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GENERALIZED PASCAL TRIANGLES AND TOEPLITZ MATRICES*

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Abstract. The purpose of this article is to study determinants of matrices which are known as generalized Pascal triangles (see R. Bacher. Determinants of matrices related to the Pascal triangle. *J. Théor. Nombres Bordeaux*, 14:19–41, 2002). This article presents a factorization by expressing such a matrix as a product of a unipotent lower triangular matrix, a Toeplitz matrix, and a unipotent upper triangular matrix. The determinant of a generalized Pascal matrix equals thus the determinant of a Toeplitz matrix. This equality allows for the evaluation of a few determinants of generalized Pascal matrices associated with certain sequences. In particular, families of quasi-Pascal matrices are obtained whose leading principal minors generate any arbitrary linear subsequences $(\mathcal{F}_{nr+s})_{n\geq 1}$ or $(\mathcal{L}_{nr+s})_{n\geq 1}$ of the Fibonacci or Lucas sequence. New matrices are constructed whose entries are given by certain linear non-homogeneous recurrence relations, and the leading principal minors of which form the Fibonacci sequence.

Key words. Determinant, Matrix factorization, Generalized Pascal triangle, Generalized symmetric (skymmetric) Pascal triangle, Toeplitz matrix, Recursive relation, Fibonacci (Lucas, Catalan) sequence, Golden ratio.

AMS subject classifications. 15A15, 11C20.

1. Introduction.

Let $P(\infty)$ be the infinite symmetric matrix with entries $P_{i,j} = \binom{i+j}{i}$ for $i, j \geq 0$. The matrix $P(\infty)$ is hence the famous Pascal triangle yielding the binomial coefficients. The entries of $P(\infty)$ satisfy the recurrence relation $P_{i,j} = P_{i-1,j} + P_{i,j-1}$. Indeed, this matrix has the following form:

$$P(\infty) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots \\ 1 & 3 & 6 & 10 & 15 & \dots \\ 1 & 4 & 10 & 20 & 35 & \dots \\ 1 & 5 & 15 & 35 & 70 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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One can easily verify that (see [2, 6]):

$$P(\infty) = L(\infty) \cdot L(\infty)^t, \tag{1.1}$$

where $L(\infty)$ is the infinite unipotent lower triangular matrix

$$L(\infty) = (L_{i,j})_{i,j\geq 0} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.2}$$

with entries $L_{i,j} = \binom{i}{j}$. We denote by $L(n)$ the finite submatrix of $L(\infty)$ with entries $L_{i,j}$, $0 \leq i, j \leq n-1$.

To introduce our result, we first present some notation and definitions. We recall that a matrix $T(\infty) = (t_{i,j})_{i,j\geq 0}$ is said to be Toeplitz if $t_{i,j} = t_{k,l}$ whenever $i - j = k - l$. Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences with $\alpha_0 = \beta_0$. We shall denote by $T_{\alpha,\beta}(\infty) = (t_{i,j})_{i,j\geq 0}$ the Toeplitz matrix with $t_{i,0} = \alpha_i$ and $t_{0,j} = \beta_j$. We also denote by $T_{\alpha,\beta}(n)$ the submatrix of $T_{\alpha,\beta}(\infty)$ consisting of the entries in its first n rows and columns.

We come now back to the (1.1). In fact, one can rewrite it as follows:

$$P(\infty) = L(\infty) \cdot I \cdot L(\infty)^t,$$

where matrix I (identity matrix) is a particular case of a Toeplitz matrix.

In [1], Bacher considers determinants of matrices generalizing the Pascal triangle $P(\infty)$. He introduces generalized Pascal triangles as follows. Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences starting with a common first term $\alpha_0 = \beta_0 = \gamma$. Then, the generalized Pascal triangle associated with α and β, is the infinite matrix $P_{\alpha,\beta}(\infty) = (P_{i,j})_{i,j\geq 0}$ with entries $P_{i,0} = \alpha_i$, $P_{0,j} = \beta_j$ $(i, j \geq 0)$ and

$$P_{i,j} = P_{i-1,j} + P_{i,j-1}, \quad \text{for} \;\; i, j \geq 1.$$

We denote by $P_{\alpha,\beta}(n)$ the finite submatrix of $P_{\alpha,\beta}(\infty)$ with entries $P_{i,j}$, $0 \leq i, j \leq n-1$. An explicit formula for entry $P_{i,j}$ of $P_{\alpha,\beta}(n)$ is also given by the following formula (see [1]):

$$P_{i,j} = \gamma\binom{i+j}{j} + \Big(\sum_{s=1}^{i}(\alpha_s - \alpha_{s-1})\binom{i+j-s}{j}\Big) + \Big(\sum_{t=1}^{j}(\beta_t - \beta_{t-1})\binom{i+j-t}{i}\Big).$$

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For an arbitrary sequence $\alpha = (\alpha_i)_{i\geq 0}$, we define the sequences $\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0}$ and $\check{\alpha} = (\check{\alpha}_i)_{i\geq 0}$ as follows:

$$\hat{\alpha}_i = \sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\alpha_k \quad \text{and} \quad \check{\alpha}_i = \sum_{k=0}^{i}\binom{i}{k}\alpha_k. \tag{1.3}$$

With these definitions we can now state our main result. Indeed, the purpose of this article is to obtain a *factorization* of the generalized Pascal triangle $P_{\alpha,\beta}(n)$ associated with the arbitrary sequences α and β, as a product of a unipotent lower triangular matrix $L(n)$, a Toeplitz matrix $T_{\hat{\alpha},\hat{\beta}}(n)$ and a unipotent upper triangular matrix $U(n)$ (see Theorem 3.1), that is

$$P_{\alpha,\beta}(n) = L(n) \cdot T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n).$$

Similarly, we show that

$$T_{\alpha,\beta}(n) = L(n)^{-1} \cdot P_{\check{\alpha},\check{\beta}}(n) \cdot U(n)^{-1}.$$

In fact, we obtain a *connection* between generalized Pascal triangles and Toeplitz matrices. In view of these factorizations, we can easily see that

$$\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n)).$$

Finally, we present *several applications* of Theorem 3.1 to some other determinant evaluations.

We conclude the introduction with notation and terminology to be used throughout the article. For convenience, we will let $\sqrt{-1}$ denote the complex number $i \in \mathbb{C}$. By $\lfloor x \rfloor$ we denote the integer part of x, i.e., the greatest integer that is less than or equal to x. We also denote by $\lceil x \rceil$ the smallest integer greater than or equal to x. Given a matrix A, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. We use the notation A^t for the transpose of A.

In general, an $n \times n$ matrix of the following form:

$$\left[\begin{array}{c|c} A & B \\ \hline C & P_{\alpha,\beta}(n-k) \end{array}\right] \quad \left(\text{resp. } \left[\begin{array}{c|c} A & B \\ \hline C & T_{\alpha,\beta}(n-k) \end{array}\right]\right)$$

where A, B and C are arbitrary matrices of order $k \times k$, $k \times (n-k)$ and $(n-k) \times k$, respectively, is called a *quasi-Pascal* (resp. *quasi-Toeplitz*) matrix.

