Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 516-529, August 2009

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PROBLEMS OF CLASSIFYING ASSOCIATIVE OR LIE ALGEBRAS OVER A FIELD OF CHARACTERISTIC NOT TWO AND FINITE METABELIAN GROUPS ARE WILD*

GENRICH BELITSKII†, ANDRII R. DMYTRYSHYN‡, RUVIM LIPYANSKI†, VLADIMIR V. SERGEICHUK§, AND ARKADY TSURKOV†

Abstract. Let $\mathbb{F}$ be a field of characteristic different from 2. It is shown that the problems of classifying

(i) local commutative associative algebras over $\mathbb{F}$ with zero cube radical,
(ii) Lie algebras over $\mathbb{F}$ with central commutator subalgebra of dimension 3, and
(iii) finite p-groups of exponent p with central commutator subgroup of order p^3

are hopeless since each of them contains

- the problem of classifying symmetric bilinear mappings $U \times U \to V$, or
- the problem of classifying skew-symmetric bilinear mappings $U \times U \to V$,

in which U and V are vector spaces over $\mathbb{F}$ (consisting of p elements for p-groups (iii)) and V is 3-dimensional. The latter two problems are hopeless since they are wild; i.e., each of them contains the problem of classifying pairs of matrices over $\mathbb{F}$ up to similarity.

Key words. Wild problems, Classification, Associative algebras, Lie algebras, Metabelian groups.

AMS subject classifications. 15A21, 16G60.

1. Introduction. A classification problem is called *wild* if it contains the problem of classifying pairs of $n \times n$ matrices over a field under similarity transformations

$$(A, B) \mapsto (S^{-1}AS,\ S^{-1}BS), \quad S \text{ is nonsingular.}$$

The latter problem is considered as hopeless: it contains the problem of classifying *any* system of linear mappings; that is, representations of any quiver, see [3, 5].

We consider

*Received by the editors December 25, 2008. Accepted for publication August 7, 2009. Handling Editor: Robert Guralnick.

†Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel (genrich@cs.bgu.ac.il, lipyansk@cs.bgu.ac.il, tsurkov@bgu.ac.il).

‡Faculty of Mechanics and Mathematics, Kiev National Taras Shevchenko University, Kiev, Ukraine (andriidmytryshyn@ukr.net).

§Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine (sergeich@imath.kiev.ua). The research was done while this author was visiting the Ben-Gurion University of the Negev.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 516-529, August 2009

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(a) the problem of classifying symmetric bilinear mappings $U \times U \to V$, and
(b) the problem of classifying skew-symmetric bilinear mappings $U \times U \to V$,

in which U and V are vector spaces over a field $\mathbb{F}$ of characteristic different from 2 and V is three-dimensional.

In Section 3, we prove that the problems (a) and (b) contain the problem of classifying pairs of matrices over $\mathbb{F}$ up to similarity. In Sections 4 and 5, we show that

(i) the problem of classifying local commutative associative algebras over $\mathbb{F}$ with zero cube radical contains (a),
(ii) the problem of classifying Lie algebras over $\mathbb{F}$ with central commutator subalgebra of dimension 3 contains (b), and
(iii) the problem of classifying finite p-groups of exponent $p \neq 2$ with central commutator subgroup of order p^3 contains the problem (a) over the field $\mathbb{F}_p$ with p elements.

Therefore, the problems (a), (b), and (i)–(iii) are wild.

Note that the wildness of (a), (b), (i) and (ii) was proved in [2] if the field $\mathbb{F}$ is algebraically closed. The purpose of our paper is to remove this restriction on $\mathbb{F}$, which admits, in particular, to prove the wildness of (iii).

In Section 2, we give two preparation lemmas. One of them is about matrix triples up to congruence; its proof is based on the method of reducing the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings, which was developed in [11] and was presented in detail in [12, Section 3]. In Section 6, we recall this reduction, restricting ourselves to the problem of classifying triples of bilinear forms.

All fields that we consider are of characteristic not 2.

2. Two lemmas. In this section, we give two lemmas that we use in later sections.

In each matrix triple that we consider, the three matrices have the same size, which we call the *size of the triple*. Triples (A, B, C) and (A', B', C') are called *equivalent* if there exist nonsingular R and S such that

$$(RAS, RBS, RCS) = (A', B', C').$$

If $R = S^T$ then the triples are *congruent*.

Define the direct sum of matrix triples:

$$(A, B, C) \oplus (A', B', C') := (A \oplus A', B \oplus B', C \oplus C').$$

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We say that a triple is *indecomposable for equivalence* if it is not equivalent to a direct sum of two triples of matrices of smaller sizes. We also say that a triple $\mathcal{U}$ is a *direct summand of a triple $\mathcal{T}$ for equivalence* if $\mathcal{T}$ is equivalent to $\mathcal{U} \oplus \mathcal{V}$ for some $\mathcal{V}$.

