$, then

$$|T_1 \cdots T_{d-1} T_{d+1}| \in \Omega(|\tilde{Q}_d|) \tag{3.7}$$

and, hence,

$$|\tilde{Q}_d| \in \mathcal{O}(|T_1 \cdots T_{d-1} T_{d+1}|) \subseteq \mathcal{O}(|T_1 \cdots T_d|). \tag{3.8}$$

Proof. It easily follows from (3.5), Lemma 6.37 and Proposition 6.39-(i). ☐

In general, it is not true that $Q_d(x)$ and its maximum (evaluated) monomial have the same order, even if we assume strict inequalities $|T_i(x)| > |T_{i+1}(x)|$ in (3.5). Nevertheless, this happens under two sets of assumptions: either if the order of T_i is strictly bigger than the order of T_{i+1} or if $q_d = \sigma_d$ and it satisfies precise requirements, as we will see in the forthcoming Theorem 3.6. In the first case, we have the following result.

PROPOSITION 3.4. *With the notation above, assume in addition that the given functions $T_1(x), \ldots, T_n(x)$ verify*

$$|T_1| \in \omega(|T_2|), \quad |T_2| \in \omega(|T_3|), \ldots, \quad |T_{n-1}| \in \omega(|T_n|).$$

Then, for any $d = 1, \ldots, n$, we have

$$\Theta(|Q_d|) = \Theta(|T_1 \cdots T_d|).$$

Proof. For $d = n$ the thesis is obvious since, in this case, $Q_d = Q_n = \alpha_{1 \cdots n} T_1 \cdots T_n$. Thus, let $d \le n-1$. To compare the orders of $T_1 \cdots T_d$ and $T_{i_1} \cdots T_{i_d}$, where $(i_1, \ldots, i_d) \neq (1, \ldots, d)$, observe that $i_1 \ge 1,\ i_2 \ge 2, \ldots,\ i_d \ge d$ and two cases may occur: either $i_1 > 1, i_2 > 2, \ldots, i_d > d$ or there exists $k \in \{1, 2, \ldots, d-1\}$ such that $i_1 = 1,\ i_2 = 2, \ldots,\ i_k = k,\ i_{k+1} > k+1, \ldots, i_d > d$.

Therefore, since the assumption also implies that $|T_i| \in \omega(0)$, $i = 1, \ldots, n-1$, we may repeatedly apply Proposition 6.32-(v) and conclude that

$$|T_1 \cdots T_d| \in \omega(|T_{i_1} \cdots T_{i_d}|).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

Now we recall that

$$|\widetilde{Q}_d| = |Q_d - \alpha_{1\cdots d}T_1\cdots T_d| \leq \sum_{(i_1,\ldots,i_d)\neq(1,\ldots,d)} |\alpha_{i_1\cdots i_d}|\cdot|T_{i_1}\cdots T_{i_d}|.$$

Thus, Proposition 6.26-(v) yields

$$|T_1\cdots T_d| \in \omega(|\widetilde{Q}_d|),$$

and, consequently, Lemma 6.38-(i) concludes the proof. ◻

In the second case ($q_d = \sigma_d$), before the main result of this section, let us prove a preliminary technical fact.

Lemma 3.5. *Let $T_1(x),\ldots,T_n(x) \in \mathcal{F} = \mathcal{F}_{x_0}(\mathcal{D})$ satisfy (3.5) eventually and let d be any integer in $\{1,\ldots,n\}$. If $|T_1\ldots T_d| \notin \mathcal{O}(|\mathrm{Sym}_d|)$, then on a suitable subdomain $\mathcal{D}' \subseteq \mathcal{D}$ it holds that*

$$\Theta(|T_1\ldots T_d|) = \Theta(|\mathrm{Sym}_d - T_1\ldots T_d|) \subseteq \omega(|\mathrm{Sym}_d|). \tag{3.9}$$

Proof. Let us apply (3.8) of Lemma 3.3 to the particular case $q_d = \sigma_d$, that is, $Q_d(x) = \mathrm{Sym}_d(x)$, obtaining

$$|\mathrm{Sym}_d - T_1\ldots T_d| \in \mathcal{O}(|T_1\cdots T_d|). \tag{3.10}$$

By assumption $|T_1\ldots T_d| \notin \mathcal{O}(|\mathrm{Sym}_d|)$. This condition, together with (3.10), allows us to apply Proposition 6.39-(ii) to the functions $f = T_1\ldots T_d$ and $g = \mathrm{Sym}_d - T_1\ldots T_d$, obtaining (3.9) on a suitable subdomain $\mathcal{D}' \subseteq \mathcal{D}$. ◻

Theorem 3.6. *Let $T_1(x),\ldots,T_n(x) \in \mathcal{F} = \mathcal{F}_{x_0}(\mathcal{D})$ satisfy (3.5) eventually and assume that*

$$|\mathrm{Sym}_i| \in \omega(0), \quad \textit{for all} \quad i = 1,\ldots,n. \tag{3.11}$$

Moreover, let s be any integer such that $1 \leq s \leq n-1$.

If, for any $i = 1,\ldots,s$, we have

$$|\mathrm{Sym}_r\mathrm{Sym}_{i-1}^{r-i}| \in \mathcal{O}(|\mathrm{Sym}_i|^{r-i+1}), \quad \textit{for all} \quad r = i+1,\ldots,n, \tag{H_i}$$

then, for all $j = 0,\ldots,s$, the following facts hold:

$$\Theta(|\mathrm{Sym}_j|) = \Theta(|T_1\cdots T_j|). \tag{K_j}$$

Proof. Let us prove (K_j) by finite induction on $j = 0,1,\ldots,s$. To this aim, observe that the condition

$$1 \in \Theta(|\mathrm{Sym}_0|) = \Theta(1), \tag{K_0}$$

is obviously true.

$(K_{j-1}) \Longrightarrow (K_j)$ for $j \geq 1$

By using (3.6) of Lemma 3.3 for $d = j$ and in the particular case $q_j = \sigma_j$, that is, $Q_j(x) = \mathrm{Sym}_j(x)$, we get

$$|T_1\ldots T_j| \in \Omega(|\mathrm{Sym}_j|).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

Hence, our goal reduces to $|T_1 \dots T_j| \in \mathcal{O}(|\mathrm{Sym}_j|)$. Assume, by contradiction, that $|T_1 \dots T_j| \notin \mathcal{O}(|\mathrm{Sym}_j|)$. Then, by Lemma 3.5, we have that the relation (3.9) holds in the case $d = j$ on a suitable subdomain $\mathcal{D}_j \subseteq \mathcal{D}$:

$$\Theta(|T_1 \dots T_j|) = \Theta(|\mathrm{Sym}_j - T_1 \dots T_j|) \subseteq \omega(|\mathrm{Sym}_j|). \tag{3.12}$$

Claim. The above fact implies that there exists a finite sequence of subdomains $\mathcal{D}_r$ with

$$\mathcal{D} \supseteq \mathcal{D}_j \supseteq \cdots \supseteq \mathcal{D}_{r-1} \supseteq \mathcal{D}_r \supseteq \cdots \supseteq \mathcal{D}_n$$

such that, for each $r \in \{j, j+1, \dots, n\}$,

$$\Theta(|T_j|) = \Theta(|T_{j+1}|) = \cdots = \Theta(|T_r|) \subseteq \omega\left(\frac{|\mathrm{Sym}_j|}{|\mathrm{Sym}_{j-1}|}\right) \text{ on } \mathcal{D}_{r'},\ r' := \max\{j; r-1\}, \tag{ZA_r}$$

and

$$\Theta(|T_1 \dots T_r|) = \Theta(|\mathrm{Sym}_r - T_1 \dots T_r|) \subseteq \omega(|\mathrm{Sym}_r|) \text{ on } \mathcal{D}_r. \tag{ZB_r}$$

Let us show the claim by finite induction on $r = j, \dots, n$.

(ZA_j) and (ZB_j)

Both easily follow from (3.12) on $\mathcal{D}_j$. In fact, (ZB_j) coincides with (3.12). Concerning (ZA_j), note instead that we are assuming (K_{j-1}), that is, $|T_1 T_2 \dots T_{j-1}| \in \Theta(|\mathrm{Sym}_{j-1}|)$ and, hence, $|T_1 T_2 \dots T_{j-1}| \in \omega(0)$ by (3.11). Therefore, by Proposition 6.24 and again by (3.12), we obtain

$$|T_j| = \frac{|T_1 T_2 \dots T_j|}{|T_1 T_2 \dots T_{j-1}|} \ \in\ \omega(|\mathrm{Sym}_j|)\Theta(|\mathrm{Sym}_{j-1}^{-1}|) = \omega(|\mathrm{Sym}_j\, \mathrm{Sym}_{j-1}^{-1}|) \text{ on } \mathcal{D}_j,$$

where the last equality is yielded by Proposition 6.32-(v).

(ZA_h) and (ZB_h), $j \le h \le r-1 \Longrightarrow (ZA_r)$ and (ZB_r)

Given (ZA_{r-1}), in order to prove (ZA_r), it is enough to verify that $\Theta(|T_{r-1}|) = \Theta(|T_r|)$. By Lemma 3.3, relation (3.7) for $d = r-1$, we have

$$|T_1 \cdots T_{r-2} T_r| \in \Omega(|\mathrm{Sym}_{r-1} - T_1 \cdots T_{r-1}|),$$

and, thus, the assumption (ZB_{r-1}) implies

$$|T_1 \cdots T_{r-2} T_r| \in \Omega(|T_1 \dots T_{r-1}|).$$

On the other hand, hypothesis (3.5) implies $|T_r| \in \mathcal{O}(|T_{r-1}|)$ and, consequently, Proposition 6.32-(i) also yields the opposite relation:

$$|T_1 \cdots T_{r-2} T_r| \in \mathcal{O}(|T_1 \dots T_{r-1}|).$$

In conclusion, we obtain

$$\Theta(|T_1 \cdots T_{r-2} T_r|) = \Theta(|T_1 \dots T_{r-1}|).$$

Now observe that, if $r = j+1$, then (K_{j-1}) and (3.11) yield $|T_1 \cdots T_{r-2}| \in \omega(0)$. Moreover, if $r \ge j+2$, the same inclusion is guaranteed by (ZB_{r-2}) on the subdomain $\mathcal{D}_{r-2}$ and hence, thanks to Proposition 6.15

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

and Corollary 6.16, also on its subdomain $\mathcal{D}_{r-1}$. So we can apply the *Cancellation Law* (see Corollary 6.35) to the above equality and obtain $\Theta(|T_r|) = \Theta(|T_{r-1}|)$, as required. Therefore, (ZA_r) holds.