Throughout this article we assume that:

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$\mathcal{F} = (\mathcal{F}_i)_{i\geq 0} = (0, 1, 1, 2, 3, 5, 8, \ldots, \mathcal{F}_i, \ldots)$ (Fibonacci numbers),

$\mathcal{F}^* = (\mathcal{F}_i)_{i\geq 1} = (1, 1, 2, 3, 5, 8, \ldots, \mathcal{F}_i, \ldots)$ (Fibonacci numbers $\neq 0$),

$\mathcal{L} = (\mathcal{L}_i)_{i\geq 0} = (2, 1, 3, 4, 7, 11, 18, \ldots, \mathcal{L}_i, \ldots)$ (Lucas numbers),

$\mathcal{C} = (\mathcal{C}_i)_{i\geq 0} = (1, 1, 2, 5, 14, 42, 132, \ldots, \mathcal{C}_i, \ldots)$ (Catalan numbers),

$\mathcal{I} = (\mathcal{I}_i)_{i\geq 0} = (0!, 1!, 2!, 3!, 4!, 5!, 6!, \ldots, \mathcal{I}_i = i!, \ldots)$

$\mathcal{I}^* = (\mathcal{I}_i)_{i\geq 1} = (1!, 2!, 3!, 4!, 5!, 6!, \ldots, \mathcal{I}_i = i!, \ldots)$

The rest of this article is organized as follows. In Section 2, we derive some preparatory results. In Section 3, we prove the main result (Theorem 3.1). Section 4 deals with applications of Theorem 3.1. In Section 5, we present two matrices whose entries are recursively defined, and we show that the leading principal minor sequence of these matrices is the sequence $\mathcal{F}^* = (\mathcal{F}_i)_{i\geq 1}$.

2. Preliminary Results. As we mentioned in the Introduction, if $\alpha = (\alpha_i)_{i\geq 0}$ is an arbitrary sequence, then we define the sequences $\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0}$ and $\check{\alpha} = (\check{\alpha}_i)_{i\geq 0}$ as in (1.3). For certain sequences α the associated sequences $\hat{\alpha}$ and $\check{\alpha}$ seem also to be of interest since they have appeared elsewhere. In Tables 1 and 2, we have presented some sequences α and the associated sequences $\hat{\alpha}$ and $\check{\alpha}$.

Table 1. Some sequences α and associated sequences $\check{\alpha}$.

α	Reference	$\check{\alpha}$
$(0, 1, -1, 2, -3, 5, -8, \ldots)$	**A**039834 in [8]	$\mathcal{F}$
$(1, 0, 1, -1, 2, -3, 5, -8, \ldots)$	**A**039834 in [8]	$\mathcal{F}^*$
$(2, -1, 3, -4, 7, -11, 18, \ldots)$	**A**061084 in [8]	$\mathcal{L}$
$(1, 0, 1, 1, 3, 6, 15, \ldots)$	**A**005043 in [8]	$\mathcal{C}$
$(1, 0, 1, 2, 9, 44, 265, \ldots)$	**A**000166 in [8]	$\mathcal{I}$
$(1, 1, 3, 11, 53, 309, 2119, \ldots)$	**A**000255 in [8]	$\mathcal{I}^*$

Table 2. Some sequences α and associated sequences $\hat{\alpha}$.

α	Reference	$\hat{\alpha}$
$(0, 1, 3, 8, 21, 55, 144, \ldots)$	**A**001906 in [8]	$\mathcal{F}$
$(1, 2, 5, 13, 34, 89, 233, \ldots)$	**A**001519 in [8]	$\mathcal{F}^*$
$(2, 3, 7, 18, 47, 123, 322, \ldots)$	**A**005248 in [8]	$\mathcal{L}$
$(1, 2, 5, 15, 51, 188, 731, \ldots)$	**A**007317 in [8]	$\mathcal{C}$
$(1, 2, 5, 16, 65, 326, 1957, \ldots)$	**A**000522 in [8]	$\mathcal{I}$
$(1, 3, 11, 49, 261, 1631, \ldots)$	**A**001339 in [8]	$\mathcal{I}^*$

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As another nice example, consider $\alpha = \mathrm{R}_i(L(\infty))$ where L is introduced as in (1.2). Then, we have

$$\check{\alpha} = \mathrm{R}_i(P(\infty)).$$

Lemma 2.1. *Let α be an arbitrary sequence. Then, we have $\hat{\check{\alpha}} = \check{\hat{\alpha}} = \alpha$.*

Proof. Suppose $\alpha = (\alpha_i)_{i\geq 0}$ and $\hat{\check{\alpha}} = (\hat{\check{\alpha}}_i)_{i\geq 0}$. Then, we have

$$\begin{aligned}
\hat{\check{\alpha}}_i &= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\check{\alpha}_k \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\sum_{s=0}^{k}\binom{k}{s}\alpha_s \\
&= \textstyle\sum_{k=0}^{i}\sum_{s=0}^{k}(-1)^{i+k}\binom{i}{k}\binom{k}{s}\alpha_s \\
&= \textstyle(-1)^i\sum_{l=0}^{i}\alpha_l\sum_{h=l}^{i}(-1)^h\binom{i}{h}\binom{h}{l} \\
&= \textstyle(-1)^i\sum_{l=0}^{i}\alpha_l\sum_{h=l}^{i}(-1)^h\binom{i}{l}\binom{i-l}{h-l} \\
&= \textstyle(-1)^i\sum_{l=0}^{i}\alpha_l\binom{i}{l}\sum_{h=l}^{i}(-1)^h\binom{i-l}{h-l}.
\end{aligned}$$

But, if $l < i$, then we have $\sum_{h=l}^{i}(-1)^h\binom{i-l}{h-l} = 0$. Therefore, we obtain

$$\hat{\check{\alpha}}_i = (-1)^i\sum_{l=i}^{i}\alpha_l\binom{i}{l}\sum_{h=l}^{i}(-1)^h\binom{i-l}{h-l} = \alpha_i,$$

and hence $\hat{\check{\alpha}} = \alpha$.

The proof of second part is similar to the previous case. □

Lemma 2.2. *Let i, j be positive integers. Then, we have*

$$\sum_{k=0}^{i-j}(-1)^k\binom{i}{k+j}\binom{k+j}{j} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The *proof* follows from the easy identity

$$\binom{i}{k+j}\binom{k+j}{j} = \binom{i}{j}\binom{i-j}{k}.$$

The following Lemma is a special case of a general result due to Krattenthaler (see Theorem 1 in [9]).

Lemma 2.3. *Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_j)_{j\geq 0}$ be two geometric sequences with $\alpha_i = \rho^i$ and $\beta_j = \sigma^j$. Then, we have*

$$\det(P_{\alpha,\beta}(n)) = (\rho + \sigma - \rho\sigma)^{n-1}.$$

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3. Main Result. Now, we are in the position to state and prove the main result of this article.

THEOREM 3.1. *Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences starting with a common first term $\alpha_0 = \beta_0 = \gamma$. Then, we have*

$$P_{\alpha,\beta}(n) = L(n) \cdot T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n), \tag{3.1}$$

and

$$T_{\alpha,\beta}(n) = L(n)^{-1} \cdot P_{\check{\alpha},\check{\beta}}(n) \cdot U(n)^{-1}, \tag{3.2}$$

where $L(n) = (L_{i,j})_{0\leq i,j<n}$ is a lower triangular matrix with

$$L_{i,j} = \begin{cases} 0 & \textit{if} \quad i < j \\ \binom{i}{j} & \textit{if} \quad i \geq j, \end{cases}$$

and $U(n) = L(n)^t$. In particular, we have $\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n))$.

