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FORTY NECESSARY AND SUFFICIENT CONDITIONS FOR REGULARITY OF INTERVAL MATRICES: A SURVEY*

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Abstract. This is a survey of forty necessary and sufficient conditions for regularity of interval matrices published in various papers over the last thirty-five years. A full list of references to the sources of all the conditions is given, and they are commented on in detail.

Key words. Interval matrix, Regularity, Singularity, Necessary and sufficient condition, Algorithm.

AMS subject classifications. 15A24, 65G40.

1. Introduction. During the last thirty-five years (1973-2008), considerable interest has been dedicated to the problem of regularity of interval matrices. It has resulted in formulations of altogether forty necessary and sufficient conditions that constitute the subject matter of this survey paper.

By definition, a square interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix). It is the purpose of this paper to show that this property can be reformulated in surprisingly many surprisingly various ways. In the main Theorem 4.1 we show that regularity of interval matrices can be characterized in terms of determinants (Theorem 4.1, condition (xxxii)), matrix inverses (xxx), linear equations (xxv), absolute value equations (v), absolute value inequalities (ii), matrix equations (xxiv), solvability in each orthant (xvi), inclusions (xxxvii), set properties (xxxvi), real spectral radius (xxxiv), P-matrices (xxix) and edge nonsingularity (xli). We do not include the proof of mutual equivalence of all the conditions since that would make for a very lengthy paper. Instead, we list in Fig. 5.1 a full list of their sources.

In Section 6, the forty conditions from Theorem 4.1 are commented on item-by-item. For clarity, they are divided into five groups handled separately in Subsections 6.1 to 6.5. The comments contain references to a lot of related results and hopefully show that regularity of interval matrices is worth further study.

2. Notations. We use the following notations. $A_{\bullet k}$ denotes the kth column of A. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. I is the unit matrix and $e = (1, \ldots, 1)^T$ is the vector of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $x \in \mathbb{R}^n$ we define its sign

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Electronic Journal of Linear Algebra ISSN 1081-3810
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vector $\mathrm{sgn}(x)$ by

$$(\mathrm{sgn}(x))_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0 \end{cases} \qquad (i = 1, \ldots, n),$$

so that $\mathrm{sgn}(x) \in Y_n$. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \mathrm{diag}\,(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix},$$

and $\mathbb{R}^n_y = \{x \mid T_y x \geq 0\}$ is the orthant prescribed by the ± 1-vector y. Finally, we introduce the *real* spectral radius of a square matrix A by

$$\varrho_0(A) = \max\{|\lambda| \mid \lambda \text{ is a real eigenvalue of } A\},$$

and we set $\varrho_0(A) = 0$ if no real eigenvalue exists.

3. Interval matrices. Given two $n \times n$ matrices A_c and Δ, $\Delta \geq 0$, the set of matrices

$$\mathbf{A} = \{A \mid |A - A_c| \leq \Delta\} \tag{3.1}$$

is called a (square) interval matrix with midpoint matrix A_c and radius matrix Δ. Since the inequality $|A - A_c| \leq \Delta$ is equivalent to $A_c - \Delta \leq A \leq A_c + \Delta$, we can also write

$$\mathbf{A} = \{A \mid \underline{A} \leq A \leq \overline{A}\} = [\underline{A}, \overline{A}],$$

where $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$ are called the bounds of $\mathbf{A}$. As it will be seen in Theorem 4.1, the notation (3.1) is preferable for our purposes.

Given an $n \times n$ interval matrix $\mathbf{A}$, we define matrices

$$A_{yz} = A_c - T_y \Delta T_z \tag{3.2}$$

for each $y \in Y_n$ and $z \in Y_n$. The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases} \qquad (i, j = 1, \ldots, n),$$

so that $A_{yz} \in \mathbf{A}$ for each $y \in Y_n$, $z \in Y_n$. Since cardinality of Y_n is 2^n, the cardinality of the set of matrices $\{A_{yz} \mid y, z \in Y_n\}$ is at most 2^{2n}. We shall write A_{-yz} instead of $A_{-y,z}$. In particular, we have $A_{ye} = A_c - T_y \Delta$ and $A_{-ye} = A_c + T_y \Delta$. The central topic of this paper is introduced in the following definition.

