Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 38, pp. 221-226, March 2022.

A NORM INEQUALITY FOR THREE MATRICES*

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Abstract. We prove a Frobenius norm inequality for three matrices, analogous to the well-known Böttcher–Wenzel inequality. The situation is also similar: standard inequalities would yield an upper bound, which however can be reduced by means of further, detailed investigations.

Key words. Frobenius norm, Böttcher–Wenzel inequality, Nonnegative polynomials.

AMS subject classifications. 15A60, 15A63, 15B57.

1. Introduction. For real matrices A, B of order $n \geq 2$, we have

$$\|AB - BA\|_F \leq \sqrt{2}\,\|A\|_F\|B\|_F, \tag{1.1}$$

conjectured by Böttcher and Wenzel [1] and proved by several authors, including the originators [2], cf. the survey [3]. Note that the immediate use of triangle inequality in conjunction with submultiplicativity would give a factor 2 on the right.

Consider now the problem of estimating from above the quantity

$$\|ABC - CBA\|_F.$$

By standard means, we get

$$\|ABC - CBA\|_F \leq \|ABC\|_F + \|CBA\|_F \leq 2\,\|A\|_F\|B\|_F\|C\|_F.$$

Surprisingly enough, the factor 2 can be replaced by 1.

Remark 1.1. The reason of why we compare ABC just with CBA is given by the following observations. There are $3! = 6$ permutations of $\{A, B, C\}$, and if we disregard ABC and CBA, the differences

$$T_1 = A(BC - CB),\ \ T_2 = (AB - BA)C,\ \ T_3 = A(BC) - (BC)A,\ \ T_4 = (AB)C - C(AB),$$

are left. Applying Böttcher and Wenzel's inequality immediately proves that

$$\max \frac{\|T_i\|_F}{\|A\|_F\|B\|_F\|C\|_F} = \sqrt{2},\quad i = 1\ldots 4,$$

since in all four cases we can find $\{-1, 0, 1\}$ matrices with equality holding. Therefore, in what follows, we can concentrate on the difference $ABC - CBA$.

*Received by the editors on August 31, 2021. Accepted for publication on March 12, 2022. Handling Editor: Fuad Kittaneh.
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2. The structure of $ABC - CBA$. The matrix

$$E = ABC - CBA,$$

depends on B linearly; therefore, one can write

$$e = Sb,$$

with suitable vectorizations e and b of E and B, resp. To be concrete, stretch out B, A, and C columnwise (as in Matlab), and E rowwise, that is, let

$$b = (b_{1,1}\ b_{2,1}, \ldots, b_{n-1,n}, b_{n,n})^T,$$

(vectors a and c are defined similarly), and let

$$e = (e_{1,1}\ e_{1,2}, \ldots, e_{n,n-1}, e_{n,n})^T.$$

Note that the $k-$th entry of vector b equals to the $(i,j)-$th entry of B for $k = i + (j-1)n$, while the $l-$th entry of vector e is equal to the $(i,j)-$th entry of E for $l = (i-1)n + j$.

Lemma 2.1. *S is skew symmetric.*

Proof. Calculate the $(i,j)-$th element of E. We get

$$e_{i,j} = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{i,k}b_{k,l}c_{l,j} - c_{i,k}b_{k,l}a_{l,j} = \sum_{k=1}^{n}\sum_{l=1}^{n} b_{k,l}(a_{i,k}c_{l,j} - a_{l,j}c_{i,k}). \tag{2.2}$$

We shall prove that $s_{p,q} = -s_{q,p}$ for $1 \le p, q \le n^2$. The unique representation

$$p = (i-1)n + j, \quad 1 \le i, j \le n,$$

defines $e_{i,j}$, while

$$q = k + (l-1)n, \quad 1 \le k, l \le n,$$

gives $b_{k,l}$, the coefficient in (2.2) of which equals $s_{p,q}$, therefore

$$s_{p,q} = a_{i,k}c_{l,j} - a_{l,j}c_{i,k}. \tag{2.3}$$

To calculate $s_{q,p}$, we have to represent q as a row index in S, hence now (l,k) is a position in E, while p is understood columnwise, that is, it points to the (j,i) position in B. In view of (2.2) this means