In order to show that (ZB_r) is true, we first observe that

$$\begin{aligned} \Theta(|T_1 \dots T_r|) &= \Theta(|T_1 \dots T_j|)\Theta(|T_{j+1}|)\cdots\Theta(|T_r|) \subseteq \\ &\subseteq \omega(|\mathrm{Sym}_j|)\,\omega\left(\frac{|\mathrm{Sym}_j^{r-j}|}{|\mathrm{Sym}_{j-1}^{r-j}|}\right) = \omega\left(\frac{|\mathrm{Sym}_j^{r-j+1}|}{|\mathrm{Sym}_{j-1}^{r-j}|}\right) \end{aligned} \tag{3.13}$$

holds on $\mathcal{D}_{r-1}$, where the first equality follows from Proposition 6.32-(iii), the inclusion from (3.12) and (ZA_r), and the last equality from Proposition 6.32-(v). On the other hand, from the assumptions (3.11) and (H_j) and by Lemma 6.25-(ii), for all $r = j+1, \dots, n$ we obtain

$$|\mathrm{Sym}_r| \in \mathcal{O}\left(\frac{|\mathrm{Sym}_j^{r-j+1}|}{|\mathrm{Sym}_{j-1}^{r-j}|}\right), \quad \text{i.e.,} \quad \frac{|\mathrm{Sym}_j^{r-j+1}|}{|\mathrm{Sym}_{j-1}^{r-j}|} \in \Omega\left(|\mathrm{Sym}_r|\right),$$

on the original domain $\mathcal{D}$. Again by Proposition 6.15 and Corollary 6.16, this fact and relation (3.13) give immediately

$$|T_1 \dots T_r| \in \omega\left(|\mathrm{Sym}_r|\right) \text{ on } \mathcal{D}_{r-1}.$$

In particular, this implies that $|T_1 \dots T_r| \notin \mathcal{O}(|\mathrm{Sym}_r|)$. Then, again by Lemma 3.5, we have that the relation (3.9) holds with $d = r$ on a suitable subdomain $\mathcal{D}_r \subseteq \mathcal{D}_{r-1}$, which is precisely (ZB_r) .

Hence, the claim is proved. In particular, this means that (ZB_n), that is,

$$\Theta(|T_1 \dots T_n|) \subseteq \omega(|\mathrm{Sym}_n|),$$

holds on a suitable subdomain $\mathcal{D}_n$. Since $\mathrm{Sym}_n = T_1 \dots T_n$ by definition, we get a contradiction and the proof of (K_j) is complete. ∎

4. Preliminary results on the eigenvalues of B^kA. In this section, we use the results of Section 3 to study products of elements of $\mathbb{R}^{n,n}$ (the space of square matrices $n \times n$ with real entries) from the point of view of their eigenvalues and eigenvectors. By $GL(n,\mathbb{R})$ we denote the general linear group, that is, the subset of $\mathbb{R}^{n,n}$ consisting of invertible matrices.

Recall that, if $M \in \mathbb{R}^{n,n}$ and $\mu_1, \dots, \mu_n \in \mathbb{C}$ are its eigenvalues, then the characteristic polynomial of M, defined to be $p_M(z) := \det(zI_n - M)$, can be expressed as:

$$\begin{aligned} p_M(z) &= (z-\mu_1)\cdots(z-\mu_n) = \\ &= z^n - (\mu_1 + \cdots + \mu_n)z^{n-1} + \cdots + (-1)^n \mu_1\mu_2\cdots\mu_n = \\ &= \sum_{j=0}^{n} (-1)^j\, \sigma_j\left(\mu_1, \dots, \mu_n\right)\, z^{n-j}, \end{aligned} \tag{4.14}$$

where σ_j denotes as usual the j-th elementary symmetric polynomial in n variables.

Definition 4.1. *If $M \in \mathbb{R}^{n,n}$ then, for each $j = 1, \dots, n$ and any j-tuple $1 \le i_1 < i_2 < \cdots < i_j \le n$, we denote by $M_{i_1 \dots i_j}$ the submatrix of M obtained by intersecting the rows $i_1, \dots, i_j$ with the columns of the same indices $i_1, \dots, i_j$. We say that* $\det(M_{i_1 \dots i_j})$ *is a principal minor of M.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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It is clear that the $j \times j$ left-upper block of M is nothing, but $M_{1\dots j}$. Let us recall an elementary fact in which this notation is used (the proof can be found, for instance, in Jacobson [7]).

Proposition 4.2. *The characteristic polynomial of a matrix $M \in \mathbb{R}^{n,n}$ has the form:*

$$p_M(z) = z^n - \alpha_1 z^{n-1} + \cdots + (-1)^n \alpha_n = \sum_{j=0}^{n} (-1)^j \alpha_j \, z^{n-j}, \tag{4.15}$$

where $\alpha_0 = 1$, α_1 is the trace of the matrix M, $\alpha_n = \det(M)$ and, in general,

$$\alpha_j = \sum_{i_1 < \cdots < i_j} \det(M_{i_1 \dots i_j}).$$

Comparing (4.14) and (4.15), we immediately obtain the following result.

Corollary 4.3. *If $M \in \mathbb{R}^{n,n}$ and $\mu_1, \dots, \mu_n \in \mathbb{C}$ are its eigenvalues then, for all $j = 0, \dots, n$,*

$$\sigma_j\left(\mu_1, \dots, \mu_n\right) = \sum_{i_1 < \cdots < i_j} \det(M_{i_1 \dots i_j}).$$

Notation 4.4. *From now on, we label the eigenvalues of M in such a way that*

$$|\mu_1| \geq \cdots \geq |\mu_n|.$$

In the sequel, we will deal with the following situation. We consider two matrices $A, B \in \mathbb{R}^{n,n}$, where $B = \mathrm{Diag}(\lambda_1, \dots, \lambda_\mathrm{n})$ and, for a positive integer k, we put

$$M := B^k A.$$

Setting $A = (a_{st})$ and $M = (m_{st})$, it is clear that $m_{st} = \lambda_s^k a_{st}$. Hence, for any $j = 1, \dots, n$ and for any j-tuple $i_1 < i_2 < \cdots < i_j$, we have

$$\det(M_{i_1 \dots i_j}) = \lambda_{i_1}^k \cdots \lambda_{i_j}^k \det(A_{i_1 \dots i_j}).$$

Therefore, Corollary 4.3 immediately gives the following fact.

Corollary 4.5. *Let $A, B \in \mathbb{R}^{n,n}$, where $B = \mathrm{Diag}(\lambda_1, \dots, \lambda_\mathrm{n})$. Let k be a positive integer, $M := B^k A$ and $\mu_1^{(k)}, \dots, \mu_n^{(k)}$ be its eigenvalues. Then, for any $j = 1, \dots, n$, we have*

$$\sigma_j\left(\mu_1^{(k)}, \dots, \mu_n^{(k)}\right) = \sum_{i_1 < \cdots < i_j} \lambda_{i_1}^k \cdots \lambda_{i_j}^k \, \det(A_{i_1 \dots i_j}).$$

Observe that the previous equality concerns polynomials either in the $\mu_i^{(k)}$'s or in the λ_{i_j}'s. In order to emphasize that they are both functions of the integer k, we slightly modify the notation:

$$\mu_i(k) := \mu_i^{(k)} \quad \text{and} \quad \Lambda_i(k) := \lambda_i^k, \quad \text{for } i = 1, \dots, n.$$

In particular, accordingly to Notation 4.4, we assume

$$|\mu_1(k)| \geq \cdots \geq |\mu_n(k)|, \quad \text{for all} \quad k \geq 1. \tag{4.16}$$

From now on, we assume the following condition on the matrix A:

$$\det(A_{1 \dots j}) \neq 0, \quad \text{for all} \quad j = 1, \dots, n. \tag{4.17}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

Remark 4.6. In the left-hand side of the equality in Corollary 4.5, we find the j-th elementary symmetric polynomial evaluated in the $\mu_i(k)$'s. So, keeping the notation of Definition 3.1, we write

$$\mathrm{Sym}_j(k) = \sigma_j(\mu_1(k), \ldots, \mu_n(k)). \tag{4.18}$$

On the right side, we have instead a quasi-symmetric polynomial of degree j in the Λ_i's so that the equality in Corollary 4.5 may be rewritten as:

$$\mathrm{Sym}_j(k) = \sum_{i_1<\cdots<i_j} \det(A_{i_1\ldots i_j})\, \Lambda_{i_1}(k)\cdots\Lambda_{i_j}(k). \tag{4.19}$$

In this framework, the domain $\mathcal{D}$ of these functions is the set $\mathbb{N}$ of natural numbers and, obviously, the accumulation point is $k_0 = +\infty$. Therefore, the symbols Θ, ω, etc., stand for $\Theta_{+\infty}$, $\omega_{+\infty}$, etc., respectively.

Proposition 4.7. *Assume in addition that $A, B \in GL(n, \mathbb{R})$, (4.17) holds and*

$$|\lambda_1| > \cdots > |\lambda_n| > 0. \tag{4.20}$$

Then the following facts hold:

(i) $|\mathrm{Sym}_j| \in \Theta(|\Lambda_1\cdots\Lambda_j|)$ *for all* $j = 1, \ldots, n$;
(ii) *for all* $j = 1, \ldots, n-1$ *we have*

$$|\mathrm{Sym}_r \mathrm{Sym}_{j-1}^{r-j}| \in o(|\mathrm{Sym}_j|^{r-j+1}) \quad \textit{for all } r = j+1, \ldots, n; \tag{$\widetilde{H}_j$}$$

(iii) $|\mu_j| \in \Theta(|\Lambda_j|)$ *for all* $j = 1, \ldots, n$.

Proof. (i) The assumption (4.20) implies $|\Lambda_i| \in \omega(|\Lambda_{i+1}|)$ for all $i = 1, \ldots, n$. This fact, together with condition (4.17), says that Sym_j, as expressed in (4.19), satisfies the assumptions of Proposition 3.4. Therefore,

$$|\mathrm{Sym}_j| \in \Theta(|\Lambda_1\cdots\Lambda_j|), \quad \text{for all} \quad j = 1, \ldots, n. \tag{4.21}$$

(ii) Now let $1 \le j \le n-1$. Using (4.21), it is clear that ($\widetilde{H}_j$) is equivalent to

$$|\Lambda_1\cdots\Lambda_r(\Lambda_1\cdots\Lambda_{j-1})^{r-j}| \in o(|\Lambda_1\cdots\Lambda_j|^{r-j+1}), \quad \text{for all} \quad r = j+1, \ldots, n,$$

or, in other words, to

$$|\Lambda_1\cdots\Lambda_j|^{r-j+1} \in \omega(|\Lambda_1\cdots\Lambda_r(\Lambda_1\cdots\Lambda_{j-1})^{r-j}|), \quad \text{for all} \quad r = j+1, \ldots, n. \tag{4.22}$$

We start by noting that

$$|\Lambda_1\cdots\Lambda_j|^{r-j+1} \in \Theta(|\Lambda_1\cdots\Lambda_{j-1}|^{r-j+1})\Theta(|\Lambda_j|^{r-j+1}).$$

On the other hand, as observed before, $|\Lambda_i| \in \omega(|\Lambda_j|)$ for all $i, j \in \{1, \ldots, n\}$ with $i < j$ and, in particular, $|\Lambda_i| \in \omega(0)$ by Proposition 6.5-(i). Consequently,

$$\Theta(|\Lambda_j|^{r-j+1}) \subseteq \Theta(|\Lambda_j|)\omega(|\Lambda_{j+1}|)\cdots\omega(|\Lambda_r|) \subseteq \omega(|\Lambda_j\cdots\Lambda_r|),$$

where the first inclusion follows from Corollary 6.10 and the second one from Proposition 6.32-(v).