Proof. First, we claim that

$$P_{\alpha,\beta}(n) = L(n) \cdot Q(n),$$

where $L(n) = (L_{i,j})_{0\leq i,j<n}$ is a lower triangular matrix with

$$L_{i,j} = \begin{cases} 0 & \text{if} \quad i < j \\ \binom{i}{j} & \text{if} \quad i \geq j, \end{cases}$$

and $Q(n) = (Q_{i,j})_{0\leq i,j<n}$ with $Q_{i,0} = \hat{\alpha}_i$, $Q_{0,i} = \beta_i$ and

$$Q_{i,j} = Q_{i-1,j-1} + Q_{i,j-1}, \qquad 1 \leq i,j < n. \tag{3.3}$$

For instance, when $n = 4$ the matrices $L(4)$ and $Q(4)$ are given by:

$$L(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix},$$

and

$$Q(4) = \begin{bmatrix} \gamma & \beta_1 & \beta_2 & \beta_3 \\ -\gamma + \alpha_1 & \alpha_1 & \beta_1 + \alpha_1 & \beta_1 + \beta_2 + \alpha_2 \\ \gamma - 2\alpha_1 + \alpha_2 & -\alpha_1 + \alpha_2 & \alpha_2 & \beta_1 + \alpha_1 + \alpha_2 \\ -\gamma + 3\alpha_1 - 3\alpha_2 + \alpha_3 & \alpha_1 - 2\alpha_2 + \alpha_3 & -\alpha_2 + \alpha_3 & \alpha_3 \end{bmatrix}.$$

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Note that the entries of $L(n)$ satisfying in the following recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 1 \leq i, j < n. \tag{3.4}$$

For the proof of the claimed factorization we compute the (i,j)-th entry of $L(n)\cdot Q(n)$, that is

$$(L(n)\cdot Q(n))_{i,j} = \sum_{k=1}^{n} L_{i,k} Q_{k,j}.$$

In fact, it suffices to show that

$$\mathrm{R}_0(L(n)\cdot Q(n)) = \mathrm{R}_0(P_{\alpha,\beta}(n)),$$
$$\mathrm{C}_0(L(n)\cdot Q(n)) = \mathrm{C}_0(P_{\alpha,\beta}(n))$$

and

$$(L(n)\cdot Q(n))_{i,j} = (L(n)\cdot Q(n))_{i,j-1} + (L(n)\cdot Q(n))_{i-1,j}, \tag{3.5}$$

for $1 \leq i, j < n$.

First, suppose that $i = 0$. Then, we obtain

$$(L(n)\cdot Q(n))_{0,j} = \sum_{k=0}^{n-1} L_{0,k} Q_{k,j} = L_{0,0} Q_{0,j} = \beta_j,$$

and so $\mathrm{R}_0(L(n)\cdot Q(n)) = \mathrm{R}_0(P_{\alpha,\beta}(n)) = (\beta_0,\ \beta_1,\ \ldots,\ \beta_{n-1})$.

Next, suppose that $i \geq 1$ and $j = 0$. In this case, we have

$$\begin{aligned} (L(n)\cdot Q(n))_{i,0} &= \textstyle\sum_{k=0}^{n-1} L_{i,k} Q_{k,0} \\ &= \textstyle\sum_{k=0}^{i} \left\{ \binom{i}{k} \sum_{l=0}^{k} (-1)^{l+k} \binom{k}{l} \beta_l \right\} \\ &= \textstyle\sum_{j=0}^{i} \beta_j \left\{ \sum_{t=0}^{i-j} (-1)^t \binom{i}{t+j} \binom{t+j}{j} \right\} \\ &= \beta_i, \quad \text{(by Lemma 2.2)} \end{aligned}$$

and so $\mathrm{C}_0(L(n)\cdot Q(n)) = \mathrm{C}_0(P_{\alpha,\beta}(n)) = (\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{n-1})^t$.

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Finally, we must establish (3.5). Therefore, we assume that $1 \le i, j < n$. In this case, we have

$$\begin{aligned}
(L(n)\cdot Q(n))_{i,j} &= \textstyle\sum_{k=0}^{n-1} L_{i,k}Q_{k,j} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=1}^{n-1} L_{i,k}Q_{k,j} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=1}^{n-1} L_{i,k}(Q_{k-1,j-1} + Q_{k,j-1}) \quad \text{(by (3.3))} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=1}^{n-1} L_{i,k}Q_{k-1,j-1} + \sum_{k=1}^{n-1} L_{i,k}Q_{k,j-1} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=1}^{n-1}(L_{i-1,k-1} + L_{i-1,k})Q_{k-1,j-1} \\
&\quad + \textstyle\sum_{k=0}^{n-1} L_{i,k}Q_{k,j-1} - L_{i,0}Q_{0,j-1} \text{ (by (3.4))} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=1}^{n-1} L_{i-1,k-1}Q_{k-1,j-1} \\
&\quad + \textstyle\sum_{k=1}^{n-1} L_{i-1,k}(Q_{k,j} - Q_{k,j-1}) + (L(n)\cdot Q(n))_{i,j-1} \\
&\quad - L_{i,0}Q_{0,j-1} \quad \text{(by (3.3))} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=0}^{n-2} L_{i-1,k}Q_{k,j-1} + \sum_{k=1}^{n-1} L_{i-1,k}Q_{k,j} \\
&\quad - \textstyle\sum_{k=0}^{n-1} L_{i-1,k}Q_{k,j-1} + L_{i-1,0}Q_{0,j-1} + (L(n)\cdot Q(n))_{i,j-1} \\
&\quad - L_{i,0}Q_{0,j-1} \\
&= L_{i,0}Q_{0,j} + \textstyle\sum_{k=0}^{n-1} L_{i-1,k}Q_{k,j-1} + \sum_{k=0}^{n-1} L_{i-1,k}Q_{k,j} \\
&\quad - L_{i-1,0}Q_{0,j} - \textstyle\sum_{k=0}^{n-1} L_{i-1,k}Q_{k,j-1} + L_{i-1,0}Q_{0,j-1} \\
&\quad + (L(n)\cdot Q(n))_{i,j-1} - L_{i,0}Q_{0,j-1} \quad \text{(note that } L_{i-1,n-1} = 0) \\
&= (L_{i,0} - L_{i-1,0})Q_{0,j} + (L(n)\cdot Q(n))_{i-1,j} \\
&\quad + (L_{i-1,0} - L_{i,0})Q_{0,j-1} + (L(n)\cdot Q(n))_{i,j-1} \\
&= (L(n)\cdot Q(n))_{i-1,j} + (L(n)\cdot Q(n))_{i,j-1}, \\
&\qquad \text{(note that } L_{i,0} = L_{i-1,0} = 1)
\end{aligned}$$

which is (3.5).

Next, we claim that

$$Q(n) = T_{\hat\alpha,\hat\beta}(n)\cdot U(n),$$

where $U(n) = L(n)^t$ and $T_{\alpha,\beta}(n) = (T_{i,j})_{0\le i,j<n}$ with $T_{i,0} = \hat\alpha_i$, $T_{0,j} = \hat\beta_j$, and

$$T_{i,j} = T_{i-1,j-1}, \quad 1 \le i, j < n.$$

Note that, we have

$$U_{i,j} = U_{i-1,j-1} + U_{i,j-1}, \quad 1 \le i, j < n. \tag{3.6}$$

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For instance, when $n = 4$ the matrices $T_{\hat{\alpha},\hat{\beta}}(4)$ and $U(4)$ are given by:

$$T_{\hat{\alpha},\hat{\beta}}(4) = \begin{bmatrix} \gamma & -\gamma+\beta_1 & \gamma-2\beta_1+\beta_2 & -\gamma+3\beta_1-3\beta_2+\beta_3 \\ -\gamma+\alpha_1 & \gamma & -\gamma+\beta_1 & \gamma-2\beta_1+\beta_2 \\ \gamma-2\alpha_1+\alpha_2 & -\gamma+\alpha_1 & \gamma & -\gamma+\beta_1 \\ -\gamma+3\alpha_1-3\alpha_2+\alpha_3 & \gamma-2\alpha_1+\alpha_2 & -\gamma+\alpha_1 & \gamma \end{bmatrix}$$

and

$$U(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As before, the proof of the claim requires some calculations. If we have $i = 0$, then

$$\begin{aligned} (T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{0,j} &= \textstyle\sum_{k=0}^{n-1} T_{0,k}U_{k,j} \\ &= \textstyle\sum_{k=0}^{j} \left\{ \left(\sum_{l=0}^{k} (-1)^{l+k} \binom{k}{l} \beta_l \right) \binom{j}{k} \right\} \\ &= \textstyle\sum_{i=0}^{j} \beta_i \left\{ \sum_{t=0}^{j-i} (-1)^t \binom{j}{t+i} \binom{t+i}{i} \right\} \\ &= \beta_j, \quad \text{(by Lemma 2.2)} \end{aligned}$$

which implies that $\mathrm{R}_0(T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n)) = \mathrm{R}_0(Q(n)) = (\beta_0,\ \beta_1,\ \ldots,\ \beta_{n-1})$. If $j = 0$, then we obtain