Lemma 2.1. *Each matrix triple over a field is equivalent to a direct sum of triples of size not 0×0 that are indecomposable for equivalence. This sum is uniquely determined, up to permutation of summands and replacement of summands by equivalent triples.*

Proof. Each triple of $m \times n$ matrices over a field $\mathbb{F}$ gives a triple of linear mappings $\mathbb{F}^n \to \mathbb{F}^m$; that is, a representation of the quiver

$$\bullet \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \bullet$$

By the Krull–Schmidt theorem [8, Section 8.2], each representation of a quiver is isomorphic to a direct sum of indecomposable representations of nonzero size, and this sum is uniquely determined up to permutation of summands and replacement of summands by isomorphic representations. $\square$

Lemma 2.2. *Let $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{U}$, $\mathcal{U}'$ be triples of square matrices over a field $\mathbb{F}$ of characteristic not 2. Let $\mathcal{T} \oplus \mathcal{T}'$ and $\mathcal{U} \oplus \mathcal{U}'$ be congruent, $\mathcal{T}$ and $\mathcal{U}$ be equivalent, and let $\mathcal{T}$ and $\mathcal{T}'$ have no common summands for equivalence of size not 0×0. Then $\mathcal{T}$ and $\mathcal{U}$ are congruent, and $\mathcal{T}'$ and $\mathcal{U}'$ are congruent.*

This lemma is proved in Section 6.

3. The wildness of the problem of classifying bilinear mappings. In this section, we prove the following theorem.

Theorem 3.1. *Let $\mathbb{F}$ be a field of characteristic different from 2, let U and V be vector spaces over $\mathbb{F}$, and let V be three-dimensional. The problem of classifying symmetric (skew-symmetric) bilinear mappings $U \times U \to V$ whose image generates the target space V is wild.*

Note that the image of a bilinear mapping into a vector space need not be a subspace of its target space. It is far from clear which subsets of the target space may be such images. For vector spaces over the real numbers, a complete classification of the images of bilinear mappings into a three-dimensional vector space is given in [4].

Let $h : U \times U \to V$ be a bilinear mapping over $\mathbb{F}$, and let $\dim V = 3$. Choose bases $e_1, \dots, e_m$ in U and f_1, f_2, f_3 in V. Then there is a unique triple (M_1, M_2, M_3) of $m \times m$ matrices over $\mathbb{F}$ such that for all $x, y \in U$, we have

$$h(x, y) = [x]^T M_1 [y] f_1 + [x]^T M_2 [y] f_2 + [x]^T M_3 [y] f_3, \tag{3.1}$$

where $[x]$ and $[y]$ are the coordinate vectors.

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The mapping h is symmetric, i.e. $h(x,y)=h(y,x)$, if and only if the matrices M_1, M_2, M_3 are symmetric. The mapping h is skew-symmetric, i.e. $h(x,y)=-h(y,x)$ if and only if the matrices M_1, M_2, M_3 are skew-symmetric. We can reduce the triple by congruence transformations

$$(M_1, M_2, M_3) \mapsto (S^TM_1S,\ S^TM_2S,\ S^TM_3S), \quad S \text{ is nonsingular,} \tag{3.2}$$

changing the basis in U, and by linear substitutions

$$(M_1, M_2, M_3) \mapsto \Big(\sum_{j=1}^{3}\gamma_{1j}M_j, \sum_{j=1}^{3}\gamma_{2j}M_j, \sum_{j=1}^{3}\gamma_{3j}M_j\Big), \quad [\gamma_{ij}] \text{ is nonsingular,} \tag{3.3}$$

changing the basis in V.

The image of h generates the target space V if and only if the triple (M_1, M_2, M_3) is *linearly independent*; that is,

$$\alpha M_1 + \beta M_2 + \gamma M_3 = 0 \quad \Longrightarrow \quad \alpha = \beta = \gamma = 0. \tag{3.4}$$

Indeed, let us choose any nonzero entry a_{ij} in M_1 and make zero the (i,j) entries in M_2 and M_3 adding γM_1 by transformations (3.3). Then we choose any nonzero entry $b_{i'j'}$ in the obtained matrix M_2 and make zero the (i',j') entries in M_1 and M_3. Finally, we choose any nonzero entry $c_{i''j''}$ in the obtained matrix M_3 and make zero the (i'',j'') entries in M_1 and M_2. By (3.1),

$$h(e_i, e_j) = a_{ij}f_1, \quad h(e_{i'}, e_{j'}) = b_{i'j'}f_2, \quad h(e_{i''}, e_{j''}) = c_{i''j''}f_3,$$

in which f_1, f_2, f_3 is the obtained basis of V.

This admits to reformulate Theorem 3.1 in the following matrix form.