Definition. A square interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 500-512, August 2009

4. Necessary and sufficient conditions. The following theorem sums up forty necessary and sufficient conditions for regularity of interval matrices.

THEOREM 4.1. *For an $n \times n$ interval matrix $\mathbf{A}$, the following assertions are equivalent:*

(i) $\mathbf{A}$ *is regular,*

(ii) the inequality

$$|A_c x| \le \Delta |x| \tag{4.1}$$

has only the trivial solution $x = 0$,

(iii) for each $d \in [0, 1]$ the equation

$$|A_c x| = d\Delta |x|$$

has only the trivial solution $x = 0$,

(iv) if $A'x' = A''x''$ for some $A', A'' \in \mathbf{A}$ and $x' \ne x''$, then there exists a j such that $A'_{\bullet j} \ne A''_{\bullet j}$ and $x'_j x''_j > 0$,

(v) for each B with $|B| \le \Delta$ and for each $b \in \mathbb{R}^n$ the equation

$$A_c x + B|x| = b \tag{4.2}$$

has a unique solution,

(vi) for each B with $|B| \le \Delta$ and for each $b \in \mathbb{R}^n$ the algorithm (Fig. 4.1) does

$$\boxed{\begin{array}{l} z = \mathrm{sgn}(A_c^{-1} b); \\ x = (A_c + BT_z)^{-1} b; \\ \textbf{while } z_j x_j < 0 \text{ for some } j \\ \quad k = \min\{j \mid z_j x_j < 0\}; \\ \quad z_k = -z_k; \\ \quad x = (A_c + BT_z)^{-1} b; \\ \textbf{end} \end{array}}$$

FIG. 4.1. *The kernel of the sign accord algorithm.*

not break down[1] *and in a finite number of steps (at most 2^n) yields the unique solution of the equation*

$$A_c x + B|x| = b,$$

(vii) for each B with $|B| \le \Delta$ and for each $b \in \mathbb{R}^n$ the sign accord algorithm (Fig. 4.2) does not break down[2] *and in a finite number of steps (at most 2^n) yields the unique solution of the equation*

$$A_c x + B|x| = b,$$

[1] I.e., all the inverses exist.

[2] I.e., all the inverses exist and the denominator of α never becomes zero.

Electronic Journal of Linear Algebra ISSN 1081-3810
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$z = \mathrm{sgn}(A_c^{-1}b);$
$x = (A_c + BT_z)^{-1}b;$
$C = -(A_c + BT_z)^{-1}B;$
while $z_j x_j < 0$ for some j
  $k = \min\{j \mid z_j x_j < 0\};$
  $z_k = -z_k;$
  $\alpha = 2z_k/(1 - 2z_k C_{kk});$
  $x = x + \alpha x_k C_{\bullet k};$
  $C = C + \alpha C_{\bullet k} C_{k\bullet};$
end

FIG. 4.2. *The sign accord algorithm.*

(viii) for each $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = y \tag{4.3}$$

has a solution,

(ix) for each $y \in Y_n$ *the equation*

$$A_c x - T_y \Delta |x| = y$$

has a unique solution,

(x) for each $b > 0$ *and for each* $y \in Y_n$ *the equation*

$$|A_c x| = \Delta |x| + b \tag{4.4}$$

has a solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,*

(xi) for each $b > 0$ *and for each* $y \in Y_n$ *the equation*

$$|A_c x| = \Delta |x| + b$$

has a unique solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,*

(xii) for each $y \in Y_n$ *the equation*

$$|A_c x| = \Delta |x| + e$$

has a solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,*

(xiii) for each $y \in Y_n$ *the equation*

$$|A_c x| = \Delta |x| + e$$

has a unique solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,*

(xiv) for each $y \in Y_n$ *the inequality*

$$|A_c x| > \Delta |x|$$

has a solution x_y *satisfying* $A_c x_y \in \mathbb{R}^n_y$*,*

Electronic Journal of Linear Algebra ISSN 1081-3810
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(xv) A_c *is nonsingular and for each* $b > 0$ *the equation*