$$(T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{i,0} = \sum_{k=0}^{n-1} T_{i,k}U_{k,1} = T_{i,0}U_{0,0} = \hat{\alpha}_i,$$

and so $\mathrm{C}_0(T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n)) = \mathrm{C}_0(Q(n)) = (\hat{\alpha}_0,\ \hat{\alpha}_1,\ \ldots,\ \hat{\alpha}_{n-1})^t$. Finally, we assume that $1 \leq i, j < n-1$ and establish (3.3). Indeed, by calculations we observe that

$$\begin{aligned} (T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{i,j} &= \textstyle\sum_{k=0}^{n-1} T_{i,k}U_{k,j} \\ &= T_{i,0}U_{0,j} + \textstyle\sum_{k=1}^{n-1} T_{i,k}U_{k,j} \\ &= T_{i,0}U_{0,j} + \textstyle\sum_{k=1}^{n-1} T_{i,k}(U_{k-1,j-1} + U_{k,j-1}) \quad \text{(by (3.6))} \\ &= T_{i,0}U_{0,j} + \textstyle\sum_{k=1}^{n-1} T_{i,k}U_{k-1,j-1} + \sum_{k=1}^{n-1} T_{i,k}U_{k,j-1} \\ &= T_{i,0}U_{0,j} + \textstyle\sum_{k=1}^{n-1} T_{i-1,k-1}U_{k-1,j-1} + \sum_{k=0}^{n-1} T_{i,k}U_{k,j-1} \\ &\quad -T_{i,0}U_{0,j-1} \quad \text{(note that, } T_{i,k} = T_{i-1,k-1}) \\ &= T_{i,0}(U_{0,j} - U_{0,j-1}) + \textstyle\sum_{k=0}^{n-1} T_{i-1,k}U_{k,j-1} \\ &\quad +(T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{i,j-1} \quad \text{(note that } U_{n-1,j-1} = 0) \\ &= (T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{i-1,j-1} + (T_{\hat{\alpha},\hat{\beta}}(n) \cdot U(n))_{i,j-1}, \end{aligned}$$

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which is (3.3). The proof of (3.1) is now complete.

To prove of (3.2), we observe that

$$\begin{aligned} L(n)^{-1} \cdot P_{\check{\alpha},\check{\beta}}(n) \cdot U(n)^{-1} &= L(n)^{-1} \cdot L(n) \cdot T_{\hat{\check{\alpha}},\hat{\check{\beta}}}(n) \cdot U(n) \cdot U(n)^{-1} \quad \text{(by (3.1))} \\ &= T_{\hat{\check{\alpha}},\hat{\check{\beta}}}(n) \\ &= T_{\alpha,\beta}(n). \quad \text{(by Lemma 2.1)} \end{aligned}$$

The proof is complete. ∎

4. Some Applications. Let $A = (a_{i,j})_{i,j\geq 0}$ be an infinite matrix and let D_n be the nth leading principal minor of A consisting of the entries in its first n rows and columns. We will mainly be interested in the sequence of leading principal minors $(D_1, D_2, D_3, \ldots)$, especially, in the case that it forms a Fibonacci or Lucas (sub)sequence.

4.1. Generalized Pascal Triangle Associated With an Arithmetic or Geometric Sequence . It is of interest to evaluate the determinant of generalized Pascal triangle $P_{\alpha,\beta}(n)$, where one of the sequences α or β is an arithmetic or geometric sequence.

Corollary 4.1. *Let $\alpha = (\alpha_i)_{i\geq 0}$ be an arithmetic sequence with $\alpha_i = a + id$, and let $\beta = (\beta_i)_{i\geq 0}$ be an arbitrary sequence with $\beta_0 = a$. We set $D_n = \det(P_{\alpha,\beta}(n))$. Then, we have*

$$D_n = \sum_{k=0}^{n-1} (-d)^k \hat{\beta}_k D_{n-k-1},$$

with $D_0 = 1$.

Proof. By Theorem 3.1, we deduce that $D_n = \det(T_{\hat{\alpha},\hat{\beta}}(n))$, where $\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0}$ with

$$\begin{aligned} \hat{\alpha}_i &= \textstyle\sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \alpha_k \\ &= \textstyle\sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \big(\beta_0 + kd\big) \\ &= \beta_0 \textstyle\sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} + d \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} k \\ &= \begin{cases} \beta_0 & \text{if} \quad i = 0, \\ d & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i > 1. \end{cases} \end{aligned}$$

Now, expanding through the first row of $T_{\hat{\alpha},\hat{\beta}}(n)$, we obtain the result. ∎

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Corollary 4.2. *Let $\alpha = (\alpha_i)_{i\geq 0}$ be an arithmetic sequence with $\alpha_i = a + id$, and let $\beta = (\beta_i)_{i\geq 0}$ be an alternating sequence with $\beta_i = (-1)^i a$. Then, we have*

$$\det(P_{\alpha,\beta}(n)) = a(2d + a)^{n-1}.$$

Proof. Let $D_n = \det(P_{\alpha,\beta}(n))$. By Corollary 4.1, we have

$$D_n = \sum_{k=0}^{n-1}(-d)^k \hat{\beta}_k D_{n-k-1} \qquad (n \geq 1),$$

with $D_0 = 1$. An easy calculation shows that

$$\begin{aligned} \hat{\beta}_k &= \textstyle\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}\beta_l \\ &= \textstyle\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}(-1)^l a \\ &= a(-1)^k \textstyle\sum_{l=0}^{k}\binom{k}{l} \\ &= a(-2)^k. \end{aligned}$$

Hence, we have

$$D_n = a\sum_{k=0}^{n-1}(2d)^k D_{n-k-1}.$$

Now, replacing n by $n-1$, we obtain

$$D_{n-1} = a\sum_{k=0}^{n-2}(2d)^{k-1} D_{n-k-2},$$

and by calculation it follows that

$$\begin{aligned} D_n - 2dD_{n-1} &= a\Big(\textstyle\sum_{k=0}^{n-1}(2d)^k D_{n-k-1} - \sum_{k=0}^{n-2}(2d)^{k-1} D_{n-k-2}\Big) \\ &= a\Big(\textstyle\sum_{k=0}^{n-1}(2d)^k D_{n-k-1} - \sum_{k=1}^{n-1}(2d)^k D_{n-k-1}\Big) \\ &= aD_{n-1}, \end{aligned}$$

or equivalently

$$D_n = (2d + a)D_{n-1}.$$

But, this implies that $D_n = a(2d + a)^{n-1}$. ∎

Corollary 4.3. *Let $\alpha = (\alpha_i)_{i\geq 0}$ be an arithmetic sequence with $\alpha_i = id$, and let $\beta = (\beta_i)_{i\geq 0}$ be the square sequence, i.e., $\beta_i = i^2$. If $D_n = \det(P_{\alpha,\beta}(n))$, then we have*

$$D_n = -dD_{n-2} + 2d^2 D_{n-3}. \tag{4.1}$$

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Proof. By Corollary 4.1, we get

$$D_n = \sum_{k=0}^{n-1} (-d)^k \hat{\beta}_k D_{n-k-1}.$$

We claim that $\hat{\beta}_0 = 0$, $\hat{\beta}_1 = 1$, $\hat{\beta}_2 = 2$ and $\hat{\beta}_k = 0$ for $k \geq 3$. Now, it is obvious that our claim implies the validity of (4.1).

