Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

THE MAXIMAL ANGLE BETWEEN 5×5 POSITIVE SEMIDEFINITE AND 5×5 NONNEGATIVE MATRICES*

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Abstract. The paper is devoted to the study of the maximal angle between the 5×5 semidefinite matrix cone and 5×5 nonnegative matrix cone. A signomial geometric programming problem is formulated in the process to find the maximal angle. Instead of using an optimization problem solver to solve the problem numerically, the method of Lagrange Multipliers is used to solve the signomial geometric program, and therefore, to find the maximal angle between these two cones.

Key words. Maximal angle, Positive semidefinite matrices, Nonnegative matrices, Signomial geometric programming.

AMS subject classifications. 15B48, 90C30.

1. Introduction.

Hiriart-Urruty and Seeger in [4] raised a question to find the maximal angle between two $n \times n$ copositive matrices. By a variational approach, Hiriart-Urruty and Seeger proved that the maximal angle between two 2×2 copositive matrices is $\frac{3\pi}{4}$. They further conjectured that the maximal angle for two $n \times n$ copositive matrices is also $\frac{3\pi}{4}$ for $n \geq 3$. In addressing this conjecture, Goldberg and Shaked-Monderer [3] constructed a sequences of pairs (P_k, N_k) such that the angle between P_k and N_k approaches π as $n_k \to \infty$, where P_k is an $n_k \times n_k$ positive semidefinite matrix and N_k is an $n_k \times n_k$ nonnegative matrix. Since the set of copositive matrices contains positive semidefinite matrices and nonnegative matrices, the construction of such a sequence disproved the conjecture of Hiriart-Urruty and Seeger.

Goldberg and Shaked-Monderer pointed out in [3] that the problem of calculating or estimating the maximal angle between an $n \times n$ positive semidefinite matrix and an $n \times n$ nonnegative matrix is interesting in its own right. In [3], Goldberg and Shaked-Monderer studied the maximal angle between an $n \times n$ semidefinite matrix and an $n \times n$ nonnegative matrix for $n = 3, 4$, and 5. They proved that for $n \leq 4$, the maximal angle between an $n \times n$ positive semidefinite matrix and an $n \times n$ nonnegative matrix is $\frac{3\pi}{4}$. For $n = 5$, a nonnegative matrix and a semidefinite matrix are constructed in [3] to show that the maximal angle is strictly greater than $\frac{3\pi}{4}$. In the proofs of the results for $n \leq 4$, an optimization problem involving a convex function was used in [3]. However, for $n \geq 5$, such an approach does not work for two reasons as mentioned in [3]: 1. Continuing this line of proof for $n \geq 5$ would require some information on the possible sets of eigenvalues of a nonnegative $n \times n$ matrix with a zero diagonal. 2. For $n = 5$, the set of all nonincreasing 5-tuples that are eigenvalues of a nonnegative trace zero matrix is not convex, complicating the relevant optimization problem. The authors in [3] further predicted that "it seems that a new approach is needed for the computation of $\gamma_n, n \geq 5$." Here, γ_n denotes the maximal angle between an $n \times n$ positive semidefinite matrix and an $n \times n$ nonnegative matrix.

By a close look at the proofs in [3], we found that an optimization problem can be avoided. Instead, we can complete the proofs by using elementary algebra. In this note, we follow this basic algebraic approach

*Received by the editors on September 30, 2021. Accepted for publication on November 3, 2021. Handling Editor: Michael Tsatsomeros.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

to study the maximal angle between a 5×5 positive semidefinite matrix and a 5×5 nonnegative matrix. Unlike the cases for 3×3 and 4×4 matrices in which no optimization problem is needed, for 5×5 matrices, a nonlinear optimization problem whose objective and constraint functions are signomial functions (called a signomial geometric programming problem in the related literature like [2]) is formulated in the process to find the maximal angle.

The paper is organized as follow: in Section 2, we will provide a minimum review regarding the maximal angle between two sets of matrices. Some results proved in [3] will be reviewed and a new algebraic proof of a result in [3] will be provided in this section. In Section 3, we will formulate a signomial geometric programming problem and provide detailed discussion on how to solve the signomial program using the method of Lagrange Multipliers. In Section 4, we will give our concluding remarks.

2. Preliminaries. Let V be an inner product space. For $u \in V$ and $v \in V$, $\langle u, v\rangle$ is used to denote the inner product of u and v. The angle between u and v is defined by $\angle(u,v) = \arccos \frac{\langle u,v\rangle}{||u||||v||}$, where $||u|| = \sqrt{\langle u, u\rangle}$. In this paper, we always take $V = \mathcal{S}_n$, the inner product space of $n \times n$ real symmetric matrices with the inner product defined by $\langle A, B\rangle = Tr(AB)$ for $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_n$. Let $\mathcal{P}_n \subset \mathcal{S}_n$ be the cone of $n \times n$ positive semidefinite matrices, and $\mathcal{N}_n \subset \mathcal{S}_n$ be the cone of $n \times n$ nonnegative matrices. As in [3], we use γ_n to represent the maximal angle between an $n \times n$ positive semidefinite matrix and an $n \times n$ nonnegative matrix, that is,

$$\gamma_n = \max_{A \in \mathcal{P}_n, B \in \mathcal{N}_n} \angle(A, B) = \max_{A \in \mathcal{P}_n, B \in \mathcal{N}_n} \arccos \frac{Tr(AB)}{||A||||B||}.$$

Of course, we can define the maximal angle between a matrix and a cone. For example, the maximal angle between a matrix $A \in \mathcal{S}_n$ and $\mathcal{P}_n$ is defined to be

$$\angle(A, \mathcal{P}_n) = \max_{B \in \mathcal{P}_n} \angle(A, B) = \max_{B \in \mathcal{P}_n} \arccos \frac{Tr(A, B)}{||A||||B||}.$$

Also we can define the maximal angle between two cones. For example, the maximal angle between $\mathcal{P}_n$ and $\mathcal{N}_n$ is defined to be

$$\angle(\mathcal{P}_n, \mathcal{N}_n) = \max_{A \in \mathcal{P}_n, B \in \mathcal{N}_n} \angle(A, B) = \max_{A \in \mathcal{P}_n, B \in \mathcal{N}_n} \arccos \frac{Tr(A, B)}{||A||||B||},$$

which of course is the same as γ_n defined above.

As pointed out in [3] that every $n \times n$ symmetric matrix A has a unique decomposition as a difference of two positive semidefinite matrices that are orthogonal to each other. In other words, $A = Q - P$, with $Q, P \in \mathcal{P}_n$ and $QP = 0$. P is called negative definite part of A. Similarly, every $A \in \mathcal{S}_n$ has a unique decomposition as a difference of two nonnegative matrices that are orthogonal to each other: $A = M - N$, with $M, N \in \mathcal{N}_n$ and $MN = 0$. N is called the negative part of A.