THEOREM 3.2. *The problem of classifying linearly independent triples of symmetric* (*skew-symmetric*) *matrices up to transformations* (3.2) *and* (3.3) *is wild.*

Proof. For a matrix triple (A,B,C) and a fixed $\varepsilon \in \{1,-1\}$, we write

$$\begin{matrix}(A,B,C)^T := (A^T, B^T, C^T),\\ (A,B,C)^{(\varepsilon)} := (A^{(\varepsilon)}, B^{(\varepsilon)}, C^{(\varepsilon)}),\end{matrix} \quad \text{where } X^{(\varepsilon)} := \begin{bmatrix} 0 & X \\ \varepsilon X^T & 0 \end{bmatrix}. \tag{3.5}$$

Let (A,B) be a pair of $n \times n$ matrices. Following [2], we define the triple of $350n$-by-$350n$ matrices

$$\begin{aligned}\mathcal{T}(A,B) := (I_n, I_n, I_n)^{(\varepsilon)} &\oplus (I_{100n}, 0_{100n}, 0_{100n})^{(\varepsilon)} \oplus (0_{50n}, I_{50n}, 0_{50n})^{(\varepsilon)} \\ &\oplus (0_{20n}, 0_{20n}, I_{20n})^{(\varepsilon)} \oplus (I_{4n},\ J_4(0_n),\ D(A,B))^{(\varepsilon)},\end{aligned} \tag{3.6}$$

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A publication of the International Linear Algebra Society
Volume 18, pp. 516-529, August 2009

in which

$$J_4(0_n) := \begin{bmatrix} 0_n & I_n & 0 & 0 \\ 0 & 0_n & I_n & 0 \\ 0 & 0 & 0_n & I_n \\ 0 & 0 & 0 & 0_n \end{bmatrix}, \qquad D(A,B) := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0_n \end{bmatrix},$$

and $\varepsilon = 1$ or -1 (respectively, the matrices of (3.6) are symmetric or skew-symmetric). The triple $\mathcal{T}(A,B)$ satisfies (3.4).

Let us prove that two pairs (A,B) and (C,D) of $n \times n$ matrices are similar if and only if $\mathcal{T}(A,B)$ reduces to $\mathcal{T}(C,D)$ by transformations (3.2) and (3.3). If (A,B) and (C,D) are similar, then the triples $\mathcal{T}(A,B)$ and $\mathcal{T}(C,D)$ are congruent since $S^{-1}(A,B)S = (C,D)$ implies

$$R^T(I_{4n},\ J_4(0_n),\ D(A,B))^{(\varepsilon)}R = (I_{4n},\ J_4(0_n),\ D(C,D))^{(\varepsilon)},$$

in which

$$R := (S^T)^{-1} \oplus (S^T)^{-1} \oplus (S^T)^{-1} \oplus (S^T)^{-1} \oplus S \oplus S \oplus S \oplus S.$$

In the remainder of this section, we prove the converse. Denote by M_1, M_2, and $M_3(A,B)$ the matrices of the triple $\mathcal{T}(A,B)$ and assume that $\mathcal{T}(A,B)$ reduces to $\mathcal{T}(C,D)$ by transformations (3.2) and (3.3). These transformations are independent; so we can produce the substitutions (3.3) reducing $\mathcal{T}(A,B) = (M_1, M_2, M_3(A,B))$ to

$$\mathcal{U}(A,B) := (\gamma_{i1}M_1 + \gamma_{i2}M_2 + \gamma_{i3}M_3(A,B))_{i=1}^3$$

in which $[\gamma_{ij}]$ is nonsingular, and then apply the remaining congruences (3.2) to $\mathcal{U}(A,B)$ and obtain $\mathcal{T}(C,D) = (M_1, M_2, M_3(C,D))$. The rank of each matrix of the triple $\mathcal{U}(A,B)$ is equal to the rank of the corresponding matrix of $\mathcal{T}(C,D)$ since the triples are congruent. This implies that $\gamma_{ij} = 0$ if $i \neq j$. Thus, $\alpha := \gamma_{11}$, $\beta := \gamma_{22}$, and $\gamma := \gamma_{33}$ are nonzero, and

$$\begin{aligned} \mathcal{U}(A,B) &= (\alpha M_1,\ \beta M_2,\ \gamma M_3(A,B)) \\ &= (\alpha I_n, \beta I_n, \gamma I_n)^{(\varepsilon)} \oplus (\alpha I_{100n}, 0_{100n}, 0_{100n})^{(\varepsilon)} \oplus (0_{50n}, \beta I_{50n}, 0_{50n})^{(\varepsilon)} \\ &\qquad \oplus (0_{20n}, 0_{20n}, \gamma I_{20n})^{(\varepsilon)} \oplus (\alpha I_{4n},\ \beta J_4(0_n),\ \gamma D(A,B))^{(\varepsilon)}. \end{aligned} \tag{3.7}$$

Write (3.6) and (3.7) in the form

$$\mathcal{T}(C,D) = (I_n, I_n, I_n)^{(\varepsilon)} \oplus \mathcal{T}', \tag{3.8}$$

$$\mathcal{U}(A,B) = (\alpha I_n, \beta I_n, \gamma I_n)^{(\varepsilon)} \oplus \mathcal{U}', \tag{3.9}$$

in which $\mathcal{T}'$ is (3.6) with (C,D) instead of (A,B) and without the first summand, and $\mathcal{U}'$ is (3.7) without the first summand.