Clearly $\hat{\beta}_0 = 0$, $\hat{\beta}_1 = 1$ and $\hat{\beta}_2 = 2$. Now, we assume that $k \geq 3$. In this case we have

$$\hat{\beta}_k = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} \beta_j = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} j^2.$$

We define the functions f and g as follows:

$$f(x) = (1-x)^k = \sum_{j=0}^{k} \binom{k}{j} (-x)^j \quad \text{and} \quad g(x) = -xf'(x).$$

Now, an easy calculation shows that

$$g'(x) = (1-x)^{k-2} \cdot (*) = \sum_{j=0}^{k} \binom{k}{j} j^2 (-x)^{j-1},$$

and putting $x = 1$, we get

$$\sum_{j=0}^{k} \binom{k}{j} j^2 (-1)^{j-1} = 0.$$

Now, by multiplying both sides by $(-1)^{k+1}$, we obtain

$$\sum_{j=0}^{k} \binom{k}{j} j^2 (-1)^{k+j} = 0,$$

as claimed. □

Corollary 4.4. *Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_j)_{j\geq 0}$ be two geometric sequences with $\alpha_i = \rho^i$ and $\beta_j = \sigma^j$. Then, we have*

$$\det(T_{\alpha,\beta}(n)) = (1-\rho\sigma)^{n-1}.$$

Proof. By Theorem 3.1, we have $\det(T_{\alpha,\beta}(n)) = \det(P_{\breve{\alpha},\breve{\beta}}(n))$. On the other hand, straightforward computations show that $\breve{\alpha} = (\breve{\alpha}_i)_{i\geq 0}$ with $\breve{\alpha}_i = (1+\rho)^i$ and similarly $\breve{\beta} = (\breve{\beta}_j)_{j\geq 0}$ with $\breve{\beta}_j = (1+\sigma)^j$. By applying Lemma 2.3, we conclude the assertion. □

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4.2. Certain Generalized Pascal Triangles.

Proposition 4.5. *Let $a, b, c \in \mathbb{C}$ and let n be a positive integer. Let $\alpha = (\alpha_i)_{i \geq 0}$ and $\beta = (\beta_j)_{j \geq 0}$ be two sequences with $\alpha_i = (2^i - 1)a + c$ and $\beta_j = (2^j - 1)b + c$. Then, we have*

$$\det(P_{\alpha,\beta}(n)) = \begin{cases} \lfloor \frac{1}{n} \rfloor c & \text{if} \quad a = b = c, \\ \big[c + a(n-1)\big](c-a)^{n-1} & \text{if} \quad a = b \neq c, \\ \frac{b}{b-a}(c-a)^n + \frac{a}{b-a}(c-b)^n & \text{if} \quad a \neq b. \end{cases}$$

Proof. By Theorem 3.1, we have $\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n))$. A straightforward computation shows that

$$\hat{\alpha} = (c, a, a, a, \ldots) \quad \text{and} \quad \hat{\beta} = (c, b, b, b, \ldots).$$

Therefore, in the notation of [7], we have $T_{\hat{\alpha},\hat{\beta}}(n) = M_n(b, a, c)$, and since $M_n(a, b, c) = M_n(b, a, c)^t$ we have

$$\det(T_{\hat{\alpha},\hat{\beta}}(n)) = \det(M_n(a, b, c)).$$

Now, the proof follows the lines in the proof of Theorem 2 in [7]. □

Proposition 4.6. *Let $a, b, c \in \mathbb{C}$ and let n be a positive integer. Let $\alpha = (\alpha_i)_{i \geq 0}$ and $\beta = (\beta_j)_{j \geq 0}$ be two sequences with $\alpha_i = 2^{i-1}(ia + 2c)$ and $\beta_j = 2^{j-1}(jb + 2c)$. Then, we have*

$$\det(P_{\alpha,\beta}(n)) = (-1)^{n+1}(a+b)^{n-2}\big[c(a+b) + (n-1)ab\big].$$

Proof. Again from Theorem 3.1, we have $\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n))$, where $\hat{\alpha}$ and $\hat{\beta}$ are two arithmetic sequences as

$$\hat{\alpha} = (\hat{\alpha}_i)_{i \geq 0} = (c, c + a, c + 2a, \ldots, c + ia, \ldots)$$

and

$$\hat{\beta} = (\hat{\beta}_j)_{j \geq 0} = (c, c + b, c + 2b, \ldots, c + jb, \ldots).$$

Now we compute the determinant of $T_{\hat{\alpha},\hat{\beta}}(n)$. To do this, we apply the following elementary column operations:

$$\mathrm{C}_j \longrightarrow \mathrm{C}_j - \mathrm{C}_{j-1}, \qquad j = n-1, n-2, \ldots, 2;$$

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and we obtain the following quasi-Toeplitz matrix:

$$\left(\begin{array}{c|cccc} c & b & b & \ldots & b \\ \hline c+a & & & & \\ c+2a & & & & \\ \vdots & & & T_{\lambda,\mu}(n-1) & \\ c+(n-1)a & & & & \end{array}\right),$$

where $\lambda = (-a, -a, -a, \ldots)$ and $\mu = (-a, b, b, \ldots)$. Again, we subtract column j from column $j+1$, $j = n-2, n-3, \ldots, 2$. It is easy to see that, step by step, the rows and columns are "emptied" until finally the determinant

$$\det(T_{\hat{\alpha},\hat{\beta}}(n)) = \det \left(\begin{array}{cc|ccc} c & b & 0 & \ldots & 0 \\ \hline c+a & -a & & & \\ c+2a & -a & & (a+b)I_{(n-2)\times(n-2)} & \\ \vdots & \vdots & & & \\ c+(n-2)a & -a & & & \\ \hline c+(n-1)a & -a & 0 & \ldots & 0 \end{array}\right),$$

is obtained. The proposition follows now immediately, by expanding the determinant along the last row. □

4.3. Fibonacci and Lucas Numbers as Leading Principal Minors of a Quasi-Pascal Matrix . There are several infinite matrices for which the leading principal minors form a Fibonacci or Lucas (sub)sequence. For instance, in [10], we have presented a family of tridiagonal matrices with the following form:

$$F_\lambda(\infty) = \begin{pmatrix} 1 & \lambda_0 & 0 & 0 & 0 & \cdots \\ -\lambda_0^{-1} & 1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & -\lambda_1^{-1} & 1 & \lambda_2 & 0 & \cdots \\ 0 & 0 & -\lambda_2^{-1} & 1 & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{pmatrix} \tag{4.2}$$

where $\lambda = (\lambda_i)_{i\geq 0}$ with $\lambda_i \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Indeed, the leading principal minors of these matrices for every λ form the sequence $\left(\mathcal{F}_{n+1}\right)_{n\geq 1}$ (Theorem 1 in [10]). Also, for the special cases $\lambda_0 = \lambda_1 = \ldots \in \{1, \sqrt{-1}\}$, see [3, 4] and [11]. In ([12], pp. 555–557), Strang presents the infinite tridiagonal (Toeplitz) matrices:

$$P = T_{(3,t,0,0,\ldots),(3,t,0,0,\ldots)}(\infty), \tag{4.3}$$

where $t = \pm 1$, and it is easy to show that the leading principal minors of T form the subsequence $\left(\mathcal{F}_{2n+2}\right)_{n\geq 1}$ from the Fibonacci sequence. As another example, the

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leading principal minors of the Toeplitz matrices:

$$Q = T_{(2,1,1,1,\ldots),(2,t,0,0,\ldots)}(\infty), \tag{4.4}$$

where $t = \pm 1$, form the sequence $\left(\mathcal{F}_{n+2}\right)_{n\geq 1}$ for $t = 1$ and the sequence $\left(\mathcal{F}_{2n+1}\right)_{n\geq 1}$ for $t = -1$ ([3], Examples 1, 2).

We can summarize the above results in the following proposition.