In the proofs of Proposition 2.6 and Theorem 2.7 in [3], optimization problems are formulated and used. We find that such optimization problems are not needed. Instead, we can complete the proofs by using basic algebra. To avoid repeating the process, we only give a proof of Theorem 2.7. For convenience of reference, we restate Proposition 2.6 and Theorem 2.7 in [3] below.

Proposition 2.1 (Proposition 2.6. in [3]). *Let $n \geq 2$, let $N \in \mathcal{N}_n$ have diag $N = 0$ and let P be its negative definite part. If rank $P = n - 1$, then $\angle(N, \mathcal{P}_n) < \frac{3\pi}{4}$.*

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THEOREM 2.2 (Theorem 2.7. in [3]). *For* $n \leq 4$, $\gamma_n = \frac{3\pi}{4}$.

Proof. It suffices to consider $\angle(N, \mathcal{P}_n)$ for $N \in \mathcal{N}_4$ with diag $N = 0$ and a negative definite part P of rank 2. Such N has a Perron eigenvalue $\rho > 0$, and its complete set of eigenvalues is $\rho \geq \mu \geq 0 > \lambda_3 \geq \lambda_4$, where $\lambda_3 + \lambda_4 = -\rho - \mu$ and $\lambda_4 \geq -\rho$. Now we let $\rho = -\lambda_4 + k$ with $k \geq 0$. We easily see that $\mu = -\lambda_3 - k$ due to $\lambda_3 + \lambda_4 = -\rho - \mu$. Therefore, $\rho^2 + \mu^2 = (-\lambda_4 + k)^2 + (-\lambda_3 - k)^2 = \lambda_3^2 + \lambda_4^2 + 2k^2 + 2k(\lambda_3 - \lambda_4) \geq \lambda_3^2 + \lambda_4^2$. Hence, $\cos\angle(N, \mathcal{P}_4) = -\frac{\sqrt{\lambda_3^2+\lambda_4^2}}{\sqrt{\rho^2+\mu^2+\lambda_3^2+\lambda_4^2}} \geq -\frac{\sqrt{\lambda_3^2+\lambda_4^2}}{\sqrt{2(\lambda_3^2+\lambda_4^2)}} = -\frac{\sqrt{2}}{2}$, which proves that $\angle(N, \mathcal{P}_4) \leq \frac{3\pi}{4}$. If we choose N to be $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ as in [3], then the eigenvalues of N are $1, 1, -1, -1$, which shows $\rho = -\lambda_4 = 1$ and $\mu = -\lambda_3 = 1$. Hence, $\cos\angle(N, \mathcal{P}_4) = -\frac{\sqrt{2}}{2}$ showing that $\angle(N, \mathcal{P}_4) = \frac{3\pi}{4}$. Therefore, $\gamma_n = \frac{3\pi}{4}$. $\square$

3. The maximal angle between $\mathcal{P}_5$ and $\mathcal{N}_5$. To find the maximal angle between $\mathcal{P}_5$ and $\mathcal{N}_5$, as in the proof of Theorem 2.7 in [3], we need to consider $\angle(N, \mathcal{P}_5)$ for $N \in \mathcal{N}_5$ with diag $N = 0$ and a negative definite part P of N of rank r with $1 \leq r \leq 4$. If $r = 1$, then N has only one negative eigenvalue, namely $-\delta$, $\delta > 0$. The Perron eigenvalue of N must be δ. Since diag N=0, we obtain that the eigenvalues of N are $\delta, 0, 0, 0, -\delta$. Therefore, $\angle(N, \mathcal{P}_5) = \arccos\left(-\frac{\sqrt{\delta^2}}{\sqrt{(-\delta)^2+\delta^2}}\right) = \frac{3\pi}{4}$. If $r = 4$, then by Proposition 2.6 in [3], we have $\angle(N, \mathcal{P}_5) < \frac{3\pi}{4}$. Therefore, to find the maximal angle between $\mathcal{P}_5$ and $\mathcal{N}_5$, we only need to consider $r = 3$ and $r = 2$, which correspond to the following two cases given that λ_i, $i = 1, 2, \ldots, 5$, are the eigenvalues of N:

1. $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4 \geq \lambda_5$.
2. $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 > \lambda_4 \geq \lambda_5$.

We work on Case 1 first. By Proposition 2.1 in [3], we know that

$$\angle(N, \mathcal{P}_5) = \arccos\left(-\frac{\sqrt{\lambda_3^2+\lambda_4^2+\lambda_5^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\right).$$

Since λ_i, $i = 1, 2, \ldots, 5$ are eigenvalues of N, we know that $\lambda_1 > 0$, $\lambda_1 \geq -\lambda_5$, and $\sum_{i=1}^5 \lambda_i = 0$. If we let $\lambda_1 = -\lambda_5 + k$, then $k \geq 0$ and $\lambda_2 = -\lambda_3 - \lambda_4 - k$. Since $\lambda_2 \geq 0$, we have $k \leq -\lambda_3 - \lambda_4$. Moreover, the assumption that $\lambda_1 \geq \lambda_2$ requires that $k \geq \frac{\lambda_5-\lambda_3-\lambda_4}{2}$. By a necessary and sufficient condition proved by Spector in [5] that α_i, $i = 1, 2, \ldots, 5$ with $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5$ are eigenvalues of a trace zero nonnegative 5×5 matrix if and only if $\alpha_1 \geq -\alpha_5$, $\sum_{i=1}^5 \alpha_i = 0$, $\alpha_2 + \alpha_5 \leq 0$, and $\sum_{i=1}^5 \alpha_i^3 \geq 0$, we have $\lambda_2 + \lambda_5 \leq 0$ and $\sum_{i=1}^5 \lambda_i^3 \geq 0$. From $\lambda_2 + \lambda_5 \leq 0$, we obtain that $k \geq \lambda_5 - \lambda_3 - \lambda_4$.