The sums (3.8) and (3.9) satisfy the conditions of Lemma 2.2 since:

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- $\mathcal{T}(C,D)$ and $\mathcal{U}(A,B)$ are congruent.
- $(I_n,I_n,I_n)^{(\varepsilon)}$ and $\mathcal{T}'$ have no common summands for equivalence. Indeed, the pairs $(I_n,I_n)^{(\varepsilon)}$ and $\mathcal{T}_2'$ formed by the first and the second matrices of these triples have no common summands for equivalence: each indecomposable summand of $(I_n,I_n)^{(\varepsilon)}$ for equivalence is equivalent to (I_1,I_1), but the form of $\mathcal{T}_2'$ ensures that if any pair of *nonzero* matrices is an indecomposable summand of $\mathcal{T}_2'$ for equivalence, then this pair is equivalent to $(I_4,J_4(0))$.
- $(I_n,I_n,I_n)^{(\varepsilon)}$ and $(\alpha I_n,\beta I_n,\gamma I_n)^{(\varepsilon)}$ are equivalent. Indeed, $\mathcal{T}(C,D)$ and $\mathcal{U}(A,B)$ are equivalent, $(I_n,I_n,I_n)^{(\varepsilon)}$ and $\mathcal{U}'$ have no common summands for equivalence, hence by Lemma 2.1 $(I_n,I_n,I_n)^{(\varepsilon)}$ is a direct summand of $(\alpha I_n,\beta I_n,\gamma I_n)^{(\varepsilon)}$ for equivalence. In like manner, $(\alpha I_n,\beta I_n,\gamma I_n)^{(\varepsilon)}$ is itself a direct summand of $(I_n,I_n,I_n)^{(\varepsilon)}$ for equivalence since $(\alpha I_n,\beta I_n,\gamma I_n)^{(\varepsilon)}$ and $\mathcal{T}'$ have no common summands for equivalence.

By Lemma 2.2, $(I_n,I_n,I_n)^{(\varepsilon)}$ and $(\alpha I_n,\beta I_n,\gamma I_n)^{(\varepsilon)}$ are congruent; that is, there exists a nonsingular matrix S such that

$$S^T I_n^{(\varepsilon)} S = \alpha I_n^{(\varepsilon)} = \beta I_n^{(\varepsilon)} = \gamma I_n^{(\varepsilon)}.$$

Write $\delta := \det S$, then $\delta^2=\alpha=\beta=\gamma$. Thus, $\mathcal{U}(A,B)=\delta\mathcal{T}(A,B)\delta$ and $\mathcal{T}(A,B)$ are congruent. Since $\mathcal{U}(A,B)$ is congruent to $\mathcal{T}(C,D)$, we have that $\mathcal{T}(A,B)$ and $\mathcal{T}(C,D)$ are congruent. Write them in the form

$$\mathcal{T}(A,B)=\mathcal{T}\oplus\mathcal{D}(A,B)^{(\varepsilon)},\qquad \mathcal{T}(C,D)=\mathcal{T}\oplus\mathcal{D}(C,D)^{(\varepsilon)}, \tag{3.10}$$

in which $\mathcal{T}$ is the direct sum (3.6) without the last summand, and

$$\mathcal{D}(X,Y) := (I_{4n},\ J_4(0_n),\ D(X,Y)).$$

Since triples (3.10) are congruent, they are equivalent and Lemma 2.1 ensures that $\mathcal{D}(A,B)^{(\varepsilon)}$ is equivalent to $\mathcal{D}(C,D)^{(\varepsilon)}$. Hence, $\mathcal{D}(A,B)\oplus\mathcal{D}(A,B)^T$ is equivalent to $\mathcal{D}(C,D)\oplus\mathcal{D}(C,D)^T$.

The triple $\mathcal{D}(A,B)^T$ is equivalent to

$$\mathcal{D}'(A,B) := P\mathcal{D}(A,B)^T P = (I_{4n},\ J_4(0_n),\ D'(A,B)),$$

where

$$P := \begin{bmatrix} 0&0&0&I_n\\ 0&0&I_n&0\\ 0&I_n&0&0\\ I_n&0&0&0 \end{bmatrix},\qquad D'(A,B) := \begin{bmatrix} 0_n&0&0&0\\ 0&B^T&0&0\\ 0&0&A^T&0\\ 0&0&0&I_n \end{bmatrix}.$$

Therefore, $\mathcal{D}(A,B)\oplus\mathcal{D}'(A,B)$ is equivalent to $\mathcal{D}(C,D)\oplus\mathcal{D}'(C,D)$.