Proposition 4.7. ([3, 4, 11, 12]) *Let n be a natural number, $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences, and let D_n be the nth leading principal minor of $T_{\alpha,\beta}(\infty)$. Then, the following hold.*

(1) *If $\alpha = \beta = (1, \sqrt{-1}, 0, 0, \ldots)$, then $D_n = \mathcal{F}_{n+1}$.*

(2) *If $\alpha = \beta = (3, t, 0, 0, 0, \ldots)$ where $t = \pm 1$, then $D_n = \mathcal{F}_{2n+2}$.*

(3) *If $\alpha = (1, -1, 0, 0, \ldots)$ and $\beta = (1, 1, 0, 0, \ldots)$, then $D_n = \mathcal{F}_{n+1}$.*

(4) *If $\alpha = (2, 1, 1, 1, \ldots)$ and $\beta = (2, -1, 0, 0, \ldots)$, then $D_n = \mathcal{F}_{2n+1}$.*

(5) *If $\alpha = (2, 1, 1, 1, \ldots)$ and $\beta = (2, 1, 0, 0, \ldots)$, then $D_n = \mathcal{F}_{n+2}$.*

Let $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio, and $\Phi = \frac{1-\sqrt{5}}{2}$, the golden ratio conjugate. The recent article of Griffin, Stuart and Tsatsomeros [7] gives the following result:

Proposition 4.8. ([7], Lemma 7) *For each positive integer n, let*

$$P(n) = T_{(1,\Phi,\Phi,\ldots),(1,\phi,\phi,\ldots)}(n), \quad \text{and} \quad Q(n) = T_{(0,-\Phi,-\Phi,\ldots),(0,-\phi,-\phi,\ldots)}(n).$$

Then, we have

$$\det(P(n)) = \mathcal{F}_{n+1}, \quad \text{and} \quad \det(Q(n)) = \mathcal{F}_{n-1}.$$

Using Propositions 4.7, 4.8 and Theorem 3.1, we immediately deduce the following corollary.

Corollary 4.9. *Let n be a natural number, $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences, and let D_n be the nth leading principal minor of $P_{\alpha,\beta}(\infty)$. Then, the following hold.*

(1) *If $\alpha_i = \beta_i = 1 + i\sqrt{-1}$, then $D_n = \mathcal{F}_{n+1}$.*

(2) *If $\alpha_i = \beta_i = 3 - i$, then $D_n = \mathcal{F}_{2n+2}$.*

(3) *If $\alpha_i = \beta_i = 3 + i$, then $D_n = \mathcal{F}_{2n+2}$.*

(4) *If $\alpha_i = 1 - i$ and $\beta_i = 1 + i$, then $D_n = \mathcal{F}_{n+1}$.*

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(5) *If* $\alpha_i = 2^i + 1$ *and* $\beta_i = 2 - i$, *then* $D_n = \mathcal{F}_{2n+1}$.

(6) *If* $\alpha_i = 2^i + 1$ *and* $\beta_i = 2 + i$, *then* $D_n = \mathcal{F}_{n+2}$.

(7) *If* $\alpha_i = (2^i - 1)\Phi + 1$ *and* $\beta_i = (2^i - 1)\phi + 1$, *then* $D_n = \mathcal{F}_{n+1}$.

(8) *If* $\alpha_i = (1 - 2^i)\Phi$ *and* $\beta_i = (1 - 2^i)\phi$, *then* $D_n = \mathcal{F}_{n-1}$.

In the sequel, we study together the sequences $\mathcal{F}$ and $\mathcal{L}$, and, in order to unify our treatment, we introduce the following useful notations. For $\varepsilon \in \{+, -\}$ we let $\mathcal{F}_n^\varepsilon = \mathcal{F}_n$ if $\varepsilon = +$; and $\mathcal{F}_n^\varepsilon = \mathcal{L}_n$ if $\varepsilon = -$.

Theorem 4.10. *Let r be a non-negative integer and s be a positive integer. Suppose that*

$$\phi_{r,s} = \left\lceil \frac{\mathcal{F}^\varepsilon_{2r+s}}{\mathcal{F}^\varepsilon_{r+s}} \right\rceil \quad \textit{and} \quad \psi_{r,s} = \sqrt{\phi_{r,s}\mathcal{F}^\varepsilon_{r+s} - \mathcal{F}^\varepsilon_{2r+s}}.$$

Then, the leading principal minors of the following infinite quasi-Pascal matrix:

$$P^{[r,s]}(\infty) = \left[\begin{array}{cc|ccc} \mathcal{F}^\varepsilon_{r+s} & \psi_{r,s} & 0 & 0 & \cdots \\ \psi_{r,s} & \phi_{r,s} & \sqrt{(-1)^r} & \sqrt{(-1)^r} & \cdots \\ \hline 0 & \sqrt{(-1)^r} & & & \\ 0 & \sqrt{(-1)^r} & & P_{\alpha,\alpha}(\infty) & \\ \vdots & \vdots & & & \end{array}\right]$$

where $\alpha = (\alpha_i)_{i\geq 0}$ is an arithmetic sequence with $\alpha_i = \mathcal{F}^-_r + i\sqrt{(-1)^r}$, form the subsequence $\{\mathcal{F}^\varepsilon_{nr+s}\}_{n=1}^\infty$ from Fibonacci or Lucas sequences.

Proof. Cahill and Narayan in [5] introduce the following quasi-Toeplitz matrices:

$$T^{[r,s]}(\infty) = \left[\begin{array}{cc|ccc} \mathcal{F}^\varepsilon_{r+s} & \psi_{r,s} & 0 & 0 & \cdots \\ \psi_{r,s} & \phi_{r,s} & \sqrt{(-1)^r} & 0 & \cdots \\ \hline 0 & \sqrt{(-1)^r} & & & \\ 0 & 0 & & T_{\beta,\beta}(\infty) & \\ \vdots & \vdots & & & \end{array}\right]$$

where $\beta = (\mathcal{F}^-_r, \sqrt{(-1)^r}, 0, 0, \ldots)$. Moreover, they show that

$$\det(T^{[r,s]}(n)) = \mathcal{F}^\varepsilon_{nr+s}. \tag{4.5}$$

Now, we decompose the matrix $T^{[r,s]}(n)$ as follows:

$$T^{[r,s]}(n) = \tilde{L}(n)P^{[r,s]}(n)\tilde{L}(n)^t, \tag{4.6}$$

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where

$$\tilde{L}(n) = I_{2\times 2} \oplus L^{-1}(n-2).$$

The proof of (4.6) is similar to the proof of Theorem 3.1 and we omit it here. Now, using (4.5) and (4.6), we easily see that

$$\det(P^{[r,s]}(n)) = \mathcal{F}^{\varepsilon}_{nr+s},$$

and the proof of theorem is complete. □

4.4. Generalized Pascal Triangle Associated With a Constant Sequence. In this subsection, we study the generalized Pascal triangle $P_{\alpha,\beta}(n)$, where one of the sequences α or β is a constant sequence. The following result is another consequence of Theorem 3.1.

COROLLARY 4.11. *Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two sequences with $\alpha_0 = \beta_0 = \gamma$. If α or β is a constant sequence, then we have $\det(P_{\alpha,\beta}(n)) = \gamma^n$.*

Proof. By Theorem 3.1, we have $\det(P_{\alpha,\beta}(n)) = \det(T_{\hat{\alpha},\hat{\beta}}(n))$. But in both cases, the Toeplitz matrix $T_{\hat{\alpha},\hat{\beta}}(n)$ is a lower triangular matrix or an upper triangular one with γ on its diagonal. This implies the corollary. □

The generalized Pascal triangle $P_{\alpha,\alpha}(\infty)$ associated with the pair of identical sequences α and α, is called the *generalized symmetric Pascal triangle* associated with α and yields symmetric matrices $P_{\alpha,\alpha}(n)$ by considering principal submatrices consisting of the first n rows and columns of $P_{\alpha,\alpha}(\infty)$. For an arbitrary sequence $\alpha = (\alpha_i)_{i\geq 0}$ with $\alpha_0 = 0$, we define $\tilde{\alpha} = (\tilde{\alpha}_i)_{i\geq 0}$ where $\tilde{\alpha}_i = (-1)^i\alpha_i$ for all i. Then, the generalized Pascal triangle $P_{\alpha,\tilde{\alpha}}(\infty)$ associated with the sequences α and $\tilde{\alpha}$, is called the *generalized skymmetric Pascal triangle* associated with α and $\tilde{\alpha}$, and yields skymmetric matrices $P_{\alpha,\tilde{\alpha}}(n)$ by considering principal submatrices consisting of the first n rows and columns of $P_{\alpha,\tilde{\alpha}}(\infty)$.