$$|x| = \Delta|A_c^{-1}x| + b \tag{4.5}$$

has a solution in each orthant,

(xvi) A_c *is nonsingular and for each* $b > 0$ *the equation*

$$|x| = \Delta|A_c^{-1}x| + b$$

has a unique solution in each orthant,

(xvii) A_c *is nonsingular and the equation*

$$|x| = \Delta|A_c^{-1}x| + e$$

has a solution in each orthant,

(xviii) A_c *is nonsingular and the equation*

$$|x| = \Delta|A_c^{-1}x| + e$$

has a unique solution in each orthant,

(xix) A_c *is nonsingular and the inequality*

$$|x| > \Delta|A_c^{-1}x| \tag{4.6}$$

has a solution in each orthant,

(xx) *there exists an* $R \in \mathbb{R}^{n\times n}$ *such that the inequality*

$$|x| > |(I - A_cR)x| + \Delta|Rx| \tag{4.7}$$

has a solution in each orthant,

(xxi) *for each* $y \in Y_n$ *the matrix equation*

$$A_cX - T_y\Delta|X| = I \tag{4.8}$$

has a solution,

(xxii) *for each* $y \in Y_n$ *the matrix equation*

$$A_cX - T_y\Delta|X| = I$$

has a unique solution X_y*,*

(xxiii) *for each* $y \in Y_n$ *the matrix equation*

$$QA_c - |Q|\Delta T_y = I \tag{4.9}$$

has a solution,

(xxiv) *for each* $y \in Y_n$ *the matrix equation*

$$QA_c - |Q|\Delta T_y = I$$

has a unique solution Q_y*,*

Electronic Journal of Linear Algebra ISSN 1081-3810
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(xxv) *for each* $y \in Y_n$ *the linear system*

$$\begin{aligned} A_{ye}x_1 - A_{-ye}x_2 &= y, \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}$$

has a solution,

(xxvi) *for each* $y \in Y_n$, A_{ye} *is nonsingular and the system*

$$\begin{aligned} A_{ye}^{-1}A_{-ye}x &> 0, \\ x &> 0 \end{aligned}$$

has a solution,

(xxvii) *for each* $y \in Y_n$, A_{ye} *and* A_{-ye} *are nonsingular and the system*

$$\begin{aligned} A_{ye}^{-1}x &> 0, \\ A_{-ye}^{-1}x &> 0 \end{aligned}$$

has a solution,

(xxviii) A_c *is nonsingular and for each* $y \in Y_n$, A_{ye} *is nonsingular and the system*

$$|A_c^{-1}T_y\Delta x| < x \tag{4.10}$$

has a solution,

(xxix) *for each* $y \in Y_n$, A_{ye} *is nonsingular and* $A_{ye}^{-1}A_{-ye}$ *is a* P*-matrix,*

(xxx) *for each* $y, z \in Y_n$, A_{yz} *is nonsingular and*

$$(A_cA_{yz}^{-1})_{ii} > \tfrac{1}{2}$$

holds for each $i \in \{1, \dots, n\}$,

(xxxi) $\det(A_c)\det(A_{yz}) > 0$ *for each* $y, z \in Y_n$,

(xxxii) $\det(A_{yz})\det(A_{y'z'}) > 0$ *for each* $y, z, y', z' \in Y_n$,

(xxxiii) $\det(A_{yz})\det(A_{y'z}) > 0$ *for each* $y, y', z \in Y_n$ *such that* y *and* y' *differ in exactly one entry,*

(xxxiv) A_c *is nonsingular and*

$$\max_{y,z \in Y_n} \varrho_0(A_c^{-1}T_y\Delta T_z) < 1$$

holds,

(xxxv) *for each interval* n*-vector* $\mathbf{b}$ *the set*

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{x \mid Ax = b \text{ for some } A \in \mathbf{A},\ b \in \mathbf{b}\} \tag{4.11}$$

is compact and connected,

(xxxvi) *there exists an interval* n*-vector* $\mathbf{b}$ *for which at least one component of the set*

$$\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{x \mid Ax = b \text{ for some } A \in \mathbf{A},\ b \in \mathbf{b}\}$$

is bounded,

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 500-512, August 2009

(xxxvii) *for each* $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, *there holds*

$$\{Ax_1 \mid A \in \mathbf{A}\} \not\subseteq \{Ax_2 \mid A \in \mathbf{A}\}, \tag{4.12}$$