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A publication of the International Linear Algebra Society
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By [2, Lemma 3], $\mathcal{D}(A,B)$ and $\mathcal{D}'(C,D)$ have no common direct summands for equivalence. Analogously, $\mathcal{D}(C,D)$ and $\mathcal{D}'(A,B)$ have no common direct summands for equivalence. Lemma 2.1 ensures that $\mathcal{D}(A,B)$ is equivalent to $\mathcal{D}(C,D)$; that is, $\mathcal{D}(A,B)S = R\mathcal{D}(C,D)$ for some nonsingular R and S. Equating the corresponding matrices of these triples gives

$$S = R = \begin{bmatrix} S_0 & S_1 & S_2 & S_3 \\ 0 & S_0 & S_1 & S_2 \\ 0 & 0 & S_0 & S_1 \\ 0 & 0 & 0 & S_0 \end{bmatrix}, \qquad D(A,B)S = RD(C,D).$$

Hence, $(A,B)S_0 = S_0(C,D)$, and so (A,B) is similar to (C,D). ◻

4. The wildness of the problems of classifying associative and Lie algebras. In this section, we prove the following theorem.

THEOREM 4.1. *Let $\mathbb{F}$ be a field of characteristic different from 2.*

(a) *The problem of classifying local commutative algebras Λ over $\mathbb{F}$ for which*

$$(\mathrm{Rad}\,\Lambda)^3 = 0 \quad \textit{and} \quad \dim(\mathrm{Rad}\,\Lambda)^2 = 3$$

is wild.

(b) *The problem of classifying Lie algebras over $\mathbb{F}$ with central commutator subalgebra of dimension 3 is wild.*

We follow the proof of Theorem 4 in [2], in which $\mathbb{F}$ is algebraically closed.

By a *semialgebra* we mean a finite-dimensional vector space R over $\mathbb{F}$ with multiplication $ab := h(a,b)$ given by a mapping

$$h : R \times R \to R \tag{4.1}$$

that is bilinear, i.e.

$$(\alpha a + \beta b)c = \alpha(ac) + \beta(bc), \qquad a(\alpha b + \beta c) = \alpha(ab) + \beta(ac)$$

for all $\alpha, \beta \in \mathbb{F}$ and all $a, b, c \in R$. A semialgebra R is *commutative* or *anti-commutative* if $ab = ba$ or, respectively, $ab = -ba$ for all $a, b \in R$. Denote by R^2 and R^3 the vector spaces spanned by all ab and, respectively, by all $(ab)c$ and $a(bc)$, in which $a, b, c \in R$.

LEMMA 4.2. *The problem of classifying commutative* (*anti-commutative*) *semialgebras R with $R^3 = 0$ and $\dim R^2 = 3$ is wild.*

Proof. Let R be a semialgebra with $R^3 = 0$ and $\dim R^2 = 3$. The multiplication on R is defined by the bilinear mapping (4.1). Since $R^3 = 0$, we have $ar = ra = 0$ for

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A publication of the International Linear Algebra Society
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all $a \in R$ and $r \in R^2$. Hence, $h(a+R^2, b+R^2) = h(a,b)$ for all $a,b \in R$, and so h induces (and is determined by) the bilinear mapping

$$\bar{h} : R/R^2 \times R/R^2 \to R^2, \tag{4.2}$$

which is symmetric or skew-symmetric if the semialgebra R is commutative or anti-commutative and whose image generates R^2.

Every symmetric or skew-symmetric bilinear mapping $g : U \times U \to V$ with $\dim V = 3$ whose image generates V can appear as (4.2). Indeed, consider the commutative or anti-commutative semialgebra $R := U \oplus V$ with multiplication given by the bilinear mapping

$$h : R \times R \to R, \qquad h(u+v, u'+v') := g(u,u')\,; \quad u,u' \in U,\ v,v' \in V.$$

Clearly, $R^3 = 0$ and $\dim R^2 = 3$. Since $V = R^2$ and the spaces U and R/V are naturally isomorphic, we can identify g and $\bar{h}$.

We have reduced the problem of classifying commutative (anti-commutative) semialgebras R with $R^3 = 0$ and $\dim R^2 = 3$ to the problem of classifying symmetric (skew-symmetric) bilinear mappings (4.2). The latter problem is wild by Theorem 3.1. $\square$

Proof of Theorem 4.1. Let R be a semialgebra with $R^3 = 0$ and $\dim R^2 = 3$.

(a) Suppose first that R is commutative. We “adjoin” the identity 1 by considering the algebra Λ consisting of the formal sums

$$\alpha 1 + a\,; \qquad \alpha \in \mathbb{F},\ a \in R,$$

with the componentwise addition and scalar multiplication and the multiplication

$$(\alpha 1 + a)(\beta 1 + b) = \alpha\beta 1 + (\alpha b + \beta a + ab).$$

This multiplication is associative since $R^3 = 0$, and so Λ is an algebra. It is commutative because R is commutative. Since R is the set of its noninvertible elements, Λ is a local algebra and R is its radical.