Since

$$\begin{aligned}\sum_{i=1}^5 \lambda_i^3 &= (-\lambda_5 + k)^3 - (\lambda_3 + \lambda_4 + k)^3 + \lambda_3^3 + \lambda_4^3 + \lambda_5^3 \\ &= -3(\lambda_3 + \lambda_4 + \lambda_5)k^2 + 3(\lambda_5^2 - \lambda_3^2 - 2\lambda_3\lambda_4 - \lambda_4^2)k - 3\lambda_3^2\lambda_4 - 3\lambda_3\lambda_4^2 \\ &\geq 0,\end{aligned}$$

noticing that $\lambda_3 + \lambda_4 + \lambda_5 < 0$, we obtain that $k^2 + (\lambda_3 + \lambda_4 - \lambda_5)k \geq -\left(\frac{\lambda_3^2\lambda_4+\lambda_3\lambda_4^2}{\lambda_3+\lambda_4+\lambda_5}\right)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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Therefore, for given $0>\lambda_3\geq\lambda_4\geq\lambda_5$, $\lambda_1=-\lambda_5+k$, $\lambda_2=-\lambda_3-\lambda_4-k$, λ_3, λ_4, and λ_5 are eigenvalues of an $N\in\mathcal{N}_5$ with diag $N=0$ and $\lambda_1\geq\lambda_2\geq 0>\lambda_3\geq\lambda_4\geq\lambda_5$ if and only if k satisfies $k\geq 0$, $k\leq -\lambda_3-\lambda_4$, $k\geq\frac{\lambda_5-\lambda_3-\lambda_4}{2}$, $k\geq\lambda_5-\lambda_3-\lambda_4$, and $k^2+(\lambda_3+\lambda_4-\lambda_5)k\geq-\left(\frac{\lambda_3^2\lambda_4+\lambda_3\lambda_4^2}{\lambda_3+\lambda_4+\lambda_5}\right)$.

Now let us get back to the maximal angle. Suppose that $0>\lambda_3\geq\lambda_4\geq\lambda_5$ are given. Since $\arccos(x)$ is a decreasing function, we need to find

$$\begin{aligned}\mu_1=\min_k -&\frac{\sqrt{\lambda_3^2+\lambda_4^2+\lambda_5^2}}{\sqrt{(-\lambda_5+k)^2+(\lambda_3+\lambda_4+k)^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\\ \text{subject to } & k\geq 0,\\ & k\leq -\lambda_3-\lambda_4,\\ & k\geq\frac{\lambda_5-\lambda_3-\lambda_4}{2},\\ & k\geq\lambda_5-\lambda_3-\lambda_4,\\ & k^2+(\lambda_3+\lambda_4-\lambda_5)k\geq-\left(\frac{\lambda_3^2\lambda_4+\lambda_3\lambda_4^2}{\lambda_3+\lambda_4+\lambda_5}\right).\end{aligned}$$

Since we assume that λ_3, λ_4, and λ_5 are given, to find μ_1, we just need to find $\nu_1=\min_k(-\lambda_5+k)^2+(\lambda_3+\lambda_4+k)^2$ subject to the same set of constraints. Since we have the constraint $k^2+(\lambda_3+\lambda_4-\lambda_5)k\geq-\left(\frac{\lambda_3^2\lambda_4+\lambda_3\lambda_4^2}{\lambda_3+\lambda_4+\lambda_5}\right)$, we know that

$$\begin{aligned}(-\lambda_5+k)^2+(\lambda_3+\lambda_4+k)^2&=2k^2+2(\lambda_3+\lambda_4-\lambda_5)k+\lambda_3^2+\lambda_4^2+\lambda_5^2+2\lambda_3\lambda_4\\ &\geq-\left(\frac{2\lambda_3^2\lambda_4+2\lambda_3\lambda_4^2}{\lambda_3+\lambda_4+\lambda_5}\right)+\lambda_3^2+\lambda_4^2+\lambda_5^2+2\lambda_3\lambda_4\\ &=\frac{2\lambda_3\lambda_4\lambda_5}{\lambda_3+\lambda_4+\lambda_5}+\lambda_3^2+\lambda_4^2+\lambda_5^2\\ &\geq\lambda_3^2+\lambda_4^2+\lambda_5^2.\end{aligned}$$

Hence, we have $\frac{\sqrt{\lambda_3^2+\lambda_4^2+\lambda_5^2}}{\sqrt{(-\lambda_5+k)^2+(\lambda_3+\lambda_4+k)^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\leq\frac{\sqrt{\lambda_3^2+\lambda_4^2+\lambda_5^2}}{\sqrt{2(\lambda_3^2+\lambda_4^2+\lambda_5^2)}}=\frac{\sqrt{2}}{2}$. Therefore, $\mu_1\geq-\frac{\sqrt{2}}{2}$ showing that the maximal angle is at most $\frac{3\pi}{4}$.

Now, we consider Case 2. If $\lambda_3=0$, then we let $\lambda_1=-\lambda_5+k$ and $\lambda_2=-\lambda_4-k$, and follow the proof for Case 1 to obtain that the maximal angle is at most $\frac{3\pi}{4}$. So we can assume that $\lambda_3>0$.

By Proposition 2.1 in [3], we know $\angle(N,\mathcal{P}_5)=\arccos\left(-\frac{\sqrt{\lambda_4^2+\lambda_5^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\right)$. We know that λ_i, $i=1,2,\ldots,5$ with $\lambda_1\geq\lambda_2\geq\lambda_3>0>\lambda_4\geq\lambda_5$ are eigenvalues of $N\in\mathcal{N}_5$ with diag N=0 if and only if $\lambda_1\geq-\lambda_5$, $\sum_{i=1}^5\lambda_i=0$, $\lambda_2+\lambda_5\leq 0$, and $\sum_{i=1}^5\lambda_i^3\geq 0$. If we let $\lambda_5=-\lambda_1+k$, then $k\geq 0$ and $\lambda_4=-\lambda_2-\lambda_3-k$. Since $\lambda_4\geq\lambda_5$, we have $k\leq\frac{\lambda_1-\lambda_2-\lambda_3}{2}$. From $\lambda_2+\lambda_5\leq 0$, we obtain that $k\leq\lambda_1-\lambda_2$.

The inequality $\sum_{i=1}^5\lambda_i^3\geq 0$ translates to

$$\begin{aligned}\sum_{i=1}^5\lambda_i^3&=\lambda_1^3+\lambda_2^3+\lambda_3^3+(-\lambda_2-\lambda_3-k)^3+(k-\lambda_1)^3\\ &=-3(\lambda_1+\lambda_2+\lambda_3)k^2+3(\lambda_1^2-\lambda_2^2-2\lambda_2\lambda_3-\lambda_3^2)k-3\lambda_2^2\lambda_3-3\lambda_2\lambda_3^2\\ &\geq 0.\end{aligned}$$

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A publication of the International Linear Algebra Society
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Noticing that $\lambda_1+\lambda_2+\lambda_3>0$, we obtain that $k^2-(\lambda_1-\lambda_2-\lambda_3)k \le -\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)$.