(xxxviii) *each matrix of the form*

$$A = A_c - dT_y \Delta T_z, \tag{4.13}$$

where $d \in [0,1]$ *and* $y, z \in Y_n$, *is nonsingular,*

(xxxix) *each matrix of the form*

$$A = A_c - T_t \Delta T_z, \tag{4.14}$$

where $|t| \leq e$ *and* $z \in Y_n$, *is nonsingular,*

(xl) *each matrix of the form*

$$A_{ij} = \begin{cases} (A_{yz})_{ij} & \text{if either } i \neq k, \text{ or } i = k \text{ and } j \in \{1, \ldots, m-1\}, \\ (A_{-yz})_{ij} & \text{if } i = k \text{ and } j \in \{m+1, \ldots, n\}, \end{cases}$$
$$A_{km} \in [\underline{A}_{km}, \overline{A}_{km}],$$

where $y, z \in Y_n$ *and* $k, m \in \{1, \ldots, n\}$, *is nonsingular,*

(xli) *each matrix of the form*

$$A_{ij} \in \begin{cases} \{\underline{A}_{ij}, \overline{A}_{ij}\} & \text{if } (i,j) \neq (k,m), \\ [\underline{A}_{ij}, \overline{A}_{ij}] & \text{if } (i,j) = (k,m) \end{cases} \qquad (i, j = 1, \ldots, n), \tag{4.15}$$

where $k, m \in \{1, \ldots, n\}$, *is nonsingular.*

5. Sources. We do not give here the proof of the mutual equivalence of all the conditions since, as the reader may expect, this would make for a lengthy and perhaps tedious paper. Instead, we list in Fig. 5.1 a full list of their sources.

6. Comments. In this section we comment on the conditions. At some places we quote related theorems; for the sake of smoothness of the exposition, they are not marked as such, but are always given in *italics*. The forty conditions can be divided into five groups: (ii)-(vii), (viii)-(xxiv), (xxv)-(xxxiv), (xxxv)-(xxxvii), and (xxxviii)-(xli).

6.1. Conditions (ii)-(vii). The conditions (ii)-(vii) sum up the basic theoretical and algorithmic facts.

(ii): This is the most important characterization, used in proofs of many other conditions. It is advantageous to read it negated: $\mathbf{A}$ *is singular if and only if the inequality (4.1) has a nontrivial solution.* If $x \neq 0$ solves (4.1), then a singular matrix $S \in \mathbf{A}$ can be constructed as $S = A_c - T_y \Delta T_z$, where $z = \mathrm{sgn}(x)$ and y is defined by $y_i = (A_c x)_i / (\Delta |x|)_i$ if $(\Delta |x|)_i > 0$ and $y_i = 1$ otherwise, $i = 1, \ldots, n$ ([6], Proposition 2.10). In particular, (ii) gives that $\max_j (|A_c^{-1}|\Delta)_{jj} \geq 1$ implies singularity of $\mathbf{A}$, see [17], Corollary 5.1; this is the most simple verifiable sufficient singularity condition (for its regularity counterpart, see the comment on condition

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 500-512, August 2009

ELA

Cond'n	**Reference**	**Cond'n**	**Reference**
(ii)	[12], Corollary 3.4.5, (iii)	(xxii)	[17], Theorem 5.1, (A3)
(iii)	[20], Corollary 2.5	(xxiii)	[25], p. 17, (v)
(iv)	[17], Theorem 1.1	(xxiv)	[25], p. 17, (v)
(v)	[25], p. 14	(xxv)	[17], Theorem 5.1, (A2)
(vi)	[25], p. 14	(xxvi)	[17], Theorem 5.1, (B2)
(vii)	[25], p. 56	(xxvii)	[17], Theorem 5.1, (B3)
(viii)	[17], Theorem 5.1, (A1)	(xxviii)	[17], Theorem 5.1, (B4)
(ix)	[17], Theorem 5.1, (A1)	(xxix)	[17], Theorem 5.1, (B1)
(x)	[27], Theorem 3.1, (ii)	(xxx)	[17], Theorem 5.1, (C4)
(xi)	[27], Theorem 3.1, (ii)	(xxxi)	[17], Theorem 5.1, (C1)
(xii)	[27], Theorem 3.1, (iii)	(xxxii)	[17], Theorem 5.1, (C1)
(xiii)	[27], Theorem 3.1, (iii)	(xxxiii)	[17], Theorem 5.1, (C2)
(xiv)	[27], Theorem 3.1, (v)	(xxxiv)	[17], Theorem 5.1, (C3)
(xv)	[27], Theorem 3.2, (ii)	(xxxv)	[4], Theorem 2, Conseq. 3
(xvi)	[27], Theorem 3.2, (ii)	(xxxvi)	[9], Theorem 5.3
(xvii)	[27], Theorem 3.2, (iii)	(xxxvii)	[15]
(xviii)	[27], Theorem 3.2, (iii)	(xxxviii)	[20], Corollary 2.3
(xix)	[27], Theorem 3.2, (v)	(xxxix)	[6], Proposition 2.10
(xx)	[27], Theorem 3.3	(xl)	[17], Theorem 5.1, (C6)
(xxi)	[17], Theorem 5.1, (A3)	(xli)	[17], Theorem 5.1, (C7)