$$s_{q,p} = a_{l,j}c_{i,k} - a_{i,k}c_{l,j} = -s_{p,q},$$

which completes the proof. ∎

Lemma 2.2. *For a skew symmetric matrix S, it holds that*

$$\|S\|_2^2 \le \tfrac{1}{2}\|S\|_F^2.$$

Proof. It is known that the eigenvalues of a real skew symmetric matrix are purely imaginary and come in conjugate pairs (and one of them is zero, if the order is odd). Ordering them monotone decreasingly as

$$|\lambda_1| = |\lambda_2| \ge |\lambda_j|, \ j = 3, 4, \ldots, n^2,$$

and using the fact that S is normal, we get

$$\|S\|_F^2 = \sum_{i=1}^{n} |\lambda_i|^2 \ge 2|\lambda_1|^2 = 2\|S\|_2^2,$$

which was to be proved. ∎

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3. The main theorem.

THEOREM 3.1. *For real square matrices A, B, C, it holds that*

$$\|ABC - CBA\|_F^2 \le \|A\|_F^2 \|B\|_F^2 \|C\|_F^2 - \|B\|_F^2 \, tr^2(A^T C),$$

where $\|\cdot\|_F$ denotes the Frobenius norm, and tr stands for the trace.

Proof. To simplify notations, we write a_q and c_q for $a_{k,l}$ and $c_{k,l}$, resp., if $q = k + (l-1)n$. Introducing the auxiliary variables

$$u_{p,q} = a_p c_q - a_q c_p,$$

Lagrange's identity [4] gives

$$\|A\|_F^2 \|C\|_F^2 - tr^2(A^T C) = \sum_{1 \le p < q \le n^2} u_{p,q}^2.$$

By virtue of Lemma 2.1 and Lemma 2.2, we have

$$\|E\|_F^2 = \|e\|_2^2 = \|Sb\|_2^2 \le \|S\|_2^2 \, \|b\|_2^2 \le \frac{1}{2} \|S\|_F^2 \, \|b\|_2^2.$$

Observing that $\|b\|_2 = \|B\|_F$, it remains to prove that

$$\frac{1}{2} \|S\|_F^2 \; \le \|A\|_F^2 \|C\|_F^2 - tr^2(A^T C).$$

This follows because the set $\{s_{p,q} : \; 1 \le p, q \le n^2\}$ is a permutation of the set $\{u_{p,q} : \; 1 \le p, q \le n^2\}$, whence

$$\|S\|_F^2 = \sum_{1 \le p,q \le n^2} s_{p,q}^2 = \frac{1}{2} \sum_{1 \le p < q \le n^2} s_{p,q}^2 = \frac{1}{2} \sum_{1 \le p < q \le n^2} u_{p,q}^2.$$

□

REMARK 3.2. For illustration, we display the matrix S for $n = 2$:

$$S = \begin{pmatrix} 0 & u_{1,3} & u_{2,1} & u_{2,3} \\ u_{3,1} & 0 & u_{4,1} & u_{4,3} \\ u_{1,2} & u_{1,4} & 0 & u_{2,4} \\ u_{3,2} & u_{3,4} & u_{4,2} & 0 \end{pmatrix}.$$

REMARK 3.3. For second-order matrices A, B, C, more is true:

$$\begin{aligned} &\|A\|_F^2 \|B\|_F^2 \|C\|_F^2 - \|ABC - CBA\|_F^2 - \|B\|_F^2 \, tr^2(A^T C) \\ &\quad = (b_1^2 + b_4^2) \, u_{1,4}^2 + (b_1 u_{2,4} - b_4 u_{1,2})^2 + (b_1 u_{3,4} - b_4 u_{1,3})^2, \end{aligned}$$

which is an identity for the matrices

$$A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}.$$

Here the middle matrix B can be assumed to be diagonal, since for orthogonal U and V we have

$$\begin{aligned} \|ABC - CBA\|_F &= \|U(ABC)V^T - U(CBA)V^T)\|_F \\ &= \|(UAV^T)(VBU^T)(UCV^T) - (UCV^T)(VBU^T)(UAV^T)\|_F, \end{aligned}$$

and VBU^T can be chosen to be *nonnegative* diagonal, due to the singular value decomposition theorem. Therefore, this identity is another proof of the theorem for 2×2 matrices.