Electronic Journal of Linear Algebra, ISSN 1081-3810
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The two relations above give immediately

$$|\Lambda_1 \cdots \Lambda_j|^{r-j+1} \in \Theta(|\Lambda_1 \cdots \Lambda_{j-1}|^{r-j+1})\omega(|\Lambda_j \cdots \Lambda_r|) = \omega(|\Lambda_1 \cdots \Lambda_{j-1}|^{r-j+1}|\Lambda_j \cdots \Lambda_r|),$$

where the equality follows again from Proposition 6.32-(v). Therefore, (4.22) is proved.

(iii) We want to apply Theorem 3.6 to the function (4.18) in the case $s = n-1$. In order to do this, observe that (4.16) guarantees the assumption (3.5) and that $(\widetilde{H}_j)$ is a stronger form than (H_j). Moreover, $|\mathrm{Sym}_i| \in \omega(0)$ for all $i = 1, \dots, n$ by (4.21) and (4.20). So we get

$$|\mu_1 \dots \mu_j| \in \Theta(|\mathrm{Sym}_j|), \quad \text{for all} \quad j = 0, 1, \dots, n-1.$$

Taking (4.21) and the obvious equality $|\mu_1 \dots \mu_n| = |\mathrm{Sym}_n|$ into account, we then obtain

$$|\mu_1 \dots \mu_j| \in \Theta(|\Lambda_1 \cdots \Lambda_j|)$$

for all $j = 0, 1, \dots, n$ or, equivalently from Proposition 6.32-(iii),

$$\Theta(|\mu_1|) \cdots \Theta(|\mu_j|) = \Theta(|\mu_1 \cdots \mu_j|) = \Theta(|\Lambda_1 \cdots \Lambda_j|) = \Theta(|\Lambda_1|) \cdots \Theta(|\Lambda_j|). \tag{4.23}$$

We have to show that $\Theta(|\mu_j|) = \Theta(|\Lambda_j|)$ for all $j = 1, \dots, n$.
Let us prove it by induction on j. Clearly, for $j = 1$, it is immediate consequence of (4.23).
Assume then that $\Theta(|\mu_t|) = \Theta(|\Lambda_t|)$ for all $t \leq j-1$. Hence, since $|\Lambda_i| \in \omega(0)$ for all i, we can apply Corollary 6.35 (Cancellation Law) to (4.23). In this way, we obtain $\Theta(|\mu_j|) = \Theta(|\Lambda_j|)$ as required. □

For an illustration of property (iii) of Proposition 4.7, we refer the reader to the forthcoming Example 5.6.

Corollary 4.8. *Let the assumptions* (4.20) *and* (4.17) *hold. Then*

$$|\mu_1| \in \omega(|\mu_2|), \quad |\mu_2| \in \omega(|\mu_3|)\,, \dots, \quad |\mu_{n-1}| \in \omega(|\mu_n|).$$

Proof. Along the proof of Proposition 4.7, we already noted that $|\Lambda_i| \in \omega(|\Lambda_{i+1}|)$ for all i. Hence, Proposition 4.7-(iii) implies that $|\mu_i| \in \omega(|\mu_{i+1}|)$ for all $i = 1, \dots, n-1$. □

5. Main result. Now we apply the asymptotic results on the eigenvalues of $\{B^k A\}_{k\geq 1}$ given in the previous section to study the corresponding eigenvectors, obtaining in this way the main result of this paper.

Let $\mathcal{E} = (v_1, \dots, v_n)$ be the canonical basis of $\mathbb{R}^n$.

From now on, each linear endomorphism of $\mathbb{R}^n$ will be identified with the $n \times n$ real matrix associated with it with respect to $\mathcal{E}$.

Notation 5.1. *Define the following subspaces of* $\mathbb{R}^n$*:*

$$V_j := \langle v_1, \dots, v_j \rangle, \quad j = 1, \dots, n,$$
$$\overline{V}_j := \langle v_{j+1}, \dots, v_n \rangle, \quad j = 1, \dots, n-1; \quad \overline{V}_n := \{0\}.$$

We also denote by p_j *(respectively* $\overline{p}_j$*) the canonical projection of* $\mathbb{R}^n$ *on the first (respectively second) summand:*

$$p_j : \mathbb{R}^n = V_j \oplus \overline{V}_j \longrightarrow V_j, \qquad \overline{p}_j : \mathbb{R}^n = V_j \oplus \overline{V}_j \longrightarrow \overline{V}_j.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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In the sequel, we have to deal with $n \times n$ matrices, so it is useful to use, instead of p_j and $\overline{p}_j$, these maps composed with the canonical embeddings e_j (respectively $\overline{e}_j$) of V_j (respectively $\overline{V}_j$) in $\mathbb{R}^n$. Hence, set

$$P_j := e_j p_j, \quad \overline{P}_j := \overline{e}_j \overline{p}_j, \quad j = 1, \ldots, n.$$

Notation 5.2. *For all integers $h, k > 1$, we denote by $\mathbb{I}_k$ the identity matrix of order k and by $\mathbb{O}_{h\times k}$ the null $h \times k$ matrix. If $h = k$, we simply write $\mathbb{O}_k$.*
Finally, if V is any vector space, by $id(V)$ and 0_V we mean, respectively, the identity and the null endomorphism of V.

Remark 5.3. With the above notation, we have that

$$\begin{aligned} \ker(P_j) = \overline{V}_j, \quad P_j(x) = x \iff x \in V_j, \\ \ker(\overline{P}_j) = V_j, \quad \overline{P}_j(x) = x \iff x \in \overline{V}_j \end{aligned} \tag{5.24}$$

and

$$P_j + \overline{P}_j = id(\mathbb{R}^n). \tag{5.25}$$

Finally, observe that, if $j = n$, then $p_n = id(\mathbb{R}^n) = P_n$, while $\overline{p}_n$ is zero everywhere and $\overline{P}_n = 0_{\mathbb{R}^n}$.

The above observations are trivial as soon as we identify all the above endomorphisms with the corresponding matrices referred to the basis $\mathcal{E}$. Namely, the $n \times n$ matrices P_j and $\overline{P}_j$ are

$$P_j = \begin{bmatrix} \mathbb{I}_j & \mathbb{O}_{j\times(n-j)} \\ \mathbb{O}_{(n-j)\times j} & \mathbb{O}_{n-j} \end{bmatrix}, \quad \overline{P}_j = \begin{bmatrix} \mathbb{O}_j & \mathbb{O}_{j\times(n-j)} \\ \mathbb{O}_{(n-j)\times j} & \mathbb{I}_{n-j} \end{bmatrix}.$$

In this framework, consider an invertible matrix $A \in GL(n, \mathbb{R})$. The natural way to associate an endomorphism of V_j to A is to consider

$$V_j \xrightarrow{e_j} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n \xrightarrow{p_j} V_j,$$

that is, $p_j A e_j$. It is immediate to see that the matrix representing $p_j A e_j$ (with respect to the basis $\mathcal{E}$) consists of the upper left $j \times j$ block of A. So, using Definition 4.1,

$$p_j A e_j = A_{1\ldots j}, \quad \text{for all} \quad j = 1, \ldots, n.$$

Conversely, if we take an invertible matrix $M \in GL(j, \mathbb{R})$, that is, a linear endomorphism of V_j, the natural way to extend it to the whole $\mathbb{R}^n$ is

$$\mathbb{R}^n \xrightarrow{p_j} V_j \xrightarrow{M} V_j \xrightarrow{e_j} \mathbb{R}^n,$$

and the resulting matrix is $e_j M p_j$.

The composition of the two operations (restriction and extension) applied to a matrix $A \in GL(n, \mathbb{R})$ clearly gives the endomorphism of $\mathbb{R}^n$:

$$\tilde{A}_j := e_j(p_j A e_j)p_j = P_j A P_j.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Since $\tilde{A}_j = e_j A_{1\ldots j} p_j$, it turns out to be the upper left $j \times j$ block of A surrounded by zeroes, that is,

$$\tilde{A}_j = \begin{bmatrix} A_{1\ldots j} & \mathbb{O}_{j\times(n-j)} \\ \mathbb{O}_{(n-j)\times j} & \mathbb{O}_{n-j} \end{bmatrix}.$$

If $A_{1\ldots j}$ is invertible, we set $\tilde{A}_j^+$ to be the $n \times n$ pseudoinverse matrix obtained surrounding by zeroes the matrix $A_{1\ldots j}^{-1}$, that is,

$$\tilde{A}_j^+ := \begin{bmatrix} A_{1\ldots j}^{-1} & \mathbb{O}_{j\times(n-j)} \\ \mathbb{O}_{(n-j)\times j} & \mathbb{O}_{n-j} \end{bmatrix}.$$

Finally, note that the following equalities hold:

$$\tilde{A}_j^+ \tilde{A}_j = P_j = \tilde{A}_j \tilde{A}_j^+, \quad \text{for all} \quad j = 1, \ldots, n. \tag{5.26}$$

Now consider a matrix $B \in GL(n, \mathbb{R})$ having n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$. Up to a suitable linear transformation, we can assume that the corresponding normalized eigenvectors of B are the elements $v_1, \ldots, v_n$ of the canonical basis $\mathcal{E}$ of $\mathbb{R}^n$. Therefore, the linear endomorphism of $\mathbb{R}^n$ associated with B with respect to $\mathcal{E}$ has a diagonal form and, without loss of generality, we assume that

$$B = \mathrm{Diag}(\lambda_1, \ldots, \lambda_\mathrm{n}).$$

Keeping Notation 5.1 into account, the following properties come from straightforward computations.