Therefore, for given $\lambda_1\ge\lambda_2\ge\lambda_3>0$, λ_1, λ_2, λ_3, $\lambda_4=-\lambda_2-\lambda_3-k$, and $\lambda_5=-\lambda_1+k$ are eigenvalues of an $N\in\mathcal{N}_5$ with diag $N=0$ and $\lambda_1\ge\lambda_2\ge\lambda_3>0>\lambda_4\ge\lambda_5$ if and only if k satisfies the following constraints $0\le k\le\frac{\lambda_1-\lambda_2-\lambda_3}{2}$, $k\le\lambda_1-\lambda_2$, and $k^2-(\lambda_1-\lambda_2-\lambda_3)k\le-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)$.

Now let us get back to the maximal angle. Suppose that $\lambda_1\ge\lambda_2\ge\lambda_3>0$ are given. We assume $\lambda_1\ge\lambda_2+\lambda_3$. Otherwise, there is no k that makes $0\le k\le\frac{\lambda_1-\lambda_2-\lambda_3}{2}$ true. Therefore, we need to find

$$\begin{aligned}
\mu_2 &= \min_k -\frac{\sqrt{\lambda_4^2+\lambda_5^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\\
&= \min_k -\frac{\sqrt{(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2}}\\
&= \min_k -\frac{1}{\sqrt{\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2}+1}}\\
&\text{subject to } 0\le k\le\frac{\lambda_1-\lambda_2-\lambda_3}{2},\\
&\qquad k\le\lambda_1-\lambda_2,\\
&\qquad k^2-(\lambda_1-\lambda_2-\lambda_3)k\le-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right).
\end{aligned}$$

Since we assume that $\lambda_1,\lambda_2,\lambda_3$ are given, to find μ_2, we just need to find $\nu_2=\max_k(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2$ subject to the same set of constraints.

Note that $(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2=2k^2+2(\lambda_2+\lambda_3-\lambda_1)k+\lambda_1^2+\lambda_2^2+\lambda_3^2+2\lambda_2\lambda_3$. Moreover, for given $\lambda_1\ge\lambda_2\ge\lambda_3>0$, if $k\le\frac{\lambda_1-\lambda_2-\lambda_3}{2}$, then $k\le\lambda_1-\lambda_2$. Therefore, we only need to solve the following optimization problem:

$$\begin{aligned}
(Prog)\qquad &\max_k k^2-(\lambda_1-\lambda_2-\lambda_3)k\\
&\text{subject to } k\ge 0,\\
&\qquad k\le\frac{\lambda_1-\lambda_2-\lambda_3}{2},\\
&\qquad k^2-(\lambda_1-\lambda_2-\lambda_3)k\le-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right). \qquad (3.1)
\end{aligned}$$

We consider the last constraint (3.1). By completing the perfect square on the left-hand side, we have

$$\left(k-\frac{\lambda_1-\lambda_2-\lambda_3}{2}\right)^2\le\frac{-4\lambda_2^2\lambda_3-4\lambda_2\lambda_3^2+(\lambda_1-\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2+\lambda_3)}{4(\lambda_1+\lambda_2+\lambda_3)}.$$

If the right-hand side of the above inequality is strictly greater than 0, then the quadratic form:

$$p(k)=\left(k-\frac{\lambda_1-\lambda_2-\lambda_3}{2}\right)^2-\frac{-4\lambda_2^2\lambda_3-4\lambda_2\lambda_3^2+(\lambda_1-\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2+\lambda_3)}{4(\lambda_1+\lambda_2+\lambda_3)}$$

has two zeros k_1 and k_2 with $k_1>\frac{\lambda_1-\lambda_2-\lambda_3}{2}>k_2$. Both zeros are greater than 0 since $p(0)>0$. Hence, the feasible set of (Prog) is $[k_2,\frac{\lambda_1-\lambda_2-\lambda_3}{2}]$ and the problem (Prog) has an optimal value $-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)$ reached

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at $k = k_2$. If the right-hand side of the above inequality is 0, then $k = \frac{\lambda_1-\lambda_2-\lambda_3}{2}$ is the only feasible solution. So the problem is also solvable with an optimal value $-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)$. Therefore, in the formulation of the optimization problem, as long as $\frac{-4\lambda_2^2\lambda_3-4\lambda_2\lambda_3^2+(\lambda_1-\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2+\lambda_3)}{4(\lambda_1+\lambda_2+\lambda_3)} \geq 0$, the optimization problem (Prog) is always solvable with an optimal value $-\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)$.

EXAMPLE 3.1. Let $\lambda_1 = 1$, and $\lambda_2 = \lambda_3 = \frac{-1+\sqrt{5}}{4}$. By a simple algebraic computation, we found that the maximum of $k^2-(\lambda_1-\lambda_2-\lambda_3)k$ subject to the constraints of (Prog) is reached at $k = \frac{3-\sqrt{5}}{4} = \frac{\lambda_1-\lambda_2-\lambda_3}{2}$. So the only trace zero nonnegative 5×5 matrices with $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \frac{-1+\sqrt{5}}{4}$ being their positive eigenvalues are the trace zero nonnegative 5×5 matrices with the five eigenvalues: $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \frac{-1+\sqrt{5}}{4}$, and $\lambda_4 = \lambda_5 = \frac{-1-\sqrt{5}}{4}$.

EXAMPLE 3.2. Let $\lambda_1 = 1$, $\lambda_2 = 0.5$, and $\lambda_3 = 0.1$. We found that the maximum of $k^2-(\lambda_1-\lambda_2-\lambda_3)k$ subject to the constraints of (Prog) is reached at $k = \frac{0.8-\sqrt{0.34}}{4}$. Although for any $k \in \left[\frac{0.8-\sqrt{0.34}}{4}, 0.2\right]$, we can find a trace zero nonnegative 5×5 matrix with its eigenvalues being $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 0.1$, $\lambda_4 = -0.6-k$, and $\lambda_5 = -1+k$, it is this $k = \frac{0.8-\sqrt{0.34}}{4}$ that makes $k^2-(\lambda_1-\lambda_2-\lambda_3)k$ to be the minimum, and therefore, determines the maximal angle between a trace zero nonnegative 5×5 matrix with three positive eigenvalues being $\lambda_1 = 1$, and $\lambda_2 = 0.5$ and $\lambda_3 = 0.1$ and a positive semidefinite 5×5 matrix. The five eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 0.1$, $\lambda_4 = \frac{-3.2+\sqrt{0.34}}{4}$, and $\lambda_5 = \frac{-3.2-\sqrt{0.34}}{4}$.