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4. On a possible generalization. The six permutations of our matrices make up two sets formed by cyclic permutations of ABC and CBA resp.; hence, it is natural to examine the difference of these three-term sums. So, let

$$E_1 = ABC + BCA + CAB, \quad E_2 = CBA + ACB + BAC,$$

and estimate from above the norm of

$$E = E_1 - E_2.$$

Introducing the abbreviations

$$X = BC - CB, \quad Y = CA - AC, \quad Z = AB - BA,$$

we examine two approaches. First, we have

$$\begin{aligned} \|E\|_F &= \|A(BC - CB) + B(CA - AC) + C(AB - BA)\|_F \\ &\equiv \|AX + BY + CZ\|_F \\ &\leq \|AX\|_F + \|BY\|_F + \|CZ\|_F \\ &\leq \|A\|_F\|X\|_F + \|B\|_F\|Y\|_F + \|C\|_F\|Z\|_F \\ &\leq 3\sqrt{2}\,\|A\|_F\|B\|_F\|C\|_F, \end{aligned} \tag{4.4}$$

where the Böttcher–Wenzel inequality (1.1) was used three times.

On the other hand, another rearrangement gives a different result, namely

$$\begin{aligned} \|E\|_F &= \|(ABC - CBA) + (BCA - ACB) + (CAB - BAC)\|_F \\ &\leq \|ABC - CBA\|_F + \|BCA - ACB\|_F + \|CAB - BAC\|_F \\ &\leq 3\|A\|_F\|B\|_F\|C\|_F, \end{aligned} \tag{4.5}$$

where we utilized our Theorem 3.1, giving therefore – even in its weaker form! – a better result.

However, some experiments show that yet more is true, allowing us to formulate our guess as follows.

Conjecture. For three real square matrices A, B, C, it holds that

$$\|E\|_F^2 \leq \tfrac{3}{2}\Big(\|A\|_F^2\|BC - CB)\|_F^2 + \|B\|_F^2\|CA - AC)\|_F^2 + \|C\|_F^2\|AB - BA)\|_F^2\Big), \tag{4.6}$$

or in short:

$$\|E\|_F^2 \leq \tfrac{3}{2}\Big(\|A\|_F^2\|X\|_F^2 + \|B\|_F^2\|Y\|_F^2 + \|C\|_F^2\|Z\|_F^2\Big).$$

Remark 4.1. The conjecture (in conjunction with Böttcher and Wenzel's inequality) would imply (4.5), and consequently also (4.4). Another argument for the strongness of the conjecture is that when estimating $\|E\|_F^2 = \|AX + BY + CZ\|_F^2$, standard calculation would give merely

$$\|E\|_F^2 \leq 3\Big(\|A\|_F^2\|X\|_F^2 + \|B\|_F^2\|Y\|_F^2 + \|C\|_F^2\|Z\|_F^2\Big).$$

Now we prove the conjecture for *second-order triangular* matrices.

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Proof. Let

$$A = \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix}.$$

Note that we write here for simplicity a_2, b_2, c_2 instead of a_4, b_4, c_4, resp. Denoting

$$\begin{aligned} F &= \|A\|_F^2 \|BC - CB)\|_F^2 + \|B\|_F^2 \|CA - AC)\|_F^2 + \|C\|_F^2 \|AB - BA)\|_F^2 \\ &= \Big(\|A\|_F^2 \|X\|_F^2 + \|B\|_F^2 \|Y\|_F^2 + \|C\|_F^2 \|Z\|_F^2 \Big), \end{aligned}$$

(4.6) is equivalent with

$$p = 3F - 2\|E\|_F^2 \geq 0,$$

where p is a sixth degree polynomial in nine variables with 78 terms. Our idea is to write p as a quadratic form

$$p = b^T Q b, \quad b = (b_1,\ b_2,\ b_3)^T, \quad Q \in \mathbb{R}^{3\times 3},$$

and show that Q is positive semidefinite. (Note that this is not a necessary condition for nonnegativity of polynomial p, due to Hilbert's 17-th problem, nevertheless, here it works.) With this end in view we examine the left upper determinants

$$d_1 = q_{1,1}, \quad d_2 = \begin{vmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{vmatrix}, \quad d_3 = det(Q).$$