(b) Suppose now that R is anti-commutative. Since $R^3 = 0$, the Jacobi identity

$$(ab)c + (bc)a + (ca)b = 0$$

holds on R, and so R is a Lie algebra. The set R^2 is the commutator subalgebra of R; it is central since $R^3 = 0$.

Thus, Theorem 4.1 follows from Lemma 4.2. $\square$

Note that the wildness of the problem of classifying local associative algebras Λ with $(\operatorname{Rad}\Lambda)^3 = 0$ and $\dim(\operatorname{Rad}\Lambda)^2 = 2$ over an algebraically closed field $\mathbb{F}$ of characteristic not 2 was proved in [1].

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A publication of the International Linear Algebra Society
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5. The wildness of the problem of classifying metabelian p-groups. The *exponent* of a finite group G is the minimal positive integer m such that $g^m = 1$ for all $g \in G$. In this section, we prove the following theorem.

THEOREM 5.1. *The problem of classifying finite p-groups of exponent $p \neq 2$ with central commutator subgroup of order p^3 is wild.*

Let G be a finite p-group of exponent $p \neq 2$ with central commutator subgroup A. Then G/A and A are abelian p-groups of type $(p, \dots, p)$. We consider G/A and A as vector spaces over the field $\mathbb{F}_p$ with p elements, but we use the multiplicative notation

$$a^\alpha b^\beta\,; \qquad \alpha, \beta \in \mathbb{F}_p, \quad a, b \in G/A \text{ or } A, \tag{5.1}$$

instead of the additive notation $\alpha a + \beta b$. Define the mapping:

$$\varphi : G/A \times G/A \to A, \qquad \varphi(gA, g'A) := [g, g']. \tag{5.2}$$

LEMMA 5.2. (a) *The mapping* (5.2) *is a skew-symmetric bilinear mapping over the field $\mathbb{F}_p$ and its image generates A.*

(b) *The mapping* (5.2) *uniquely determines the group G, up to isomorphism.*

(c) *Let A and B be vector spaces over $\mathbb{F}_p$, $p \neq 2$. Let $\psi : B \times B \to A$ be a skew-symmetric bilinear mapping whose image generates A. Then there exist*

- *a finite p-group G of exponent p with central commutator subgroup A, and*
- *a linear bijection $B \to G/A$ that transforms ψ to* (5.2).

Proof. (a) The mapping (5.2) is bilinear and skew-symmetric since

$$[gh, x] = h^{-1}g^{-1}x^{-1}ghx = h^{-1}[g, x]x^{-1}hx = [g, x][h, x], \qquad [g, h] = [h, g]^{-1}$$

for all $g, h, x \in G$.

(b) The group G/A can be decomposed into a direct product of cyclic groups of order p; let $g_1A, \dots, g_tA$ be their generators. Then $g_i^p = 1$ and each element of G is uniquely represented in the form

$$g_1^{\alpha_1} \cdots g_t^{\alpha_t} a\,; \qquad 0 \le \alpha_i < p, \quad a \in A. \tag{5.3}$$

The multiplication of two elements of G that are written in the form (5.3) is fully determined by the mapping (5.2) since

$$\begin{aligned} g_1^{\alpha_1} \cdots g_t^{\alpha_t} a \cdot g_1^{\beta_1} \cdots g_t^{\beta_t} b &= g_1^{\alpha_1+\beta_1} g_2^{\alpha_2} \cdots g_t^{\alpha_t} \cdot g_2^{\beta_2} \cdots g_t^{\beta_t} ab[g_2^{\alpha_2} \cdots g_t^{\alpha_t}, g_1^{\beta_1}] \\ &= g_1^{\alpha_1+\beta_1} \cdots g_t^{\alpha_t+\beta_t} ab[g_2^{\alpha_2} \cdots g_t^{\alpha_t}, g_1^{\beta_1}][g_3^{\alpha_3} \cdots g_t^{\alpha_t}, g_2^{\beta_2}] \cdots [g_t^{\alpha_t}, g_{t-1}^{\beta_{t-1}}]. \end{aligned} \tag{5.4}$$

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A publication of the International Linear Algebra Society
Volume 18, pp. 516-529, August 2009

(c) Let $\psi : B \times B \to A$ be a skew-symmetric bilinear mapping on vector spaces A and B over $\mathbb{F}_p$ and let the image of ψ generate A. We write the operations in A and B in the multiplicative notation, as in (5.1). Choose a basis $b_1, \dots, b_t$ in B. Denote by G the set of all formal expressions of the form (5.3) and extend the multiplication on B to G by analogy with (5.4):

$$g_1^p = \cdots = g_t^p = 1, \qquad g_1^{\alpha_1} \cdots g_t^{\alpha_t} a \cdot g_1^{\beta_1} \cdots g_t^{\beta_t} b$$
$$= g_1^{\alpha_1+\beta_1} \cdots g_t^{\alpha_t+\beta_t} ab\psi(b_2^{\alpha_2} \cdots b_t^{\alpha_t}, b_1^{\beta_1})\psi(b_3^{\alpha_3} \cdots b_t^{\alpha_t}, b_2^{\beta_2}) \cdots \psi(b_t^{\alpha_t}, b_{t-1}^{\beta_{t-1}}).$$