Previously, we assume that λ_1, λ_2, and λ_3 are given with $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ and $\lambda_1 \geq \lambda_2+\lambda_3$. We formulate an optimization problem to find k such that λ_1, λ_2, λ_3, $\lambda_4 = -\lambda_2-\lambda_3-k$, and $\lambda_5 = -\lambda_1+k$ are eigenvalues of a trace zero matrix in $\mathcal{N}_5$, which forms the maximal angle between a trace zero nonnegative 5×5 matrix with three positive eigenvalues λ_1, λ_2, and λ_3 and a positive semidefinite 5×5 matrix. Now we turn our attention to the problem of finding the maximal angle between the 5×5 nonnegative matrix cone and 5×5 positive semidefinite matrix cone. Noting that in the optimization problem (Prog), the maximum of $k^2-(\lambda_1-\lambda_2-\lambda_3)k$ is reached and the optimal value is $-\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}$ as long as $\lambda_1 \geq \lambda_2+\lambda_3$ and $\frac{-4\lambda_2^2\lambda_3-4\lambda_2\lambda_3^2+(\lambda_1-\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2+\lambda_3)}{4(\lambda_1+\lambda_2+\lambda_3)} \geq 0$. We plug this optimal value into the original optimization problem to find the maximal angle. We have

$$
\begin{aligned}
\mu_2 &= \min_k - \frac{1}{\sqrt{\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{(\lambda_2+\lambda_3+k)^2+(-\lambda_1+k)^2}+1}} \\
&= \min_k - \frac{1}{\sqrt{\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{2k^2-2(\lambda_1-\lambda_2-\lambda_3)k+2\lambda_2\lambda_3+\lambda_1^2+\lambda_2^2+\lambda_3^2}+1}} \\
&= - \frac{1}{\sqrt{\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{-2\left(\frac{\lambda_2^2\lambda_3+\lambda_2\lambda_3^2}{\lambda_1+\lambda_2+\lambda_3}\right)+2\lambda_2\lambda_3+\lambda_1^2+\lambda_2^2+\lambda_3^2}+1}} \\
&= - \frac{1}{\sqrt{\frac{1}{\frac{2\lambda_1\lambda_2\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2)}+1}+1}} \\
&= - \frac{1}{\sqrt{\frac{1}{\frac{1}{\frac{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2)}{2\lambda_1\lambda_2\lambda_3}}+1}+1}}.
\end{aligned}
$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

Therefore, to find the maximal angle between $\mathcal{N}_5$ and $\mathcal{P}_5$, we need to solve the following optimization problem:

$$\begin{aligned}
&\min_{\lambda_1,\lambda_2,\lambda_3} \frac{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2)}{\lambda_1\lambda_2\lambda_3}\\
&\text{subject to } \lambda_1>0, \lambda_2>0, \lambda_3>0,\\
&\qquad\qquad \lambda_1 \geq \lambda_2+\lambda_3,\\
&\qquad\qquad \frac{-4\lambda_2^2\lambda_3-4\lambda_2\lambda_3^2+(\lambda_1-\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2+\lambda_3)}{4(\lambda_1+\lambda_2+\lambda_3)} \geq 0.
\end{aligned}$$

By rewriting this optimization problem, we actually find it to be a signomial geometric programming problem. We call it Program (SGP):

$$\begin{aligned}
&\inf_{\lambda_1,\lambda_2,\lambda_3} \lambda_1^2\lambda_2^{-1}\lambda_3^{-1}+\lambda_2\lambda_3^{-1}+\lambda_2^{-1}\lambda_3+\lambda_1\lambda_3^{-1}+\lambda_1^{-1}\lambda_2^2\lambda_3^{-1}+\lambda_1^{-1}\lambda_3+\lambda_1\lambda_2^{-1}+\lambda_1^{-1}\lambda_2+\lambda_1^{-1}\lambda_2^{-1}\lambda_3^2\\
&\text{subject to } \lambda_1>0, \lambda_2>0, \lambda_3>0,\\
&\qquad\qquad \lambda_1^{-1}\lambda_2+\lambda_1^{-1}\lambda_3 \leq 1,\\
&\qquad\qquad \lambda_1^3+\lambda_2^3+\lambda_3^3 \geq \lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_1\lambda_2^2+2\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_3^2+\lambda_1^2\lambda_2+\lambda_1^2\lambda_3.
\end{aligned} \tag{3.2}$$

Remark 3.3. We may simply formulate an optimization problem to find the maximal angle between a positive semidefinite and a nonnegative matrix, which can be stated for each of the two cases. For example, the following optimization problem is formulated for Case 2:

$$\begin{aligned}
&\min_k -\frac{\sqrt{\lambda_4^2+\lambda_5^2}}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2}}\\
&\text{subject to } \lambda_1\geq\lambda_2\geq\lambda_3\geq 0>\lambda_4\geq\lambda_5,\\
&\qquad\qquad \lambda_1 \geq -\lambda_5,\\
&\qquad\qquad \lambda_2+\lambda_5 \leq 0,\\
&\qquad\qquad \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=0,\\
&\qquad\qquad \lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3+\lambda_5^3 \geq 0.
\end{aligned}$$

This program is more complicated and harder to solve than Program (SGP) because it has more variables and more constraints compared to Program (SGP).

Actually, the last constraint of Program (SGP) can be written as:

$$\lambda_1^{-3}\lambda_2^2\lambda_3+\lambda_1^{-3}\lambda_2\lambda_3^2+\lambda_1^{-2}\lambda_2^2+2\lambda_1^{-2}\lambda_2\lambda_3+\lambda_1^{-2}\lambda_3^2+\lambda_1^{-1}\lambda_2+\lambda_1^{-1}\lambda_3-\lambda_1^{-3}\lambda_2^3-\lambda_1^{-3}\lambda_3^3\leq 1$$

making Program (SGP) to be in the standard form of signomial programming [2]. Also if $(\lambda_1,\lambda_2,\lambda_3)$ is a feasible solution of Program (SGP), then any multiple is also feasible and all these feasible solutions share the same value of the objective function. So we can simply assume that $\lambda_1=1$. Program (SGP) becomes

$$\begin{aligned}
&\inf_{\lambda_2,\lambda_3} \lambda_2^{-1}\lambda_3^{-1}+\lambda_2\lambda_3^{-1}+\lambda_2^{-1}\lambda_3+\lambda_3^{-1}+\lambda_2^2\lambda_3^{-1}+\lambda_3+\lambda_2^{-1}+\lambda_2+\lambda_2^{-1}\lambda_3^2\\
&\text{subject to } \lambda_2>0, \lambda_3>0,\\
&\qquad\qquad \lambda_2+\lambda_3\leq 1,\\
&\qquad\qquad 1+\lambda_2^3+\lambda_3^3 \geq \lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_2^2+2\lambda_2\lambda_3+\lambda_3^2+\lambda_2+\lambda_3.
\end{aligned} \tag{3.3}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