LEMMA 5.4. *For all $j = 1, \ldots, n$, the subspaces V_j and $\overline{V}_j$ of $\mathbb{R}^n$ are invariant under the map B. More precisely,*

$$B(V_j) = V_j, \qquad B(\overline{V}_j) = \overline{V}_j.$$

Moreover, for all $k \in \mathbb{N}$:

$$P_j B^k = P_j B^k P_j = B^k P_j, \tag{5.27}$$

$$\overline{P}_j B^k = \overline{P}_j B^k \overline{P}_j = B^k \overline{P}_j, \tag{5.28}$$

$$P_j B^k \overline{P}_j = 0 = \overline{P}_j B^k P_j. \tag{5.29}$$

Finally, we are in a position to state the main result of this paper.

THEOREM 5.5. *Let $A, B \in GL(n, \mathbb{R})$ and assume that B has $\lambda_1, \ldots, \lambda_n$ as (real) eigenvalues and that (4.17) and (4.20) hold. Moreover, let $v_1, \ldots, v_n$ be the normalized eigenvectors of B (i.e., the canonical basis of $\mathbb{R}^n$).*

As usual, for each positive integer k, let $\mu_1(k), \ldots, \mu_n(k)$ be the eigenvalues (possibly complex) of $B^k A$, where

$$|\mu_1(k)| \geq \cdots \geq |\mu_n(k)|.$$

Then we have:

(i) *there exists an integer $\overline{k}$ such that, for all $k \geq \overline{k}$, the eigenvalues $\mu_1(k), \ldots, \mu_n(k)$ are distinct;*

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(ii) *for all* $k \geq \overline{k}$, *denote by* $u_1^{(k)}, \ldots, u_n^{(k)}$ *the normalized eigenvectors of the matrix* $B^k A$ *such that* $\operatorname{vers}(\tilde{A}_j^+ v_j)^T u_j^{(k)} \geq 0$ *(if equality occurs, either of the two options may be selected). Then the sequence* $\{u_j^{(k)}\}_k$ *is convergent for each* $j = 1, \ldots, n$ *and*

$$\lim_{k\to\infty} u_j^{(k)} = \operatorname{vers}(\tilde{A}_j^+ v_j).$$

Proof. (i) It immediately follows from Corollary 4.8.

(ii) Let $k \geq \overline{k}$, so that all the eigenvalues of $B^k A$ are distinct by part (i) and, hence, so are the corresponding eigenvectors are well.

Now choose an index $j \in \{1, \ldots, n\}$. Note that, being $\|u_j^{(k)}\| = 1$, there exists a converging subsequence of $\{u_j^{(k)}\}_k$. So we consider any of such converging subsequences which, for the sake of simplicity, we still denote by $\{u_j^{(k)}\}_k$. We also denote by u_j its limit, which is a normalized vector.

We split the proof into three parts.

Part 1: we prove that $\overline{P}_j u_j = 0$ and $P_j u_j = u_j$.

If $j = n$, the equalities $\overline{P}_n u_n = 0$ and $P_n u_n = u_n$ are trivial.

Thus, we are left to consider $j \in \{1, \ldots, n-1\}$. Possibly by scaling both the matrices A and B by λ_j^{-1}, we can assume that $\lambda_j = 1$; so $|\lambda_i| > 1$ for all $i \leq j-1$ and $|\lambda_i| < 1$ for all $i \geq j+1$. Therefore, we have

$$\lim_{k\to\infty} B^k \overline{P}_j = 0 \quad \text{and} \quad \lim_{k\to\infty} B^{-k} P_j = B_j^{-\infty}, \tag{5.30}$$

where we denote by $B_j^{-\infty}$ the matrix (referred to the basis $\mathcal{E}$) which is null everywhere but the element (j, j) which is 1. In particular, for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, observe that

$$\lim_{k\to\infty} B^{-k} P_j x = B_j^{-\infty} x = x_j v_j = (0, \ldots, 0, x_j, 0, \ldots, 0)^T. \tag{5.31}$$

Moreover, from (5.30), it immediately follows that

$$\lim_{k\to\infty} \tilde{A}_j^+ B^{-k} P_j = \tilde{A}_j^+ B_j^{-\infty}. \tag{5.32}$$

Finally recall that, by assumption,

$$\mu_j(k) u_j^{(k)} = B^k A u_j^{(k)}. \tag{5.33}$$

Since we are assuming $\lambda_j = 1$, by Proposition 4.7, it holds that $|\mu_j| \in \Theta(\Lambda_j) = \Theta(1)$. In particular, $|\mu_j| \in \Omega(1)$ so that the function $\mu_j(k)$ is eventually lower bounded away from zero, that is, there exists a real number $D > 0$ and $\tilde{k} \geq \overline{k}$ such that

$$D \leq |\mu_j(k)|, \quad \text{for} \quad k \geq \tilde{k}. \tag{5.34}$$

In order to show that $\overline{P}_j u_j = 0$, we project the vectors in (5.33) on $\overline{V}_j$ via $\overline{P}_j$ and we obtain

$$\mu_j(k) \overline{P}_j u_j^{(k)} = \overline{P}_j B^k A u_j^{(k)} = B^k \overline{P}_j A u_j^{(k)},$$

where the last equality follows from (5.28). Since $\{A u_j^{(k)}\}$ is bounded, taking the limit of the equality above and using (5.30), we finally obtain

$$\lim_{k\to\infty} \mu_j(k) \overline{P}_j u_j^{(k)} = \lim_{k\to\infty} B^k \overline{P}_j A u_j^{(k)} = 0.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Using (5.34), we get $\lim_{k\to\infty} \overline{P}_j u_j^{(k)} = 0$ and, since $u_j = \lim_{k\to\infty} u_j^{(k)}$, also $\overline{P}_j u_j = 0$. Consequently, (5.25) yields $P_j u_j = u_j$.

Part 2: we prove that $\mathrm{vers}(\tilde{A}_j^+ v_j)$ is the limit of all the converging subsequences considered in Part 1.

From $P_j u_j = u_j$, using (5.24) we obtain that $u_j \in V_j$.
In order to complete the argument, we project the vectors in (5.33) on V_j via P_j and obtain

$$\mu_j(k) P_j u_j^{(k)} = P_j B^k A u_j^{(k)}.$$

Therefore, taking (5.25) into account and applying (5.27), we get

$$\begin{aligned}\mu_j(k) P_j u_j^{(k)} &= P_j B^k A (P_j + \overline{P}_j) u_j^{(k)} = B^k P_j A (P_j + \overline{P}_j) u_j^{(k)} \\ &= B^k \tilde{A}_j u_j^{(k)} + B^k P_j A \overline{P}_j u_j^{(k)},\end{aligned}$$

where the last equality follows from $P_j A P_j = \tilde{A}_j$. Composing both sides with the operator $\tilde{A}_j^+ B^{-k}$ and taking (5.26) into account, we finally obtain

$$\mu_j(k) \tilde{A}_j^+ B^{-k} P_j u_j^{(k)} = P_j u_j^{(k)} + \tilde{A}_j^+ P_j A \overline{P}_j u_j^{(k)}. \tag{5.35}$$

Taking the limit of the equality above and using the equalities $\overline{P}_j u_j = 0$ and $P_j u_j = u_j$, we have that the right-hand side converges to u_j, clearly nonzero. Therefore, also the left-hand side does the same.

Note that, by (5.32), the sequence $\tilde{A}_j^+ B^{-k} P_j u_j^{(k)}$ converges to $\tilde{A}_j^+ B_j^{-\infty} u_j$. Thus, also the sequence $\{\mu_j(k)\}_k$ converges to a nonzero limit and we can set

$$\mu_j^\infty := \lim_{k\to\infty} \mu_j(k) \neq 0.$$

Therefore, the limit of (5.35) is

$$\mu_j^\infty \tilde{A}_j^+ B_j^{-\infty} u_j = u_j.$$

So, taking (5.31) into account, we have $B_j^{-\infty} u_j = \alpha v_j$, where $\alpha = (u_j)_j$. In this way, we finally obtain

$$\mu_j^\infty \alpha \tilde{A}_j^+ v_j = u_j,$$

and, since $\mathrm{vers}(\tilde{A}_j^+ v_j)^T u_j^{(k)} \geq 0$ by assumption and since $\|u_j\| = 1$, we get $u_j = \mathrm{vers}(\tilde{A}_j^+ v_j)$, as required.

Part 3: Conclusion.

In the previous part, we have proved that any converging subsequence of $\{u_j^{(k)}\}_k$ has a limit equal to the normalized vector $u_j = \mathrm{vers}(\tilde{A}_j^+ v_j)$. Now assume by contradiction that the whole sequence is not converging to this limit. Then there exists a subsequence, $\{\tilde{u}_j^{(k)}\}_k$ say, and a constant $K > 0$ such that $\|\tilde{u}_j^{(k)} - u_j\| > K$ eventually. Clearly, also the elements of such subsequence are normalized and, consequently, by using again the previous arguments, we obtain the existence of a subsequence of $\{\tilde{u}_j^{(k)}\}_k$ that converges to u_j, making the absurde. □

Example 5.6. We consider the pair of 2×2-matrices

$$A = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} \\ -\frac{5}{2} & \frac{11}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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already introduced in Example 3 of [4], which satisfy conditions (4.17) and (4.20). In fact,

$$\det(A_1) = -\frac{2}{9}, \quad \det(A_{12}) = \det(A) = \frac{1}{6},$$

and

$$\lambda_1 = 3, \quad \lambda_2 = \frac{1}{3}.$$

It turns out that

$$\tilde{A}_1^+ = \begin{bmatrix} -\frac{9}{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_2^+ = A^{-1} = \begin{bmatrix} 33 & -\frac{10}{3} \\ 15 & -\frac{4}{3} \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which imply

$$u_1 = \mathrm{vers}(\tilde{A}_1^+ v_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \mathrm{vers}(\tilde{A}_2^+ v_2) = \begin{bmatrix} \frac{5}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \end{bmatrix} = \begin{bmatrix} 0.928476690885259... \\ 0.371390676354103... \end{bmatrix}.$$

In order to illustrate the validity of property (iii) of Proposition 4.7, we compute

$$\begin{aligned} |\mu_1(1)/\lambda_1| &= 0.333333333333333..., & |\mu_2(1)/\lambda_2| &= 0.333333333333333..., \\ |\mu_1(2)/\lambda_1^2| &= 0.139579470420236..., & |\mu_2(2)/\lambda_2^2| &= 1.194062895960829..., \\ |\mu_1(3)/\lambda_1^3| &= 0.213607341785639..., & |\mu_2(3)/\lambda_2^3| &= 0.780247838269162..., \\ |\mu_1(4)/\lambda_1^4| &= 0.221269131148016..., & |\mu_2(4)/\lambda_2^4| &= 0.753230537861045..., \\ |\mu_1(5)/\lambda_1^5| &= 0.222116371870562..., & |\mu_2(5)/\lambda_2^5| &= 0.750357415183206..., \end{aligned}$$

whose behavior suggests that $|\mu_1(k)/\lambda_1^k|$ and $|\mu_2(k)/\lambda_2^k|$ have a limit as $k \to \infty$.