EXAMPLE 4.12. Let $n \geq 2$ be a natural number. Then, we have:

(i) The generalized symmetric Pascal triangle $P_{\mathcal{F},\mathcal{F}}(n)$ has determinant -2^{n-2}.

(ii) The generalized skymmetric Pascal triangle $P_{\mathcal{F},\tilde{\mathcal{F}}}(n)$ has determinant 2^{n-2}.

All assertions in this example follow from Theorem 3.1 of [1]. However, we will reprove them independently.

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Proof. (i) Consider the generalized symmetric Pascal triangle

$$P_{\mathcal{F},\mathcal{F}}(n) = \begin{pmatrix} 0 & 1 & 1 & 2 & 3 & \dots & \mathcal{F}_{n-1} \\ 1 & 2 & 3 & 5 & 8 & \dots & \mathcal{F}_{n+1} \\ 1 & 3 & 6 & 11 & 19 & \dots & . \\ 2 & 5 & 11 & 22 & 41 & \dots & . \\ 3 & 8 & 19 & 41 & 82 & \dots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & . \\ \mathcal{F}_{n-1} & \mathcal{F}_{n+1} & . & . & . & \dots & . \end{pmatrix}$$

Now, we apply the following elementary row operations:

$$\mathrm{R}_i \longrightarrow \mathrm{R}_i - \mathrm{R}_{i-1} - \mathrm{R}_{i-2}, \qquad i = n-1, n-2, \dots, 2.$$

It is easy to see that

$$\det\big(P_{\mathcal{F},\mathcal{F}}(n)\big) = \det \left(\begin{array}{cc|ccccc} 0 & 1 & 1 & 2 & 3 & \dots & \mathcal{F}_{n-1} \\ 1 & 2 & 3 & 5 & 8 & \dots & \mathcal{F}_{n+1} \\ \hline 0 & 0 & 2 & 4 & 8 & \dots & 2(\mathcal{F}_n - 1) \\ 0 & 0 & 2 & 6 & 14 & \dots & * \\ 0 & 0 & 2 & 8 & 22 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 2 & * & * & \dots & * \end{array}\right)$$

$$= \det \left(\begin{array}{cc|ccccc} 0 & 1 & 1 & 2 & 3 & \dots & \mathcal{F}_{n-1} \\ 1 & 2 & 3 & 5 & 8 & \dots & \mathcal{F}_{n+1} \\ \hline 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & P_{\lambda,\mu}(n-2) & & \\ 0 & 0 & & & & & \end{array}\right)$$

where $\lambda = (2, 2, 2, \dots)$ and $\mu = (2(\mathcal{F}_3 - 1), 2(\mathcal{F}_4 - 1), 2(\mathcal{F}_5 - 1), 2(\mathcal{F}_6 - 1), \dots)$. Now, by Corollary 4.11, we get

$$\det(P_{\mathcal{F},\mathcal{F}}(n)) = \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \cdot \det\big(P_{\lambda,\mu}(n-2)\big) = -2^{n-2},$$

as desired.

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(ii) Here, we consider the generalized skymmetric Pascal triangle

$$P_{\mathcal{F},\tilde{\mathcal{F}}}(n) = \begin{pmatrix} 0 & -1 & 1 & -2 & 3 & \dots & (-1)^{n-1}\mathcal{F}_{n-1} \\ 1 & 0 & 1 & -1 & 2 & \dots & . \\ 1 & 1 & 2 & 1 & 3 & \dots & . \\ 2 & 3 & 5 & 6 & 9 & \dots & . \\ 3 & 6 & 11 & 17 & 26 & \dots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & . \\ \mathcal{F}_{n-1} & . & . & . & . & \dots & . \end{pmatrix}$$

Similarly, we apply the following elementary column operations:

$$\mathrm{C}_j \longrightarrow \mathrm{C}_j + \mathrm{C}_{j-1} - \mathrm{C}_{j-2}, \qquad j = n-1, n-2, \dots, 2;$$

and we obtain

$$\det(P_{\mathcal{F},\tilde{\mathcal{F}}}(n)) \quad = \quad \det \left(\begin{array}{cc|ccccc} 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \hline 1 & 1 & 2 & 2 & 2 & \dots & 2 \\ 2 & 3 & 6 & 8 & 10 & \dots & * \\ 3 & 6 & 14 & 22 & 32 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & * \\ \mathcal{F}_{n-1} & \mathcal{F}_{n+1}-2 & * & * & * & \dots & * \end{array}\right)$$

$$= \quad \det \left(\begin{array}{cc|cccccc} 0 & -1 & 0 & 0 & 0 & & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & & \dots & 0 \\ \hline 1 & 1 & & & & & & \\ 2 & 3 & & & & & & \\ \vdots & \vdots & & & & P_{\nu,\lambda}(n-2) & & \\ \mathcal{F}_{n-1} & \mathcal{F}_{n+1}-2 & & & & & & \end{array}\right)$$

where $\lambda = (2, 2, 2, \dots)$. Again, by Corollary 4.11, we get

$$\det(P_{\mathcal{F},\tilde{\mathcal{F}}}(n)) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \det\left(P_{\nu,\lambda}(n-2)\right) = 2^{n-2},$$

as desired. $\square$

EXAMPLE 4.13. Let $n \geq 2$ be a natural number. Then, the generalized Pascal triangle $P_{\mathcal{F}^*,\mathcal{I}^*}(n)$ has determinant $(-1)^n$.

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Proof. Consider the following generalized Pascal triangle:

$$P_{\mathcal{F}^*,\mathcal{I}^*}(n)=\begin{pmatrix} 1 & 2 & 6 & 24 & 120 & \dots & n! \\ 1 & 3 & 9 & 33 & 153 & \dots & . \\ 2 & 5 & 14 & 47 & 200 & \dots & . \\ 3 & 8 & 22 & 69 & 269 & \dots & . \\ 5 & 13 & 35 & 104 & 373 & \dots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & . \\ \mathcal{F}_{n-1} & \mathcal{F}_{n+1} & . & . & . & \dots & . \end{pmatrix}$$

Again, we use the similar elementary row operations as Example 1(i):

$$\mathrm{R}_i \longrightarrow \mathrm{R}_i - \mathrm{R}_{i-1} - \mathrm{R}_{i-2}, \qquad i = n-1, n-2, \dots, 2.$$

Therefore, we deduce that

$$\det\left(P_{\mathcal{F}^*,\mathcal{I}^*}(n)\right) = \det\left(\begin{array}{cc|ccccc} 1 & 2 & 6 & 24 & 120 & \dots & n! \\ 1 & 3 & 9 & 33 & 153 & \dots & * \\ \hline 0 & 0 & -1 & -10 & -73 & \dots & * \\ 0 & 0 & -1 & -11 & -84 & \dots & * \\ 0 & 0 & -1 & -12 & -96 & \dots & * \\ 0 & 0 & -1 & -13 & -109 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & -1 & * & * & \dots & * \end{array}\right)$$

$$= \det\left(\begin{array}{cc|ccccc} 1 & 2 & 6 & 24 & 120 & \dots & n! \\ 1 & 3 & 9 & 33 & 153 & \dots & * \\ \hline 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & P_{\lambda,\mu}(n-2) & & \\ 0 & 0 & & & & & \end{array}\right)$$

where $\lambda = (-1,-1,-1,\dots)$ and $\mu = (-1,-10,-73,\dots)$. Now, by Corollary 4.11, we get

$$\det\left(P_{\mathcal{F}^*,\mathcal{I}^*}(n)\right) = \det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \cdot \det\left(P_{\lambda,\mu}(n-2)\right) = (-1)^{n-2} = (-1)^n,$$

as desired. ∎

5. New Matrices Whose Leading Principal Minors Form the Fibonacci Sequence. In this section, we present two matrices whose entries are recursively

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defined, and we show that the leading principal minor sequence of these matrices is the sequence $\mathcal{F}^* = (\mathcal{F}_n)_{n\geq 1}$. It is worth mentioning that to construct these matrices we use non-homogeneous recurrence relations.