Fig. 5.1. *Sources of the conditions.*

(xxxiv) below). Moreover, the condition (ii) is the main tool for the proof of co-NP-completeness of checking regularity (*checking regularity of interval matrices of the form* $[A_c - ee^T, A_c + ee^T]$ *is co-NP-complete in the class of nonnegative symmetric positive definite rational matrices* A_c; short proof in [6], Theorem 2.33, original proof in [14]). This complexity result sheds its light on all the subsequent conditions: if the conjecture P≠NP is true, then none of them is verifiable in polynomial time.

(iii): This condition shows that if (4.1) holds for some $x \neq 0$, then it also holds "uniformly" for some $x' \neq 0$. It has a nontrivial consequence. Let us define a nonnegative number d to be an *absolute eigenvalue* of a real matrix $A \in \mathbb{R}^{n\times n}$ if $|Ax| = d|x|$ holds for some real $x \neq 0$. Surprisingly enough, *each square real matrix has an absolute eigenvalue* ([24], Theorem 5). Even more, the smallest absolute eigenvalue $d_{\min}$ can be given explicitly by $d_{\min} = \inf\{\, \varepsilon \geq 0 \mid [A - \varepsilon I, A + \varepsilon I] \text{ is singular} \,\}$ ([20], Theorem 2.2).

(iv): This is a lemma-type assertion: it is of little use as such, but it is indispensable for the proofs of the subsequent important conditions (v) and (vi). In fact, Theorem 1.1 in [17] states only necessity of (iv). But sufficiency follows easily by contradiction: if $\mathbf{A}$ is singular, then $A'x' = 0$ for some $A' \in \mathbf{A}$ and $x' \neq 0$, hence $A'x' = A'x'' = 0$ for $x'' = 0$, but $x'_j x''_j = 0$ for each j, a contradiction.

(v): This is probably the most important single result among the forty conditions, as it asserts existence and uniqueness of solution of any equation of the type (4.2),

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 500-512, August 2009

thereby enabling us to establish existence of certain uniquely defined objects (as the matrices X_y, Q_y in conditions (xxii), (xxiv), etc.).

(vi): The algorithm uses a **while** loop which is finite under the regularity assumption, although this is not obvious from the algorithm description (theoretically, it could return to the same z, x and cycle infinitely; in fact, this can appear if $\mathbf{A}$ is singular ([17], Example 5.4)). Finiteness is proved by a sophisticated combinatorial argument based on condition (iv) (cycling would imply singularity). When the algorithm terminates in a finite number of steps, it is not difficult to see that the resulting x is a solution of (4.2), and its uniqueness follows from the condition (ii); this gives a constructive proof of (v).

(vii): The algorithm in condition (vi) requires explicit computation of $x = (A + BT_z)^{-1}b$ at each step. In condition (vii), a Sherman-Morrison rank-one update is used for this purpose; the updated values always satisfy $x = (A + BT_z)^{-1}b$, $C = -(A + BT_z)^{-1}B$ for the current z. We present the algorithm in both forms because the simple, although less effective form in (vi) better reveals its basic sign-accord-oriented mechanism (z, x are said to be in sign accord if they satisfy $z_jx_j \geq 0$ for each j).