For this signomial geometric programming problem, we may use an optimization solver to numerically solve it. However, this optimization problem only has two variables. So we can use the method of Lagrange Multipliers. To this end, we assume that $\lambda_2 = e^{x_2}$ and $\lambda_3 = e^{x_3}$. We further let $f(x) = e^{-x_2-x_3} + e^{2x_2-x_3} + e^{-x_2+2x_3}$ and $g(x) = e^{x_2-x_3} + e^{-x_2+x_3} + e^{-x_2} + e^{x_2} + e^{-x_3} + e^{x_3}$. So that if we divide the constraint (3.3) by $\lambda_2\lambda_3$ and use the functions $f(x)$ and $g(x)$, the constraint becomes $-f(x) + g(x) + 2 \leq 0$. Therefore, the optimization problem becomes

$$\begin{aligned}
&\inf_{x_2,x_3} f(x) + g(x)\\
&\text{subject to } x_2 \in \mathbb{R}, x_3 \in \mathbb{R},\\
&\qquad\qquad e^{x_2} + e^{x_3} - 1 \leq 0,\\
&\qquad\qquad -f(x) + g(x) + 2 \leq 0.
\end{aligned}$$

The Lagrangian of this optimization problem is

$$L(x, \mu, \nu) = f(x) + g(x) + \mu(e^{x_2} + e^{x_3} - 1) + \nu(-f(x) + g(x) + 2).$$

If x is an optimal solution, then x must satisfy the following:

$$\begin{aligned}
&e^{x_2} + e^{x_3} - 1 \leq 0,\\
&-f(x) + g(x) + 2 \leq 0,\\
&L_{x_2}(x, \mu, \nu) = 0,\\
&L_{x_3}(x, \mu, \nu) = 0,\\
&\mu(e^{x_2} + e^{x_3} - 1) = 0,\\
&\nu(-f(x) + g(x) + 2) = 0,\\
&\mu \geq 0 \text{ and } \nu \geq 0.
\end{aligned}$$

We actually know that $\mu = 0$. Otherwise, $e^{x_2} + e^{x_3} = 1$, which gives $-f(x)+g(x)+2 > 0$ by a straightforward computation, contradicting $-f(x) + g(x) + 2 \leq 0$. We claim that $\nu \neq 0$. Otherwise, $L_{x_2}(x, \mu, \nu) = 0$ and $L_{x_3}(x, \mu, \nu) = 0$ becomes

$$\begin{aligned}
&-e^{-x_2-x_3} + 2e^{2x_2-x_3} - e^{-x_2+2x_3} + e^{x_2-x_3} - e^{-x_2+x_3} + e^{x_2} - e^{-x_2} = 0,\\
&-e^{-x_2-x_3} - e^{2x_2-x_3} + 2e^{-x_2+2x_3} - e^{x_2-x_3} + e^{-x_2+x_3} + e^{x_3} - e^{-x_3} = 0,
\end{aligned}$$

which when we use λ_2 and λ_3, become

$$-\lambda_2^{-1}\lambda_3^{-1} + 2\lambda_2^2\lambda_3^{-1} - \lambda_2^{-1}\lambda_3^2 + \lambda_2\lambda_3^{-1} - \lambda_2^{-1}\lambda_3 + \lambda_2 - \lambda_2^{-1} = 0, \tag{3.4}$$

$$-\lambda_2^{-1}\lambda_3^{-1} - \lambda_2^2\lambda_3^{-1} + 2\lambda_2^{-1}\lambda_3^2 - \lambda_2\lambda_3^{-1} + \lambda_2^{-1}\lambda_3 + \lambda_3 - \lambda_3^{-1} = 0. \tag{3.5}$$

Subtracting the second equation from the first one and manipulate it algebraically, we obtain

$$\begin{aligned}
0 &= \lambda_2^{-1}\lambda_3^{-1}(3\lambda_2^3 - 3\lambda_3^3 + 2\lambda_2^2 - 2\lambda_3^2 + \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2 + \lambda_2 - \lambda_3)\\
&= \lambda_2^{-1}\lambda_3^{-1}(\lambda_2 - \lambda_3)(3\lambda_2^2 + 4\lambda_2\lambda_3 + 3\lambda_3^2 + 2\lambda_2 + 2\lambda_3 + 1).
\end{aligned}$$

Because $\lambda_2 > 0$ and $\lambda_3 > 0$, we obtain that $\lambda_2 = \lambda_3$. Using (3.4) or (3.5), we have $0 = 2\lambda_2^3 - \lambda_2 - 1 = (\lambda_2 - 1)(2\lambda_2^2 + 2\lambda_2 + 1)$, which gives $\lambda_2 = 1$. Hence, $(\lambda_2, \lambda_3) = (1, 1)$, which is not a feasible solution of the optimization problem since it violates the constraint $\lambda_2 + \lambda_3 \leq 1$. This shows that $\nu \neq 0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

$L_{x_2}(x,\mu,\nu)=0$ and $L_{x_3}(x,\mu,\nu)=0$ can be written as:

$$f_{x_2}(x)+g_{x_2}(x)+\nu(-f_{x_2}(x)+g_{x_2}(x))=0, \tag{3.6}$$

$$f_{x_3}(x)+g_{x_3}(x)+\nu(-f_{x_3}(x)+g_{x_3}(x))=0. \tag{3.7}$$

From (3.6), we obtain that $\nu=-\frac{f_{x_2}(x)+g_{x_2}(x)}{-f_{x_2}(x)+g_{x_2}(x)}$, which plugging into (3.7) gives

$$(f_{x_3}(x)+g_{x_3}(x))-\frac{(f_{x_2}(x)+g_{x_2}(x))(-f_{x_3}(x)+g_{x_3}(x))}{-f_{x_2}(x)+g_{x_2}(x)}=0.$$

Simplifying the equation, we obtain $f_{x_2}(x)g_{x_3}(x)=f_{x_3}(x)g_{x_2}(x)$. Note that

$$\begin{aligned}
f_{x_2}&=-e^{-x_2-x_3}+2e^{2x_2-x_3}-e^{-x_2+2x_3}=-\lambda_2^{-1}\lambda_3^{-1}+2\lambda_2^2\lambda_3^{-1}-\lambda_2^{-1}\lambda_3^2\\
g_{x_2}&=e^{x_2-x_3}-e^{-x_2+x_3}+e^{x_2}-e^{-x_2}=\lambda_2\lambda_3^{-1}-\lambda_2^{-1}\lambda_3+\lambda_2-\lambda_2^{-1}\\
f_{x_3}&=-e^{-x_2-x_3}-e^{2x_2-x_3}+2e^{-x_2+2x_3}=-\lambda_2^{-1}\lambda_3^{-1}-\lambda_2^2\lambda_3^{-1}+2\lambda_2^{-1}\lambda_3^2\\
g_{x_3}&=-e^{x_2-x_3}+e^{-x_2+x_3}+e^{x_3}-e^{-x_3}=-\lambda_2\lambda_3^{-1}+\lambda_2^{-1}\lambda_3+\lambda_3-\lambda_3^{-1},
\end{aligned}$$