Theorem 5.1. *Let $a, b \in \mathbb{C}$ and let n be a natural number. Let $(r_{i,j})_{i,j\geq 0}$ be the doubly indexed sequence given by the recurrence*

$$r_{i,j} = r_{i,j-1} + r_{i-1,j} + ai + bj,$$

for $i, j \geq 1$, and the initial conditions $r_{i,0} = r_{0,j} = 1$, $i, j \geq 0$. Let

$$D_n = \det(r_{i,j})_{0\leq i,j\leq n-1}.$$

Then, D_n satisfies the following recursion:

$$\begin{cases} D_1 = 1, \\ D_2 = 1 + a + b, \\ D_n = (1 + a + b)D_{n-1} - abD_{n-2} \quad (n \geq 3). \end{cases} \tag{5.1}$$

In particular, we have $D_n = \mathcal{F}_n$ if and only if $(a, b) \in \{(1, -1), (-1, 1)\}$.

Proof. Let $R(n)$ denote the matrix $(r_{i,j})_{0\leq i,j\leq n-1}$. First, we claim that

$$R(n) = L(n) \cdot T(n) \cdot U(n),$$

where $U(n) = L(n)^t$ and

$$T(n) = \left[\begin{array}{c|cccccc} 1 & & & & & \\ \hline & \omega & b & & & \\ & a & \omega & b & & \\ & & a & \omega & \ddots & \\ & & & \ddots & \ddots & b \\ & & & & a & \omega \end{array}\right]_{n\times n},$$

where $\omega = 1 + a + b$. Now it is easy to see that $D_n = \det(T(n))$ and by expansion through the last row of $T(n)$, we obtain (5.1).

In what follows, for convenience, we will let $R = R(n)$, $L = L(n)$, $T = T(n)$ and $U = U(n)$. Now, for the proof of the claimed factorization we compute the (i, j)-entry of $L \cdot T \cdot U$, that is

$$(L \cdot T \cdot U)_{i,j} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}T_{r,s}U_{s,j}. \tag{5.2}$$

In fact, so as to prove the theorem, we should establish

$$\mathrm{R}_0(L \cdot T \cdot U) = \mathrm{R}_0(R) = (1, 1, \ldots, 1),$$

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$$\mathrm{C}_0(L\cdot T\cdot U)=\mathrm{C}_0(A)=(1,1,\ldots,1),$$

and

$$(L\cdot T\cdot U)_{i,j}=(L\cdot T\cdot U)_{i-1,j-1}+(L\cdot T\cdot U)_{i-1,j}+ai+bj, \tag{5.3}$$

for $1\leq i,j\leq n-1$.

Let us do the required calculations. First, suppose that $i=0$. Then, we have

$$(L\cdot T\cdot U)_{0,j}=\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}L_{0,r}T_{r,s}U_{s,j}=\sum_{s=0}^{n-1}T_{0,s}U_{s,j}=U_{0,j}=1,$$

and so $\mathrm{R}_0(L\cdot T\cdot U)=\mathrm{R}_0(R)=(1,1,\ldots,1)$.

Next, assume that $j=0$. In this case, we obtain

$$(L\cdot T\cdot U)_{i,0}=\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}L_{i,r}T_{r,s}U_{s,0}=\sum_{r=0}^{n-1}L_{i,r}T_{r,0}=L_{i,0}=1,$$

and hence we have $\mathrm{C}_0(L\cdot T\cdot U)=\mathrm{C}_0(R)=(1,1,\ldots,1)$.

Finally, we must establish (5.3). At the moment, let us assume that $1\leq i,j\leq n-1$. In this case we have

$$\begin{aligned}
(L\cdot T\cdot U)_{i,j} &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1}L_{i,r}T_{r,s}U_{s,j}\\
&= \sum_{r=0}^{n-1}L_{i,r}T_{r,0}U_{0,j}+\sum_{r=0}^{n-1}\sum_{s=1}^{n-1}L_{i,r}T_{r,s}U_{s,j}\\
&= \sum_{r=0}^{n-1}L_{i,r}T_{r,0}U_{0,j}+\sum_{r=0}^{n-1}\sum_{s=1}^{n-1}L_{i,r}T_{r,s}\big(U_{s-1,j-1}+U_{s,j-1}\big)\\
&\quad \big(\text{by (3.6)}\big)\\
&= \sum_{r=0}^{n-1}L_{i,r}T_{r,0}U_{0,j}+\sum_{r=0}^{n-1}\sum_{s=1}^{n-1}L_{i,r}T_{r,s}U_{s-1,j-1}\\
&\quad +\sum_{r=0}^{n-1}\sum_{s=1}^{n-1}L_{i,r}T_{r,s}U_{s,j-1}\\
&= \sum_{r=0}^{n-1}L_{i,r}T_{r,0}U_{0,j}+\sum_{r=1}^{n-1}\sum_{s=1}^{n-1}L_{i,r}T_{r,s}U_{s-1,j-1}\\
&\quad +\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}L_{i,r}T_{r,s}U_{s,j-1}\\
&\quad +\sum_{s=1}^{n-1}L_{i,0}T_{0,s}U_{s-1,j-1}-\sum_{r=0}^{n-1}L_{i,r}T_{r,0}U_{0,j-1}
\end{aligned}$$

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$$
\begin{aligned}
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\big(L_{i-1,r-1}+L_{i-1,r}\big)T_{r,s}U_{s-1,j-1} + (L\cdot T\cdot U)_{i,j-1} \\
& + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \quad \big(\text{by (3.4) and (5.2)}\big) \\
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}T_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s-1,j-1} \\
& + (L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \\
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}T_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} \\
& + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s-1,j-1} + (L\cdot T\cdot U)_{i,j-1} \\
& + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \\
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}T_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} \\
& + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}\big(U_{s,j}-U_{s,j-1}\big) \\
& + (L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \quad \big(\text{by (3.6)}\big) \\
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}T_{r-1,s-1}U_{s-1,j-1} \\
& + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} \\
& + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j} - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j-1} \\
& + (L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \\
& \big(\text{by the structure of T}\big) \\
= \; & \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} \\
& + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}T_{r,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}T_{r,0}U_{0,j} \\
& - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}T_{r,s}U_{s,j-1} + (L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} \\
& - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1}\big(\text{note that } L_{i-1,n-1}=U_{n-1,j-1}=0\big)
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}T_{r,s}U_{s,j} - \sum_{s=0}^{n-1} L_{i-1,0}T_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}T_{r,0}U_{0,j} \\
&\quad +(L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \\
&= \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} \\
&\quad +(L\cdot T\cdot U)_{i-1,j} - \sum_{s=0}^{n-1} L_{i-1,0}T_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}T_{r,0}U_{0,j} \\
&\quad +(L\cdot T\cdot U)_{i,j-1} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} \\
&\quad - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1} \quad \big(\text{by } (5.2)\big) \\
&= (L\cdot T\cdot U)_{i-1,j} + (L\cdot T\cdot U)_{i,j-1} + \Psi_{i,j},
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_{i,j} &= \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j} + \sum_{r=1}^{n-1} L_{i-1,r-1}T_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}T_{1,s}U_{s-1,j-1} \\
&\quad - \sum_{s=0}^{n-1} L_{i-1,0}T_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}T_{r,0}U_{0,j} + \sum_{s=1}^{n-1} L_{i,0}T_{0,s}U_{s-1,j-1} \\
&\quad - \sum_{r=0}^{n-1} L_{i,r}T_{r,0}U_{0,j-1}.
\end{aligned}
$$

But by an easy calculation one can show that

$$\Psi_{i,j} = ai + bj,$$

which implies the first part of theorem.

The second part of theorem is now obvious. □

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