Moreover, the validity of (ii) of Theorem 5.5 is supported by the computed eigenvectors:

$$u_1^{(1)} = \begin{bmatrix} 0.707106781186547... \\ 0.707106781186547... \end{bmatrix}, \quad u_2^{(1)} = \begin{bmatrix} 0.894427190999915... \\ 0.447213595499957... \end{bmatrix},$$

$$u_1^{(2)} = \begin{bmatrix} 0.989115991126860... \\ 0.147137881244529... \end{bmatrix}, \quad u_2^{(2)} = \begin{bmatrix} 0.936801009986018... \\ 0.349862641173897... \end{bmatrix},$$

$$u_1^{(3)} = \begin{bmatrix} 0.999879791491405... \\ 0.015504920738377... \end{bmatrix}, \quad u_2^{(3)} = \begin{bmatrix} 0.929092626659850... \\ 0.369847118531833... \end{bmatrix},$$

$$u_1^{(4)} = \begin{bmatrix} 0.999998528423443... \\ 0.001715561408985... \end{bmatrix}, \quad u_2^{(4)} = \begin{bmatrix} 0.928542842067220... \\ 0.371225255667942... \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

$$u_1^{(5)} = \begin{bmatrix} 0.999999981849039... \\ \\ 0.000190530629530... \end{bmatrix}, \quad u_2^{(5)} = \begin{bmatrix} 0.928484013966713... \\ \\ 0.371372368127006... \end{bmatrix},$$

which clearly seem to converge to u_1 and u_2, respectively, as $k \to \infty$.

6. Appendix on Bachmann–Landau symbols. The topic considered in this note has been widely studied in the literature, starting from the works by Bachmann [1] and Landau [9]. We also mention Hardy [6] and the important successive contribution by Knuth [8].

However, there are some slight differences among the various treatments, both in the notation and in the basic definitions themselves. Therefore, although most of the properties and results are simple to prove and extensively used, here we collect them in a systematic way and give the proofs that are less immediate.

We first establish the notation and the definitions we are intended to use.

NOTATION 6.1. *Let $\mathcal{D}$ be a subset of $\mathbb{R}$ and let $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ be an accumulation point of $\mathcal{D}$. We denote by $\mathcal{F}^+_{x_0}(\mathcal{D})$, or simply $\mathcal{F}^+$, the set of the real nonnegative functions:*

$$\mathcal{F}^+ := \{f : \mathcal{D} \setminus \{x_0\} \longrightarrow \mathbb{R}^+\},$$

defined on $\mathcal{D} \setminus \{x_0\}$, where $\mathbb{R}^+$ is the set of the nonnegative real numbers.

DEFINITION 6.2. *If $f, g \in \mathcal{F}^+$, we say that $f(x) \leq g(x)$ ($<, \geq, >, =$, respectively) eventually (near x_0) if there exists a neighborhood U of x_0 such that $f(x) \leq g(x)$ ($<, \geq, >, =$, respectively) for all $x \in U \cap (\mathcal{D} \setminus \{x_0\})$.*

DEFINITION 6.3. *For all $f \in \mathcal{F}^+$, we define the following subsets of $\mathcal{F}^+$:*

(a) *$\Theta_{x_0}(f)$, or simply $\Theta(f)$, is the set*

$$\Theta(f) := \{g \in \mathcal{F}^+ \mid \text{there exist } c_1, c_2 > 0 \text{ such that } c_1 f(x) \leq g(x) \leq c_2 f(x) \text{ eventually}\};$$

(b) *$\mathcal{O}_{x_0}(f)$, or simply $\mathcal{O}(f)$, is the set*

$$\mathcal{O}(f) := \{g \in \mathcal{F}^+ \mid \text{there exists } c > 0 \text{ such that } g(x) \leq c f(x) \text{ eventually}\};$$

(c) *$o_{x_0}(f)$, or simply $o(f)$, is the set*

$$o(f) := \{g \in \mathcal{F}^+ \mid \text{for all } c > 0 \text{ it holds that } g(x) < c f(x) \text{ eventually}\};$$

(d) *$\Omega_{x_0}(f)$, or simply $\Omega(f)$, is the set*

$$\Omega(f) := \{g \in \mathcal{F}^+ \mid \text{there exists } c > 0 \text{ such that } g(x) \geq c f(x) \text{ eventually}\};$$

(e) *$\omega_{x_0}(f)$, or simply $\omega(f)$, is the set*

$$\omega(f) := \{g \in \mathcal{F}^+ \mid \text{for all } c > 0 \text{ it holds that } g(x) > c f(x) \text{ eventually}\}.$$

The symbols Θ, $\mathcal{O}$, Ω, o, and ω are called Bachmann–Landau symbols.

REMARK 6.4. Denoting by "0" the everywhere zero function, it holds that $\Omega(0) = \mathcal{F}^+$, while $\Theta(0) = \mathcal{O}(0)$ is the set of the eventually zero functions. Furthermore, $o(0) = \emptyset$, while $\omega(0) = \{f \in \mathcal{F}^+ \mid f(x) > 0 \text{ eventually}\}$ is the set of the eventually positive functions.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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The following results regarding the set $\omega(0)$ of the eventually positive functions are immediate to be checked.

PROPOSITION 6.5. *For all $f \in \mathcal{F}^+$ it holds that:*

(i) $\omega(f) \subseteq \omega(0)$;
(ii) $o(f) \neq \emptyset \iff f \in \omega(0)$.

REMARK 6.6. It is easy to see that the Θ symbol determines an equivalence relation on $\mathcal{F}^+$, that is, for all $f, g \in \mathcal{F}^+$ it holds that

$$f \in \Theta(g) \iff g \in \Theta(f) \iff \Theta(f) = \Theta(g).$$

DEFINITION 6.7. *Given $f \in \mathcal{F}^+$, the equivalence class $\Theta(f)$ is said to be the order of f for $x \longrightarrow x_0$ on the domain $\mathcal{D}$.*

Moreover, if $f, g \in \mathcal{F}^+$ are such that $\Theta(f) = \Theta(g)$, then we say that they have the same order.

The next list of properties of the Bachmann–Landau symbols is an easy consequence of Definition 6.3.

PROPOSITION 6.8. *For all $f, g \in \mathcal{F}^+$ it holds that:*

(i) $\mathcal{O}(f) \cap \Omega(f) = \Theta(f)$, $\Theta(f) \subsetneq \Omega(f)$ *and, if* $f \notin \Theta(0)$, *then* $\Theta(f) \subsetneq \mathcal{O}(f)$;
(ii) $\Theta(f) = \Theta(g) \iff \mathcal{O}(f) = \mathcal{O}(g) \iff \Omega(f) = \Omega(g)$;
(iii) $\mathcal{O}(f) \subseteq \mathcal{O}(g) \iff \Omega(f) \supseteq \Omega(g)$;
(iv) $g \in \mathcal{O}(f) \iff \Theta(g) \subseteq \mathcal{O}(f) \iff \Theta(f) \subseteq \Omega(g) \iff f \in \Omega(g)$;
(v) $g \in o(f) \iff f \in \omega(g)$;
(vi) $g \in o(f) \iff \Theta(g) \subseteq o(f) \iff \mathcal{O}(g) \subseteq o(f)$;
(vii) $g \in \omega(f) \iff \Theta(g) \subseteq \omega(f) \iff \Omega(g) \subseteq \omega(f)$;
(viii) $\Theta(f) = \Theta(g) \Longrightarrow \{o(f) = o(g)$ *and* $\omega(f) = \omega(g)\}$;
(ix) $o(f) \subsetneq \mathcal{O}(f)$ *and, dually,* $\omega(f) \subsetneq \Omega(f)$;
(x) $\mathcal{O}(f) \cap \omega(f) = \emptyset$ *and, dually,* $o(f) \cap \Omega(f) = \emptyset$.

NOTATION 6.9. *By writing $BL(f)$, we mean any Bachmann–Landau symbol of a function $f \in \mathcal{F}^+$.*

The next result is an obvious corollary to Remark 6.6 and Proposition 6.8-(iv,vi,vii).

COROLLARY 6.10. *For all $f, g \in \mathcal{F}^+$ and for all symbols BL it holds that*

$$g \in BL(f) \iff \Theta(g) \subseteq BL(f).$$

REMARK 6.11. It is easy to see that the $\mathcal{O}$ and o symbols determine a partial order and a strict partial order relation, respectively, with the latter stronger than the former, on the quotient set of the equivalence relation determined by the Θ symbol. In fact, for all $f, g, h \in \mathcal{F}^+$ it holds that:

(i) $\{\Theta(f) \subseteq \mathcal{O}(g)$ and $\Theta(g) \subseteq \mathcal{O}(h)\} \Longrightarrow \Theta(f) \subseteq \mathcal{O}(h)$;
(ii) $o(f) \cap \Theta(f) = \emptyset$;
(iii) $\{\Theta(f) \subseteq o(g)$ and $\Theta(g) \subseteq o(h)\} \Longrightarrow \Theta(f) \subseteq o(h)$;
(iv) $\{\Theta(f) \subseteq \mathcal{O}(g)$ and $\Theta(g) \subseteq o(h)\} \Longrightarrow \Theta(f) \subseteq o(h)$;
(v) $\{\Theta(f) \subseteq o(g)$ and $\Theta(g) \subseteq \mathcal{O}(h)\} \Longrightarrow \Theta(f) \subseteq o(h)$.

Moreover, the dual symbols Ω and ω determine order relations, in some sense "opposite", such that:

(vi) $\{\Theta(f) \subseteq \Omega(g)$ and $\Theta(g) \subseteq \Omega(h)\} \Longrightarrow \Theta(f) \subseteq \Omega(h)$;

Electronic Journal of Linear Algebra, ISSN 1081-3810
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(vii) $\omega(f) \cap \Theta(f) = \emptyset$;
(viii) $\{\Theta(f) \subseteq \omega(g) \text{ and } \Theta(g) \subseteq \omega(h)\} \Longrightarrow \Theta(f) \subseteq \omega(h)$;
(ix) $\{\Theta(f) \subseteq \Omega(g) \text{ and } \Theta(g) \subseteq \omega(h)\} \Longrightarrow \Theta(f) \subseteq \omega(h)$;
(x) $\{\Theta(f) \subseteq \omega(g) \text{ and } \Theta(g) \subseteq \Omega(h)\} \Longrightarrow \Theta(f) \subseteq \omega(h)$.

NOTATION 6.12. *In the sequel, by $\mathcal{D}'$ we mean a subdomain of $\mathcal{D}$ having the same accumulation point x_0 and, for any function $f \in \mathcal{F}^+$, we still denote its restriction $f_{|\mathcal{D}'}$ by f.*

The following result is obvious by Definition 6.2.