It is easy to check that G is a group, A is its central commutator subgroup, and the vector space G/A is isomorphic to B. The exponent of G is p because if $x, y \in G$ and $x^p = y^p = 1$ then

$$\begin{aligned}(xy)^p &= x^2y^2xy \cdots xy[y,x] = x^3y^3xy \cdots xy[y,x][y^2,x] \\ &= x^py^p[y,x][y^2,x] \cdots [y^{p-1},x] = [y,x]^{1+2+\cdots+(p-1)} \\ &= [y,x]^{(p-1)p/2} = 1. \qquad \square\end{aligned}$$

Proof of Theorem 5.1. By Lemma 5.2, the problem of classifying finite p-groups of exponent p with central commutator subgroup of order p^3 reduces to the problem of classifying skew-symmetric bilinear mappings over $\mathbb{F}_p$ whose images generate the target spaces, which is wild by Theorem 3.1. $\square$

The problem of classifying finite p-groups with central commutator subgroup G' of order p^2 is wild both for the groups in which G' is cyclic and for the groups in which G' is of type (p,p); see [9]. Finite p-groups with central commutator subgroup of order p are classified in [7, 10].

Note that

> if U and V are vector spaces over a field $\mathbb{F}$ of characteristic not 2 and $\dim V = 3$, then each *skew-symmetric* bilinear mapping $h : U \times U \to V$ whose image generates V is surjective. (5.5)

Indeed, $d := \dim U \geq 3$. Let $d = 3$ (the case $d > 3$ is considered analogously). Represent h in the form (3.1). Using transformations (3.3), we have

$$M_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

and obtain

$$h(x,y) = (x_1y_2 - x_2y_1,\ x_1y_3 - x_3y_1,\ x_2y_3 - x_3y_2),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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in which x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of x and y. It is easy to verify that the system

$$x_1y_2 - x_2y_1 = a, \quad x_1y_3 - x_3y_1 = b, \quad x_2y_3 - x_3y_2 = c$$

is solvable for all $a, b, c \in \mathbb{F}$.

By Lemma 5.2(c), the statement (5.5) over $\mathbb{F} = \mathbb{F}_p$ is a special case of the following theorem [6, Theorem B]: Let $p > 3$ and let G be a finite group whose commutator subgroup G' is an abelian p-group generated by at most 3 elements. Then G' coincides with the set of all commutators; moreover, the proof is valid for $p = 3$ under the assumption that G is nilpotent of class 2.

The statement (5.5) does not extend to *symmetric* bilinear mappings: the mapping $f : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}^3$ over any field $\mathbb{F}$ given by the formula

$$f((x_1, x_2), (y_1, y_2)) = (x_1y_1, x_1y_2, x_2y_1)$$

is not surjective (since $(0, 1, 1)$ has no preimage) though its image generates $\mathbb{F}^3$ (since each (a, b, c) with $a \neq 0$ has a preimage); see [4, Example 1].

6. A proof of Lemma 2.2. Lemma 2.2 follows from [11, Theorem 1] (or see [12, Theorem 3.1]), in which the problem of classifying systems of forms and linear mappings over a field of characteristic not 2 was reduced to the problem of classifying systems of linear mappings. This reduction is presented in detail in [12, Section 3]; we recall it restricting ourselves to the problem of classifying triples of bilinear forms.

For a fixed $\varepsilon \in \{1, -1\}$ and each matrix triple (A, B, C), define the *adjoint triple*

$$(A, B, C)^\circ = \varepsilon(A^T, B^T, C^T).$$

Suppose we know a maximal set $\operatorname{ind}(Q)$ of nonequivalent indecomposable matrix triples (this means that every matrix triple that is indecomposable for equivalence is equivalent to exactly one triple from $\operatorname{ind}(Q)$). Replace each triple in $\operatorname{ind}(Q)$ that is equivalent to a selfadjoint triple by one that is *actually* selfadjoint, and denote the set of these selfadjoint triples by $\operatorname{ind}_0(Q)$. Then in each two- or one-element subset $\{\mathcal{T}, \mathcal{U}\} \subset \operatorname{ind}(Q) \smallsetminus \operatorname{ind}_0(Q)$ such that $\mathcal{T}^\circ$ is equivalent to $\mathcal{U}$, select one triple and denote the set of selected triples by $\operatorname{ind}_1(Q)$. (If $\mathcal{T}^\circ$ is equivalent to $\mathcal{U}$ then $\{\mathcal{T}, \mathcal{U}\}$ consists of one triple and we take it.)