we get

$$\begin{aligned}
f_{x_2}(x)g_{x_3}(x)&=(-\lambda_2^{-1}\lambda_3^{-1}+2\lambda_2^2\lambda_3^{-1}-\lambda_2^{-1}\lambda_3^2)(-\lambda_2\lambda_3^{-1}+\lambda_2^{-1}\lambda_3-\lambda_3^{-1}+\lambda_3)\\
&=\lambda_3^{-2}-\lambda_2^{-2}+\lambda_2^{-1}\lambda_3^{-2}-\lambda_2^{-1}-2\lambda_2^3\lambda_3^{-2}\\
&\qquad+2\lambda_2-2\lambda_2^2\lambda_3^{-2}+2\lambda_2^2+\lambda_3-\lambda_2^{-2}\lambda_3^3+\lambda_2^{-1}\lambda_3-\lambda_2^{-1}\lambda_3^3,
\end{aligned}$$

and

$$\begin{aligned}
f_{x_3}(x)g_{x_2}(x)&=(-\lambda_2^{-1}\lambda_3^{-1}-\lambda_2^2\lambda_3^{-1}+2\lambda_2^{-1}\lambda_3^2)(\lambda_2\lambda_3^{-1}-\lambda_2^{-1}\lambda_3-\lambda_2^{-1}+\lambda_2)\\
&=-\lambda_3^{-2}+\lambda_2^{-2}+\lambda_2^{-2}\lambda_3^{-1}-\lambda_3^{-1}-\lambda_2^3\lambda_3^{-2}+\lambda_2\\
&\qquad+\lambda_2\lambda_3^{-1}-\lambda_2^3\lambda_3^{-1}+2\lambda_3-2\lambda_2^{-2}\lambda_3^3-2\lambda_2^{-2}\lambda_3^2+2\lambda_3^2.
\end{aligned}$$

In view of $f_{x_2}(x)g_{x_3}(x)=f_{x_3}(x)g_{x_2}(x)$, we obtain

$$\begin{aligned}
&2\lambda_3^{-2}-2\lambda_2^{-2}+\lambda_2^{-1}\lambda_3^{-2}-\lambda_2^{-2}\lambda_3^{-1}-\lambda_2^{-1}+\lambda_2^{-1}\lambda_3-\lambda_2^{-1}\lambda_3^3-\lambda_2^3\lambda_3^{-2}+\lambda_2\\
&\quad-2\lambda_2^2\lambda_3^{-2}+2\lambda_2^2+\lambda_3^{-1}-\lambda_2\lambda_3^{-1}+\lambda_2^3\lambda_3^{-1}-\lambda_3+\lambda_2^{-2}\lambda_3^3+2\lambda_2^{-2}\lambda_3^2-2\lambda_3^2=0.
\end{aligned} \tag{3.8}$$

By $\nu(-f(x)+g(x)+2)=0$ and $\nu\neq 0$, we have $-f(x)+g(x)+2=0$, which is the same as

$$1+\lambda_2^3+\lambda_3^3=\lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_2^2+2\lambda_2\lambda_3+\lambda_3^2+\lambda_2+\lambda_3. \tag{3.9}$$

Hence,

$$\begin{aligned}
-\lambda_2^{-1}+\lambda_2^{-1}\lambda_3-\lambda_2^{-1}\lambda_3^3&=-\lambda_2^{-1}(1-\lambda_3+\lambda_3^3)\\
&=-\lambda_2^{-1}(-\lambda_2^3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_2^2+2\lambda_2\lambda_3+\lambda_3^2+\lambda_2)\\
&=\lambda_2^2-\lambda_2\lambda_3-\lambda_3^2-\lambda_2-2\lambda_3-\lambda_2^{-1}\lambda_3^2-1
\end{aligned}$$

and

$$\begin{aligned}
\lambda_3^{-1}-\lambda_2\lambda_3^{-1}+\lambda_2^3\lambda_3^{-1}&=\lambda_3^{-1}(1-\lambda_2+\lambda_2^3)\\
&=\lambda_3^{-1}(-\lambda_3^3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_2^2+2\lambda_2\lambda_3+\lambda_3^2+\lambda_3)\\
&=-\lambda_3^2+\lambda_2^2+\lambda_2\lambda_3+\lambda_2^2\lambda_3^{-1}+2\lambda_2+\lambda_3+1.
\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

Plugging the above into (3.8) and simplifying, we have

$$\begin{aligned}
0 &= 2\lambda_3^{-2} - 2\lambda_2^{-2} + \lambda_2^{-1}\lambda_3^{-2} - \lambda_2^{-2}\lambda_3^{-1} - \lambda_2^{3}\lambda_3^{-2} + 2\lambda_2 - 2\lambda_3 + \lambda_2^{-2}\lambda_3^{3} + \lambda_2^{2}\lambda_3^{-1} - \lambda_2^{-1}\lambda_3^{2} \\
&\qquad - 2\lambda_2^{2}\lambda_3^{-2} + 2\lambda_2^{-2}\lambda_3^{2} + 4\lambda_2^{2} - 4\lambda_3^{2} \\
&= \lambda_2^{-2}\lambda_3^{-2}\left(2\lambda_2^2 - 2\lambda_3^2 + \lambda_2 - \lambda_3 - \lambda_2^5 + 2\lambda_2^3\lambda_3^2 - 2\lambda_2^2\lambda_3^3 + \lambda_3^5 + \lambda_2^4\lambda_3 - \lambda_2\lambda_3^4\right. \\
&\qquad \left. -2\lambda_2^4 + 2\lambda_3^4 + 4\lambda_2^4\lambda_3^2 - 4\lambda_2^2\lambda_3^4\right) \\
&= \lambda_2^{-2}\lambda_3^{-2}\left[2(\lambda_2^2 - \lambda_3^2) + (\lambda_2 - \lambda_3) - \lambda_2^3(\lambda_2^2 - \lambda_3^2) + \lambda_2^2\lambda_3^2(\lambda_2 - \lambda_3) - \lambda_3^3(\lambda_2^2 - \lambda_3^2)\right. \\
&\qquad \left. +\lambda_2\lambda_3(\lambda_2^3 - \lambda_3^3) - 2(\lambda_2^4 - \lambda_3^4) + 4\lambda_2^2\lambda_3^2(\lambda_2^2 - \lambda_3^2)\right] \\
&= \lambda_2^{-2}\lambda_3^{-2}(\lambda_2 - \lambda_3)\left[2(\lambda_2 + \lambda_3) + 1 - \lambda_2^3(\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3^2 - \lambda_3^3(\lambda_2 + \lambda_3)\right. \\
&\qquad \left. +\lambda_2\lambda_3(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2) - 2(\lambda_2^2 + \lambda_3^2)(\lambda_2 + \lambda_3) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3)\right].
\end{aligned}$$