PROPOSITION 6.13. *Let $f, g \in \mathcal{F}^+$, and $g \in BL(f)$. Then $g \in BL(f)$ on $\mathcal{D}'$ for all subdomains $\mathcal{D}' \subseteq \mathcal{D}$.*

LEMMA 6.14. *Let $f, h \in \mathcal{F}^+$, $\mathcal{D}' \subseteq \mathcal{D}$, and $h \in BL(f)$ on the subdomain $\mathcal{D}'$. Moreover, if $BL = o$, we also assume that $f \in \omega(0)$. Then there exists $\tilde{h} \in \mathcal{F}^+$ such that $\tilde{h}_{|\mathcal{D}'} = h_{|\mathcal{D}'}$ and $\tilde{h} \in BL(f)$ on the whole domain $\mathcal{D}$.*

Proof. First note that $BL(f) \neq \emptyset$ on the whole domain $\mathcal{D}$ also in the case $BL = o$, since we assume $f \in \omega(0)$ and thanks to Proposition 6.5-(ii). Therefore, in any case, we can consider $k \in \mathcal{F}^+$ such that $k \in BL(f)$ and extend $h_{|\mathcal{D}'}$ to the whole domain $\mathcal{D}$ by setting

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \in \mathcal{D}' \\ k(x) & \text{if } x \notin \mathcal{D}'. \end{cases}$$

Now, it is immediate to check that $\tilde{h} \in BL(f)$ on $\mathcal{D}$, concluding the proof. □

We can conclude with the following result.

PROPOSITION 6.15. *Let $f, g \in \mathcal{F}^+$, and $BL_1(f) \subseteq BL_2(g)$, where BL_1 and BL_2 are two, possibly equal to each other, Bachmann–Landau symbols. Moreover, if $BL_1 = o$, we also assume that $f \in \omega(0)$. Then it also holds that $BL_1(f) \subseteq BL_2(g)$ on $\mathcal{D}'$ for all $\mathcal{D}' \subseteq \mathcal{D}$.*

Proof. First note that, like in Lemma 6.14, our assumptions imply that $BL_1(f) \neq \emptyset$ on the whole domain $\mathcal{D}$ (and, consequently, on any $\mathcal{D}' \subseteq \mathcal{D}$ by Proposition 6.13) also in the case $BL_1 = o$. Then, for a given subdomain $\mathcal{D}'$, we consider $h \in \mathcal{F}^+$ such that $h \in BL_1(f)$ on $\mathcal{D}'$ (note that the values attained by h outside $\mathcal{D}'$ do not matter).

By Lemma 6.14, there exists $\tilde{h} \in \mathcal{F}^+$ such that $\tilde{h}_{|\mathcal{D}'} = h_{|\mathcal{D}'}$ and $\tilde{h} \in BL_1(f) \subseteq BL_2(g)$ on the whole domain $\mathcal{D}$. Now Proposition 6.13 allows us to conclude that $\tilde{h} \in BL_2(g)$ on $\mathcal{D}'$, that is, $h \in BL_2(g)$ on $\mathcal{D}'$, as well. Therefore, the arbitrariness of h concludes the proof. □

The next corollary is straightforward.

COROLLARY 6.16. *Let $f, g \in \mathcal{F}^+$, and $BL(f) = BL(g)$. Moreover, if $BL = o$, we also assume that $f, g \in \omega(0)$. Then $BL(f) = BL(g)$ on $\mathcal{D}'$ for all $\mathcal{D}' \subseteq \mathcal{D}$.*

Although Proposition 6.8-(x) tells us that the sets $\mathcal{O}(f)$ and $\omega(f)$ are disjoint (and, analogously, the sets $\Omega(f)$ and $o(f)$), they do not form a partition of $\mathcal{F}^+$. Nevertheless, now we shall prove that they do this in a "weak sense".

PROPOSITION 6.17. *Let $f, g \in \mathcal{F}^+$. Then the following statements hold:*

(i) *$g \notin \mathcal{O}(f) \iff g \in \omega(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$;*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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(ii) *$g \notin \Omega(f) \iff g \in o(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$;*
(iii) *$g \notin o(f) \iff g \in \Omega(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$;*
(iv) *$g \notin \omega(f) \iff g \in \mathcal{O}(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$.*

Proof. (i) If $g \notin \mathcal{O}(f)$ then, for each $c > 0$ and for each neighborhood I_{x_0}, there exists a point $x_c \in I_{x_0}$ such that $g(x_c) > cf(x_c)$. In particular, one can find a sequence $(x_n)_n$, $x_n \longrightarrow x_0$, such that $g(x_n) > M_n f(x_n)$, where the M_n's are chosen in such a way that $M_n \longrightarrow +\infty$ and $(M_n)_n$ is a strictly increasing sequence. Therefore, $g \in \omega(f)$ on $\mathcal{D}' := (x_n)_n$.

Conversely, if $g \in \omega(f)$ on $\mathcal{D}'$, then by Proposition 6.8-(x) it holds that $g \notin \mathcal{O}(f)$ on $\mathcal{D}'$. Therefore, by Proposition 6.13, $g \notin \mathcal{O}(f)$ on $\mathcal{D}$ either.

(ii) It follows from the previous point (i) and Proposition 6.8-(iv,v).

(iii) If $g \notin o(f)$, then there exists $c > 0$ such that, for each neighborhood I_{x_0}, there exists a point $x_I \in I_{x_0}$ such that $g(x_I) \geq cf(x_I)$. Hence, there exists a sequence $(x_n)_n$, $x_n \longrightarrow x_0$, such that $g(x_n) \geq cf(x_n)$. Therefore, we get $g \in \Omega(f)$ on $\mathcal{D}' := (x_n)_n$.

Conversely, if $g \in \Omega(f)$ on $\mathcal{D}'$, then by Proposition 6.8-(x) it holds that $g \notin o(f)$ on $\mathcal{D}'$. Therefore, by Proposition 6.13, $g \notin o(f)$ on $\mathcal{D}$ either.

(iv) It follows from the previous point (iii) and Proposition 6.8-(iv,v). □

Corollary 6.18. *Let $f, g \in \mathcal{F}^+$. Then it holds that:*

(i) *if $g \in \mathcal{O}(f) \setminus o(f)$, then $g \in \Theta(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$;*
(ii) *if $g \in \Omega(f) \setminus \omega(f)$, then $g \in \Theta(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$.*

The next three results suitably extend Proposition 6.8-(ii, iii) to the o and ω symbols.

Proposition 6.19. *Let $f, g \in \mathcal{F}^+$. Then it holds that:*

(i) *$\omega(f) \supseteq \omega(g) \Longrightarrow o(f) \subseteq o(g)$;*
(ii) *if $f \in \omega(0)$, then $o(f) \subseteq o(g) \iff \omega(f) \supseteq \omega(g)$;*
(iii) *if $f, g \in \omega(0)$, then $o(f) = o(g) \iff \omega(f) = \omega(g)$.*

Proof. (i) If $f \notin \omega(0)$, then $o(f) = \emptyset$ by Proposition 6.5-(ii) and, hence, the statement is clearly true. Thus, we assume that $f \in \omega(0)$ and consider $h \in o(f)$. If $h \notin o(g)$, then $h \in \Omega(g)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$ by Proposition 6.17-(iii) and, consequently, $g \in \mathcal{O}(h)$ on $\mathcal{D}'$. Therefore, Remark 6.11-(iv) and Proposition 6.13 yield $g \in o(f)$ on $\mathcal{D}'$ and so, by Proposition 6.8-(v), $f \in \omega(g)$ on $\mathcal{D}'$. Finally, Proposition 6.15 assures that $\omega(g) \subseteq \omega(f)$ also on $\mathcal{D}'$, leading us to the the absurd inclusion $f \in \omega(f)$ on $\mathcal{D}'$ (see Remark 6.11-(vii)). Thus, we can conclude that, necessarily, $h \in o(g)$.

(ii) We need to prove the opposite implication to (i). To this purpose, we consider $h \in \omega(g)$ and, assuming that $h \notin \omega(f)$, by using the dual version of the previous arguments, we arrive at $g \in o(f)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$. Since $f \in \omega(0)$, this time Proposition 6.15 assures that $o(f) \subseteq o(g)$ also on $\mathcal{D}'$, leading us to the the absurd inclusion $g \in o(g)$ on $\mathcal{D}'$ (see Remark 6.11-(ii)). Thus, we can conclude that, necessarily, $h \in \omega(f)$.

(iii) Since also $g \in \omega(0)$, we can interchange the roles of f and g in (ii) and get the desired result. □

Proposition 6.20. *Let $f, g \in \mathcal{F}^+$. Then*

$$\Theta(f) = \Theta(g) \iff \omega(f) = \omega(g).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Proof. Clearly, it is sufficient to prove the opposite implication to Proposition 6.8-(viii). Thus, we assume that $\omega(f) = \omega(g)$. If $f \notin \Theta(g)$, then necessarily either $f \notin \mathcal{O}(g)$ or $f \notin \Omega(g)$. In the first case, by Proposition 6.17-(i), there exists a suitable $\mathcal{D}' \subseteq \mathcal{D}$ where $f \in \omega(g)$. In the second case, by Proposition 6.17-(ii), there exists a suitable $\mathcal{D}'' \subseteq \mathcal{D}$ where $f \in o(g)$ and this means that $g \in \omega(f)$ on $\mathcal{D}''$ (see Proposition 6.8-(v)). On the other hand, Corollary 6.16 assures that $\omega(g) = \omega(f)$ also on $\mathcal{D}'$ and $\mathcal{D}''$, yielding the absurd inclusion $f \in \omega(f)$ in the first case and $g \in \omega(g)$ in the second case (see Remark 6.11-(vii)). Thus, we can conclude that it must be $f \in \Theta(g)$ and, consequently, $\Theta(f) = \Theta(g)$. ∎

Corollary 6.21. *Let $f, g \in \omega(0)$. Then*

$$\Theta(f) = \Theta(g) \iff o(f) = o(g).$$

Proof. The result is immediately obtained by combining Proposition 6.20 and Proposition 6.19-(iii). ∎

Remark 6.22. Observe that the hypothesis $f \in \omega(0)$ in Proposition 6.19-(ii) is essential. In fact, if $f \notin \omega(0)$, Proposition 6.5-(ii) implies that $o(f) = \emptyset \subseteq o(0) = \emptyset$. On the other hand, if $f \notin \Theta(0)$ either, then Proposition 6.20 yields $\omega(f) \neq \omega(0)$ and, consequently, Proposition 6.5-(i) implies $\omega(f) \not\supseteq \omega(0)$.

Observe that in the points (c) and (e) of Definition 6.3, we impose the strict inequalities. This allows us to introduce the following notion.