Let $\mathcal{T} = (A, B, C) \in \operatorname{ind}_0(Q)$. A matrix pair (R, S) for which

$$(AS, BS, CS) = (RA, RB, RC)$$

is called an *endomorphism* of $\mathcal{T}$; the set $\operatorname{End}(\mathcal{T})$ of endomorphisms is a ring. Since $\mathcal{T}$ is indecomposable for equivalence, the subset

$$\operatorname{Rad}(\mathcal{T}) := \{(R, S) \in \operatorname{End}(\mathcal{T}) \mid R \text{ or } S \text{ is singular}\}$$

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A publication of the International Linear Algebra Society
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is the radical and the factor ring

$$\mathbb{T}(\mathcal{T}) := \operatorname{End}(\mathcal{T}) / \operatorname{Rad}(\mathcal{T})$$

is a field or skew field, see [11]. Since $\mathcal{T}$ is selfadjoint, for each endomorphism (R,S) of $\mathcal{T}$ we can define the *adjoint endomorphism*

$$(R,S)^\circ := (S^T, R^T).$$

The mapping $(R,S) \mapsto (R,S)^\circ$ is an involution on $\operatorname{End}(\mathcal{T})$, which induces the involution

$$(f + \operatorname{Rad}(\mathcal{T}))^\circ = f^\circ + \operatorname{Rad}(\mathcal{T}) \tag{6.1}$$

on $\mathbb{T}(\mathcal{T})$. For each selfadjoint automorphism

$$f := (R,S) = (R,S)^\circ \in \operatorname{End}(\mathcal{T}) \smallsetminus \operatorname{Rad}(\mathcal{T}),$$

define the selfadjoint matrix triple

$$\mathcal{T}^f := R\mathcal{T} = (RA, RB, RC).$$

Lemma 6.1. *Over a field $\mathbb{F}$ of characteristic different from 2, every triple of symmetric (skew-symmetric) matrices of the same size is congruent to a direct sum*

$$\mathcal{T}_1^{(\varepsilon)} \oplus \cdots \oplus \mathcal{T}_p^{(\varepsilon)} \oplus \bigoplus_{i=1}^{q} (\mathcal{U}_i^{f_{i1}} \oplus \cdots \oplus \mathcal{U}_i^{f_{il_i}}),$$

in which $\varepsilon = 1$ (respectively, $\varepsilon = -1$),

$$\mathcal{T}_1, \dots, \mathcal{T}_p \in \operatorname{ind}_1(Q), \qquad \mathcal{U}_i \in \operatorname{ind}_0(Q),$$

$\mathcal{T}_i^{(\varepsilon)}$ is defined in (3.5), $\mathcal{U}_i \neq \mathcal{U}_{i'}$ if $i \neq i'$, and each f_{ij} is a selfadjoint automorphism of $\mathcal{U}_i$. This sum is uniquely determined by the initial triple, up to permutation of summands and replacement of f_{ij} by g_{ij} such that the Hermitian forms

$$x_1^\circ(f_{i1} + \operatorname{Rad}(\mathcal{U}_i))x_1 + \cdots + x_{l_i}^\circ(f_{il_i} + \operatorname{Rad}(\mathcal{U}_i))x_{l_i}$$

and

$$x_1^\circ(g_{i1} + \operatorname{Rad}(\mathcal{U}_i))x_1 + \cdots + x_{l_i}^\circ(g_{il_i} + \operatorname{Rad}(\mathcal{U}_i))x_{l_i}$$

are equivalent over the field or skew field $\mathbb{T}(\mathcal{U}_i)$ with involution (6.1).

Proof. This lemma is a special case of [12, Theorem 3.1] (which was deduced from [11, Theorem 1]). Indeed, each triple of $n \times n$ symmetric or skew-symmetric

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A publication of the International Linear Algebra Society
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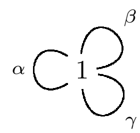
matrices over $\mathbb{F}$ defines the triple of symmetric or skew-symmetric bilinear forms on $\mathbb{F}^n$. Following [12, Section 3], we can consider the latter triple as a "representation of the graph with relations"

$G:$



$$\alpha = \varepsilon\alpha^*, \quad \beta = \varepsilon\beta^*, \\ \gamma = \varepsilon\gamma^*$$

in which $\varepsilon = 1$ or $\varepsilon = -1$, respectively (compare the graph G with the graph (7) in [11], whose representations are pairs of symmetric or skew-symmetric bilinear forms). The "quiver with involution" of G (see [12, Section 3]) is

$$Q: \qquad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha=\varepsilon\alpha^*"] \arrow[r, "\beta=\varepsilon\beta^*"] \arrow[r, bend right, "\gamma=\varepsilon\gamma^*"'] & 1^* \end{tikzcd}$$

Each triple of $m\times n$ matrices can be considered as a triple of linear mappings $\mathbb{F}^n \to \mathbb{F}^m$; that is, a representation of Q. Respectively, $\mathrm{ind}(Q)$ is a maximal set of nonisomorphic indecomposable representations of Q and we can apply [12, Theorem 3.1]. $\square$

Lemma 2.2 follows from Lemma 6.1.

Acknowledgment. The authors are very grateful to the referee for valuable remarks and suggestions.

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