Therefore, either $\lambda_2 - \lambda_3 = 0$ or

$$\begin{aligned}
2(\lambda_2 + \lambda_3) + 1 - \lambda_2^3(\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3^2 - \lambda_3^3(\lambda_2 + \lambda_3) + \lambda_2\lambda_3(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2)& \\
- 2(\lambda_2^2 + \lambda_3^2)(\lambda_2 + \lambda_3) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3) = 0.&
\end{aligned}$$

Since

$$\begin{aligned}
&2(\lambda_2 + \lambda_3) + 1 - \lambda_2^3(\lambda_2 + \lambda_3) + \lambda_2^2\lambda_3^2 - \lambda_3^3(\lambda_2 + \lambda_3) + \lambda_2\lambda_3(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2) \\
&\qquad - 2(\lambda_2^2 + \lambda_3^2)(\lambda_2 + \lambda_3) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3) \\
&= 2(\lambda_2 + \lambda_3) + 1 - \lambda_2^3(\lambda_2 + \lambda_3) - \lambda_3^3(\lambda_2 + \lambda_3) + \lambda_2\lambda_3(\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2) \\
&\qquad - 2(\lambda_2^2 + \lambda_3^2)(\lambda_2 + \lambda_3) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3) \\
&= 2(\lambda_2 + \lambda_3) + 1 - \lambda_2^3(\lambda_2 + \lambda_3) - \lambda_3^3(\lambda_2 + \lambda_3) + \lambda_2\lambda_3(\lambda_2 + \lambda_3)^2 \\
&\qquad - 2(\lambda_2^2 + \lambda_3^2)(\lambda_2 + \lambda_3) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3) \\
&= 2(\lambda_2 + \lambda_3)(1 - (\lambda_2^2 + \lambda_3^2)) + (1 - \lambda_2^3(\lambda_2 + \lambda_3)) \\
&\qquad\qquad + \lambda_3(\lambda_2 + \lambda_3)(\lambda_2(\lambda_2 + \lambda_3) - \lambda_3^2) + 4\lambda_2^2\lambda_3^2(\lambda_2 + \lambda_3) \\
&> 0 \text{ (due to } \lambda_2 \geq \lambda_3 > 0 \text{ and } \lambda_2 + \lambda_3 \leq 1),
\end{aligned}$$

we know that $\lambda_2 = \lambda_3$. By (3.9), we get $4\lambda_2^2 + 2\lambda_2 - 1 = 0$. Solving this equation, we find that $\lambda_2 = \frac{-1+\sqrt{5}}{4}$. Therefore, the three positive eigenvalues are given by $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \frac{-1+\sqrt{5}}{4}$. The two negative eigenvalues are those given in Example 1, which are $\lambda_4 = \lambda_5 = \frac{-1-\sqrt{5}}{4}$. Hence, the maximal angle between the 5×5 positive semidefinite matrix cone and the 5×5 nonnegative matrix cone is $\angle(\mathcal{P}_5, \mathcal{N}_5) = \arccos\left(-\frac{1+1/\sqrt{5}}{2}\right) > \frac{3\pi}{4}$.

REMARK 3.4. In [3], an example was given to show that the maximal angle between the positive semidefinite matrix cone and nonnegative cone is more than $\frac{3\pi}{4}$. Specifically, the maximal angle between the adjacency matrix of the 5-cycle, which is a trace zero nonnegative 5×5 matrix, and the positive semidefinite matrix cone was obtained that is $\arccos\left(-\frac{1+1/\sqrt{5}}{2}\right)$. The eigenvalues of the adjacency matrix of the 5-cycle are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = \frac{-1+\sqrt{5}}{2}$, and $\lambda_4 = \lambda_5 == \frac{-1-\sqrt{5}}{2}$, which is a multiple of the eigenvalues of the matrix N in the above discussion. Therefore, while in [3] the authors calculated the maximal angle between the adjacency matrix of the 5-cycle and the positive semidefinite matrix cone, which is $\arccos\left(-\frac{1+1/\sqrt{5}}{2}\right)$, we prove in this paper that such an angle is actually the maximal angle between the 5×5 positive semidefinite matrix cone and 5×5 nonnegative matrix cone.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 37, pp. 698-708, November 2021.

4. Concluding remarks. We have provided a different proof for a theorem in [3] about the maximal angle between the $n \times n$ positive semidefinite matrix cone and $n \times n$ nonnegative matrix cone for $n \leq 4$ by using basic algebra. Such an approach has been extended to study the maximal angle between these two matrix cones for $n = 5$. Using the method of Lagrange Multipliers, we have shown in this paper that the maximal angle between the 5×5 positive semidefinite matrix cone and 5×5 nonnegative matrix cone is $\angle(\mathcal{P}_5, \mathcal{N}_5) = \arccos\left(-\frac{1+1/\sqrt{5}}{2}\right)$. We hope that the approach used in this paper will provide some insight about the problem to find the maximal angles in the copositive cone and/or in $Cone(\mathcal{P}_n + \mathcal{N}_n)$ for $n \geq 3$, which is still an open problem as mentioned in [1].

Acknowledgment. The author would like to thank Professor Dennis Bricker for his time to solve the signomial program numerically. Professor Bricker's numerical solution, which is obtained by using sequential posynomial Geometric Programming approximations, matches the solution obtained by the method of Lagrange Multipliers. The author is also grateful to an anonymous referee for helpful comments and suggestions.

REFERENCES

[1] A. Berman, M. Dür, and N. Shaked-Monderer. Open problems in the theory of completely positive and copositive matrices. *Electron. J. Linear Algebra,* 29:46–58, 2015.

[2] S. Boyd, S. Kim, L. Vandenberghe, and A. Hassibi. A tutorial on geometric programming. *Optim. Eng.*, 8:67–127, 2007.

[3] F. Goldberg and N. Shaked-Monderer. On the maximal angle between copositive matrices. *Electron. J. Linear Algebra*, 27:837–850, 2014.

[4] J.-B. Hiriart-Urruty and A. Seeger. A variational approach to copositive matrices. *SIAM Rev.*, 52:593–629, 2010.

[5] O. Spector. A characterization of trace zero symmetric nonnegative 5×5 matrices. *Linear Algebra Appl.*, 434:1000–1017, 2011.