Definition 6.23. *If $f \in \omega(0)$, we define*

$$f^{-1}(x) := \begin{cases} (f(x))^{-1} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

and, since $\Theta(f) \subseteq \omega(0)$, we also define

$$(\Theta(f))^{-1} := \{h^{-1} \mid h \in \Theta(f)\}.$$

Observe that defining $f^{-1}(x) := 0$ whenever $f(x) = 0$ allows us to conclude that $(f^{-1})^{-1}(x) = f(x)$ for all $x \in \mathcal{D}$, although, in principle, this is an inconsistent definition. Anyway, this fact has no consequences at all on the order of f^{-1} as both f and f^{-1} are eventually positive and their values are insignificant outside a neighborhood of the accumulation point x_0 of $\mathcal{D}$.

Proposition 6.24. *Let $f \in \omega(0)$. Then*

$$\Theta(f^{-1}) = (\Theta(f))^{-1}.$$

Proof. Note first that $f^{-1} \in \omega(0)$ as well. Hence, $\Theta(f^{-1}) \subseteq \omega(0)$ by Proposition 6.8-(vii). On the other hand, by definition, it is also clear that $(\Theta(f))^{-1} \subseteq \omega(0)$.

Now observe that $h \in \Theta(f^{-1})$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 f^{-1}(x) \leq h(x) \leq c_2 f^{-1}(x)$$

eventually. But this is equivalent to

$$\frac{f(x)}{c_2} \leq \frac{1}{h(x)} \leq \frac{f(x)}{c_1}$$

eventually, which, in turn, is equivalent to $h^{-1} \in \Theta(f)$, that is, $h \in (\Theta(f))^{-1}$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

The following result is easy to prove.

LEMMA 6.25. *Let $f \in \omega(0)$, $h \in \Theta(f)$ and $g \in \mathcal{F}^+$. Then the following facts hold for all symbols BL:*

(i) $k \in BL(g) \Longrightarrow hk \in BL(fg)$;
(ii) $k \in BL(fg) \Longrightarrow h^{-1}k \in BL(g)$.

Concerning the "sum" of Bachmann–Landau symbols (intended in the natural sense), for any BL it is straightforward to see that

$$BL(f) + BL(g) \subseteq BL(f+g). \tag{6.36}$$

To always give a meaning to the left-hand side of (6.36), in the case $BL = o$, we assume that $f, g \in \omega(0)$, so that $o(f) \neq \emptyset$ and $o(g) \neq \emptyset$ (see Proposition 6.5-(ii)).

Indeed, also the opposite inclusions hold, as is illustrated by the next Proposition 6.26.

PROPOSITION 6.26. *If $f, g \in \mathcal{F}^+$, then the following statements hold:*

(i) $\mathcal{O}(f+g) = \mathcal{O}(f) + \mathcal{O}(g)$;
(ii) $\Omega(f+g) = \Omega(f) + \Omega(g) = \Theta(f) + \Omega(g) = \Omega(f) + \Theta(g)$;
(iii) $\Theta(f+g) = \Theta(f) + \Theta(g)$;
(iv) *if $f, g \in \omega(0)$, then* $o(f+g) = o(f) + o(g)$;
(v) $\omega(f+g) = \omega(f) + \omega(g)$.

Proof. Observe that, in all cases, it is sufficient to prove the opposite inclusions to (6.36).

(i) Let $h \in \mathcal{O}(f+g)$. So there exists $c > 0$ such that $h(x) \leq c(f(x) + g(x))$ eventually. If we define $F(x) := \max\{0, h(x) - cg(x)\}$ and $G(x) := h(x) - F(x)$, we clearly have that $F \in \mathcal{O}(f)$, $G \in \mathcal{O}(g)$ and $h(x) = F(x) + G(x)$ for all x, concluding the proof.

(ii) Except for exchanging the roles of f and g and taking Proposition 6.8-(i) into account, we only need to show that $\Omega(f+g) \subseteq \Theta(f) + \Omega(g)$. Therefore, let us consider $h \in \Omega(f+g)$. So there exists $c > 0$ such that $h(x) \geq c(f(x) + g(x))$ eventually. If we define $F(x) := cf(x)$ and $G(x) := h(x) - cf(x)$, we clearly have that $F \in \Theta(f)$, $G \in \Omega(g)$ and $h(x) = F(x) + G(x)$ for all x. Hence, $h \in \Theta(f) + \Omega(g)$, as required.

(iii) Consider $h \in \Theta(f+g)$. So there exist two constants $c_1, c_2 > 0$ such that

$$c_1\,(f(x) + g(x)) \leq h(x) \leq c_2\,(f(x) + g(x)),$$

eventually. If we define

$$F(x) := \begin{cases} c_1 f(x) & \text{if } f(x) \leq g(x) \\ h(x) - c_1 g(x) & \text{if } f(x) > g(x) \end{cases} \quad \text{and} \quad G(x) := \begin{cases} h(x) - c_1 f(x) & \text{if } f(x) \leq g(x) \\ c_1 g(x) & \text{if } f(x) > g(x) \end{cases}$$

we obviously get $h(x) = F(x) + G(x)$ for all x. In order to show that $F \in \Theta(f)$ and $G \in \Theta(g)$, it is enough to observe that, eventually, we have

$$c_1 f(x) \leq F(x) = h(x) - c_1 g(x) \leq (2c_2 - c_1) f(x)$$

whenever $f(x) > g(x)$ and

$$c_1 g(x) \leq G(x) = h(x) - c_1 f(x) \leq (2c_2 - c_1) g(x)$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

whenever $f(x) \leq g(x)$. This proves that $h \in \Theta(f) + \Theta(g)$.

(iv) Let $h \in o(f+g)$. Since $f, g \in \omega(0)$, we have that $f+g \in \omega(0)$ as well. Therefore, $(f+g)^{-1}h \in o(1)$ by Lemma 6.25-(ii) and, in turn, $F := (f+g)^{-1}hf \in o(f)$ and $G := (f+g)^{-1}hg \in o(g)$ by Lemma 6.25-(i). Moreover, $h(x) = F(x) + G(x)$ eventually, implying that $h \in o(f) + o(g)$.

(v) Since $\omega(f+g) \subseteq \omega(f) \cap \omega(g)$, if $h \in \omega(f+g)$, then $h = \frac{1}{2}h + \frac{1}{2}h \in \omega(f) + \omega(g)$. ∎

The following corollary to Proposition 6.26-(ii), which regards the symbol Ω, is straightforward.

Corollary 6.27. *For all $f \in \mathcal{F}^+$, it holds that*

$$\Theta(f) + \Omega(f) = \Omega(f).$$

As for the other symbols, the situation is illustrated by the next proposition, whose proof is once again rather easy.

Proposition 6.28. *For all $f \in \mathcal{F}^+$ the following properties hold:*

(i) $\Theta(f) + \mathcal{O}(f) = \Theta(f)$;
(ii) *if* $f \in \omega(0)$, *then* $\Theta(f) + o(f) = \Theta(f)$ *and* $\mathcal{O}(f) + o(f) = \mathcal{O}(f)$;
(iii) $\Theta(f) + \omega(f) = \Omega(f) + \omega(f) = \omega(f)$.

Regarding the symbol ω, as a complement to Propositions 6.26-(v) and 6.28-(iii), we have the following result.

Proposition 6.29. *For all $f, g \in \mathcal{F}^+$, it holds that*

$$\omega(f+g) \subseteq \Theta(f) + \omega(g) \quad \textit{and, simmetrically,} \quad \omega(f+g) \subseteq \omega(f) + \Theta(g).$$

Proof. Except for exchanging the roles of f and g, we only need to show the former inclusion. Therefore, let us consider $h \in \omega(f+g)$. So for all $c > 0$, we have that $h(x) > c(f(x) + g(x))$ eventually. If we define $F(x) := f(x)$ and $G(x) := h(x) - f(x)$, we clearly have that $h(x) = F(x) + G(x)$ for all x, $F \in \Theta(f)$ and, for all $c > 1$, $G(x) > (c-1)f(x) + cg(x) > cg(x)$ eventually, that is, $G \in \omega(g)$. Hence, we can conclude that $h \in \Theta(f) + \omega(g)$, as required. ∎

Remark 6.30. In general, the opposite inclusions are not true. To see this, we assume that $f \in \omega(g)$ and consider $h \in \omega(g) \cap o(f)$ (we choose, for instance, $g \in \omega(0)$ and $h = \sqrt{fg}$). Then Proposition 6.28-(ii) yields $f + h \in \Theta(f)$ and, consequently, Proposition 6.8-(x) implies that $f + h \notin \omega(f)$. Since $\omega(f+g) \subseteq \omega(f)$, we conclude that $\Theta(f) + \omega(g) \not\subseteq \omega(f+g)$.

Concerning the "product" of Bachmann–Landau symbols (intended in the natural sense), for any BL it is straightforward to see that

$$BL(f)\ BL(g) \subseteq BL(fg). \tag{6.37}$$

Once again, to always give a meaning to the left-hand side of (6.37); in the case $BL = o$, we assume that $f, g \in \omega(0)$ so that $o(f) \neq \emptyset$ and $o(g) \neq \emptyset$.

In this framework, Lemma 6.25 may be equivalently restated as follows.

Lemma 6.31. *Let $f \in \omega(0)$ and $g \in \mathcal{F}^+$. Then the following facts hold for all symbols BL:*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

(i) $\Theta(f)\, BL(g) \subseteq BL(fg)$;
(ii) $\Theta(f^{-1})\, BL(fg) \subseteq BL(g)$.

Indeed, also the opposite inclusions to (6.37) and to Lemma 6.31-(i) hold, as will be illustrated by the next Proposition 6.32.

Proposition 6.32. *If $f, g \in \mathcal{F}^+$, then the following equalities hold:*

(i) $\mathcal{O}(fg) = \mathcal{O}(f)\mathcal{O}(g) = \Theta(f)\mathcal{O}(g) = \mathcal{O}(f)\Theta(g)$;
(ii) $\Omega(fg) = \Omega(f)\Omega(g)$ *and, if* $f \in \omega(0)$, *then* $\Omega(fg) = \Theta(f)\Omega(g)$;
(iii) $\Theta(fg) = \Theta(f)\Theta(g)$;
(iv) *if* $f, g \in \omega(0)$, *then* $o(fg) = o(f)o(g) = \Theta(f)o(g) = o(f)\Theta(g)$;
(v) $\omega(fg) = \omega(f)\omega(g)$ *and, if* $f \in \omega(0)$, *then* $\omega(fg) = \Theta(f)\omega(g)$.