1. d_1, a fourth degree polynomial is a quadratic form of the variables $\{a_1c_3,\ a_3c_1,\ a_2c_3,\ a_3c_2\}$ with coefficient matrix

$$\begin{pmatrix} 6 & -3 & -3 & 3 \\ -3 & 6 & 3 & -3 \\ -3 & 3 & 4 & -1 \\ 3 & -3 & -1 & 4 \end{pmatrix},$$

   if we omit the surplus term $a_3^2c_3^2$. Its characteristic polynomial in x is $(x^2 - 14x + 9)(x - 3)^2$ with positive zeros; hence, d_1 is positive definite.
2. d_2 is of eighth degree, however, fortunately enough, it can be factored as

$$d_2 = 3\,K^2\,\delta_2,$$

   with

$$K = a_1c_3 - a_3c_1 + a_3c_2 - a_2c_3.$$

   As for δ_2, the term containing $a_3^2c_3^2$ can be dropped again, and the corresponding coefficient matrix is

$$\begin{pmatrix} 5 & -1 & -3 & 3 \\ -1 & 5 & 3 & -3 \\ -3 & 3 & 5 & -1 \\ 3 & -3 & -1 & 5 \end{pmatrix},$$

   with characteristic polynomial $x(x - 12)(x - 4)^2$; hence, δ_2 (and thus d_2) is positive semidefinite.

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3. The determinant $|Q|$ itself is of degree 12; however, the fourth power of K can be factored out, so that we get
$$d_3 = 18\, K^4\, \delta_3.$$
The factor δ_3 is a quartic again and can be handled similarly to the former ones. The basis elements are
$$\{a_1c_1,\, a_1c_2,\, a_1c_3,\, a_2c_1,\, a_2c_2,\, a_2c_3, a_3c_1,\, a_3c_2,\, a_3c_3\}.$$
Here, however, one must utilize the basic identities
$$\begin{aligned} &a_1c_1 \cdot a_3c_3 = a_1c_3 \cdot a_3c_1, \quad a_2c_2 \cdot a_3c_3 = a_2c_3 \cdot a_3c_2,\\ &a_1c_2 \cdot a_3c_3 = a_1c_3 \cdot a_3c_2, \quad a_2c_1 \cdot a_3c_3 = a_2c_3 \cdot a_3c_2, \end{aligned} \tag{4.7}$$
to get rid of a_3c_3 and obtain thus the 9×9 matrix
$$\begin{pmatrix} 6 & -3 & 0 & -3 & 0 & 0 & 0 & 0 & 0\\ -3 & 4 & 0 & 2 & -3 & 0 & 0 & 0 & 0\\ 0 & 0 & 10 & 0 & 0 & -6 & -1 & 3 & 0\\ -3 & 2 & 0 & 4 & -3 & 0 & 0 & 0 & 0\\ 0 & -3 & 0 & -3 & 6 & 0 & 0 & 0 & 0\\ 0 & 0 & -6 & 0 & 0 & 10 & 3 & -1 & 0\\ 0 & 0 & -1 & 0 & 0 & 3 & 10 & -6 & 0\\ 0 & 0 & 3 & 0 & 0 & -1 & 6 & 10 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}.$$
Its characteristic polynomial is $x(x-20)(x-2)^2(x-6)^2(x-12)^3$, having nonnegative zeros; hence, this matrix is positive semidefinite. Although d_2 and d_3 can vanish, this problem can be handled easily by calculating the missing three subdeterminants, mentioned in Sylvester's criterion. They prove to be nonnegative as well.
To sum up, the original matrix Q is positive semidefinite, and the conjecture is in this special case proved. ∎

Remark 4.2. Changing as in (4.7) the basis elements in case of need, is an essential part of semidefinite programming. (Without doing that one would have here a negative eigenvalue ≈ -0.3614.)

Also note that a similar proof for second-order full matrices seems to be too difficult, since the number of terms in p is then 243, and over this, the subdeterminants of matrix Q cannot be factored.

Hence, we need to raise the following problem.

Problem. Prove (or disprove) the above conjecture for real (or complex) matrices of arbitrary order!

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