Proof. Observe that, in all cases, it is sufficient to prove the opposite inclusions to (6.37). To this purpose, for given $f, g, h \in \mathcal{F}^+$, we define the functions:

$$F(x) := \begin{cases} f(x) & \text{if } f(x) \neq 0 \\ \frac{h(x)}{g(x)} & \text{if } f(x) = 0,\ g(x) \neq 0 \\ \sqrt{h(x)} & \text{if } f(x) = g(x) = 0 \end{cases} \quad \text{and} \quad G(x) := \begin{cases} \frac{h(x)}{f(x)} & \text{if } f(x) \neq 0 \\ g(x) & \text{if } f(x) = 0,\ g(x) \neq 0 \\ \sqrt{h(x)} & \text{if } f(x) = g(x) = 0 \end{cases}$$

which are such that $h(x) = F(x)G(x)$ for all x.

(i) Except for exchanging the roles of f and g and taking Proposition 6.8-(i) into account, we only need to show that $\mathcal{O}(fg) \subseteq \Theta(f)\mathcal{O}(g)$. To this purpose, let us consider $h \in \mathcal{O}(fg)$, that is, for some $c > 0$ we have $h(x) \leq cf(x)g(x)$ eventually. Therefore, $f(x)g(x) = 0$ implies $h(x) = 0$ eventually and hence, on the one hand, $F(x) = f(x)$ eventually (so that $F \in \Theta(f)$) and, on the other hand, $G(x) \leq \max\{c, 1\}g(x)$ eventually (so that $G \in \mathcal{O}(g)$). Hence, $h \in \Theta(f)\mathcal{O}(g)$, as required.

(ii) Consider $h \in \Omega(fg)$. In this case, we clearly have that $F \in \Omega(f)$ and $G \in \Omega(g)$. Thus, $h \in \Omega(f)\Omega(g)$, as required.

Furthermore, if $f \in \omega(0)$, it turns out that $F(x) = f(x)$ eventually so that $F \in \Theta(f)$.

(iii) Consider $h \in \Theta(fg)$. Like in case (i), we clearly have again that $F(x) = f(x)$ eventually so that $F \in \Theta(f)$. Moreover, this time it turns out that $G \in \Theta(g)$. Thus, $h \in \Theta(f)\Theta(g)$, as required.

(iv) Since $f, g \in \omega(0)$, it holds that $o(f) \neq \emptyset$, $o(g) \neq \emptyset$ and $o(fg) \neq \emptyset$ (see Proposition 6.5-(ii)). Then we consider $h \in o(fg)$, which easily implies $\sqrt{h} \in o(\sqrt{fg})$. Therefore, by Lemma 6.25, we immediately get

$$\Phi := \sqrt{g^{-1}fh} \in o(f) \quad \text{and} \quad \Gamma := \sqrt{f^{-1}gh} \in o(g).$$

Clearly, it holds that $h(x) = \Phi(x)\Gamma(x)$ eventually and, therefore, $h \in o(f)o(g)$, as required.

To complete this case, except for exchanging the roles of f and g, we are left to show that $o(fg) = \Theta(f)o(g)$. The inclusion $\Theta(f)o(g) \subseteq o(fg)$ is granted by Lemma 6.31-(i). In order to prove the opposite one, we consider $h \in o(fg)$ and, once again, the functions F and G defined above. Since $f \in \omega(0)$, as usual we have that $F \in \Theta(f)$. Moreover, this time it immediately turns out that $G \in o(g)$. Thus, $h \in \Theta(f)o(g)$, as required.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

(v) Consider $h \in \omega(fg)$ and another function $u \in \omega(1)$ such that $u(x) > 0$ for all x. If we define

$$\Phi(x) := \begin{cases} \frac{\sqrt{f(x)}\sqrt{h(x)}}{\sqrt{g(x)}} & \text{if } f(x) \neq 0,\ g(x) \neq 0 \\ \frac{h(x)}{g(x)u(x)} & \text{if } f(x) = 0,\ g(x) \neq 0 \\ f(x)u(x) & \text{if } f(x) \neq 0,\ g(x) = 0 \\ \sqrt{h(x)} & \text{if } f(x) = g(x) = 0 \end{cases},$$

and

$$\Gamma(x) := \begin{cases} \frac{\sqrt{g(x)}\sqrt{h(x)}}{\sqrt{f(x)}} & \text{if } f(x) \neq 0,\ g(x) \neq 0 \\ g(x)u(x) & \text{if } f(x) = 0,\ g(x) \neq 0 \\ \frac{h(x)}{f(x)u(x)} & \text{if } f(x) \neq 0,\ g(x) = 0 \\ \sqrt{h(x)} & \text{if } f(x) = g(x) = 0 \end{cases},$$

we clearly have that $\Phi \in \omega(f)$, $\Gamma \in \omega(g)$ and $h(x) = \Phi(x)\Gamma(x)$ for all x. Thus, $h \in \omega(f)\omega(g)$, as required.

To complete this case, we need to assume that $f \in \omega(0)$ and show that $\omega(fg) = \Theta(f)\omega(g)$. Again, the inclusion $\Theta(f)\omega(g) \subseteq \omega(fg)$ is granted by Lemma 6.31-(i), whereas the opposite one is obtained by considering $h \in \omega(fg)$ and the functions F and G defined above. Since $f \in \omega(0)$, as usual we have that $F \in \Theta(f)$. Moreover, this time it immediately turns out that $G \in \omega(g)$. Thus, $h \in \Theta(f)\omega(g)$, as required. ∎

Remark 6.33. Observe that, once again, the hypothesis $f \in \omega(0)$ in Proposition 6.32 is essential. In fact, if $f \notin \omega(0)$, the situation is quite different. Namely, it holds that $\Theta(f)\omega(g) \cap \omega(fg) = \emptyset$, $\Theta(f)o(g) \cap o(fg) = \emptyset$ and $\Theta(f)\Omega(g) \subsetneq \Omega(fg)$ for all $g \in \mathcal{F}^+$.

Clearly, Proposition 6.32 can be immediately resumed as follows.

Corollary 6.34. *Let $f, g \in \omega(0)$. Then, for any symbol BL, it holds that*

$$BL(f)\, BL(g) = BL(fg) = \Theta(f)\, BL(g).$$

Corollary 6.35 (Cancellation Law). *Let $f, g \in \omega(0)$ such that $\Theta(f) = \Theta(g)$ and let $h, k \in \mathcal{F}^+$. Then, for any symbol BL, it holds that*

$$BL(fh) = BL(gk) \quad \Longrightarrow \quad BL(h) = BL(k).$$

Proof. Using Definition 6.23 and Corollary 6.34, we can write

$$BL(h) = BL(f^{-1}fh) = \Theta(f^{-1})BL(fh) \quad \text{and} \quad BL(k) = BL(g^{-1}gk) = \Theta(g^{-1})BL(gk).$$

Since $\Theta(f) = \Theta(g)$, Proposition 6.24 implies $\Theta(f^{-1}) = \Theta(g^{-1})$ as well. Therefore, the assumption $BL(fh) = BL(gk)$ lets us conclude that $BL(h) = BL(k)$. ∎

In order to study also the order of functions with complex values, we introduce a larger set than $\mathcal{F}^+$.

Notation 6.36. *Let $\mathcal{D}$ be a subset of $\mathbb{R}$ and let $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ be an accumulation point of $\mathcal{D}$. We set*

$$\mathcal{F} = \mathcal{F}_{x_0}(\mathcal{D}) := \{f : \mathcal{D} \setminus \{x_0\} \longrightarrow \mathbb{C}\}.$$

Observe that, if $f \in \mathcal{F}$, then $|f| \in \mathcal{F}^+$. Therefore, we can define the *order* of f for $x \longrightarrow x_0$ on the domain $\mathcal{D}$ as the order of $|f|$ (see Definition 6.7).

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 38, pp. 377-403, August 2022.

LEMMA 6.37. *If $f, g \in \mathcal{F}$, then it holds that $\mathcal{O}(|f+g|) \subseteq \mathcal{O}(|f|) + \mathcal{O}(|g|)$ and, dually, $\Omega(|f+g|) \supseteq \Omega(|f|) + \Omega(|g|)$.*

Proof. It is a consequence of Proposition 6.26-(i,ii) and the fact that $|f(x)+g(x)| \leq |f(x)| + |g(x)|$. ∎

LEMMA 6.38. *Let $f, g \in \mathcal{F}$; then the following facts hold:*

(i) $|f| \in \omega(|g|) \Longrightarrow \Theta(|f+g|) = \Theta(|f|)$;
(ii) $|f| \in \omega(|f+g|) \Longrightarrow |g| \in \Omega(|f|)$.

Proof. Recall first that

$$|f(x)| - |g(x)| \leq |f(x)+g(x)| \leq |f(x)| + |g(x)|. \tag{6.38}$$

(i) If $|f| \in \omega(|g|)$ then, for all $c > 1$, it eventually holds that $|f(x)| > c|g(x)|$. Therefore, (6.38) implies

$$\frac{c-1}{c}\, |f(x)| < |f(x)+g(x)| < \frac{c+1}{c}\, |f(x)|$$

where, clearly, $\frac{c-1}{c} > 0$. So $\Theta(|f+g|) = \Theta(|f|)$.

(ii) The assumption means that, for all $c > 1$, $|f(x)| > c|f(x)+g(x)|$ eventually. Hence, from (6.38), we obtain $|f(x)| > c(|f(x)| - |g(x)|)$, that is, $|g(x)| > \frac{c-1}{c}\, |f(x)|$, and this implies $|g| \in \Omega(|f|)$. ∎

PROPOSITION 6.39. *Let $f, g \in \mathcal{F}$ and assume that $|g| \in \mathcal{O}(|f|)$. Then it holds that:*

(i) $\Theta(|f|) \subseteq \Omega(|f+g|)$;
(ii) *if $|f| \notin \mathcal{O}(|f+g|)$, then $\Theta(|f|) = \Theta(|g|) \subseteq \omega(|f+g|)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$.*

Proof.

(i) By assumption and by Remark 6.11-(i), we have $\mathcal{O}(|g|) \subseteq \mathcal{O}(|f|)$ and so, by Proposition 6.8-(iii), we obtain $\Omega(|f|) \subseteq \Omega(|g|)$. Therefore, by using Proposition 6.26-(ii), we obtain

$$\Theta(|f|) \subseteq \Omega(|f|) = \Omega(2|f|) = \Omega(|f|) + \Omega(|f|) \subseteq \Omega(|f|) + \Omega(|g|) \subseteq \Omega(|f+g|),$$

where the last inclusion follows from Lemma 6.37.

(ii) Clearly, Proposition 6.17-(i) and Corollary 6.10 imply that $\Theta(|f|) \subseteq \omega(|f+g|)$ on a suitable $\mathcal{D}' \subseteq \mathcal{D}$. Consequently, Lemma 6.38-(ii), the assumption $|g| \in \mathcal{O}(|f|)$, Proposition 6.13, and Remark 6.6 allow us to conclude that $\Theta(|f|) = \Theta(|g|)$ on the same subdomain $\mathcal{D}'$. ∎

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