Comparison of (v) and (ii) gives rise to the *first theorem of alternatives* for real (noninterval) data ([24], Thms. 1 and 2): *for each $A, B \in \mathbb{R}^{n\times n}$, exactly one of the alternatives (a), (b) holds: (a) For each B' with $|B'| \leq |B|$ and for each $b \in \mathbb{R}^n$ the equation $Ax + B'|x| = b$ has a unique solution, (b) the inequality $|Ax| \leq |B||x|$ has a nontrivial solution.* Here, (a) corresponds to regularity and (b) to singularity of $[A - |B|, A + |B|]$.

6.2. Conditions (viii)-(xxiv). Conditions (viii)-(xxiv) concern equations and inequalities containing absolute values.

(viii)-(ix): The main contribution here consists in the fact that solvability of finitely many equations (4.3) implies unique solvability of infinitely many equations of the form (4.2). Notice that (viii) and (ix) differ only in the word "unique". The same holds for several subsequent pairs of conditions ((x) and (xi), (xii) and (xiii), etc.).

(x)-(xiv): The strongest assertion here is (xi) which shows that for each $b > 0$ the set $\{ A_cx \mid |A_cx| = \Delta|x| + b \}$ intersects all the orthants. The weakest one is (xiv), and the proof of "(xiv)⇒(i)" [27] requires use of a little known existence theorem for linear equations [18].

(xv)-(xx): (xv)-(xix) are counterparts of (x)-(xiv) formed by replacing the equation (4.4) by (4.5). This results in *unique solvability in each orthant*, which is certainly a valuable property. The additional condition (xx) shows that the inequality (4.6) can be replaced by (4.7), thereby escaping the use of the exact inverse A_c^{-1}. This is a single example of using an approximate inverse among all the forty conditions.

(xxi)-(xxii): For a regular interval matrix $\mathbf{A}$, its inverse interval matrix is defined as $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$, where $\underline{B} = \min\{ A^{-1} \mid A \in \mathbf{A} \}$, $\overline{B} = \max\{ A^{-1} \mid A \in \mathbf{A} \}$ (componentwise). Theorem 6.2 in [17] asserts that the bounds of $\mathbf{A}^{-1}$ can be expressed by finite means as $\underline{B} = \min_{y\in Y_n} X_y$, $\overline{B} = \max_{y\in Y_n} X_y$ (componentwise), where X_y is the unique solution of (4.8).

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 500-512, August 2009

(xxiii)-(xxiv): These conditions follow from (xxi), (xxii) when applied to the transpose interval matrix $\mathbf{A}^T = \{ A^T \mid A \in \mathbf{A} \} = [A_c^T - \Delta^T, A_c^T + \Delta^T]$. The matrices Q_y, $y \in Y_n$ are used in the VERINTERVALHULL.M function [1] of VERSOFT [3] for computing a verified interval hull of the solution set of interval linear equations (based on [25], Sections 4.3 and 7.11).

Comparison of (xi) and (ii) gives rise to the *second theorem of alternatives* (unpublished): *for each $A, B \in \mathbb{R}^{n\times n}$, exactly one of the alternatives (a), (b) holds: (a) For each $b > 0$ and for each orthant $\mathcal{O}$ of $\mathbb{R}^n$ the equation $|Ax| - |B||x| = b$ has a unique solution $x_{\mathcal{O}}$ satisfying $Ax_{\mathcal{O}} \in \mathcal{O}$, (b) the equation $|Ax| - |B||x| = b$ has a nontrivial solution for some $b \le 0$.* Here, (a) corresponds to regularity and (b) to singularity of $[A - |B|, A + |B|]$.

Comparison of (xiv) and (ii) gives rise to the *third theorem of alternatives* ([27], Thm. 4.1): *for each $A, B \in \mathbb{R}^{n\times n}$, exactly one of the alternatives (a), (b) holds: (a) For each orthant $\mathcal{O}$ of $\mathbb{R}^n$ the inequality $|Ax| > |B||x|$ has a solution $x_{\mathcal{O}}$ satisfying $Ax_{\mathcal{O}} \in \mathcal{O}$, (b) the inequality $|Ax| \le |B||x|$ has a nontrivial solution.* Again, (a) corresponds to regularity and (b) to singularity of $[A - |B|, A + |B|]$.

6.3. Conditions (xxv)-(xxxiv). Conditions (xxv)-(xxxiv) concern equations, inequalities and other expressions not containing absolute values[3].

(xxv): This condition is formulated in terms of $n \times 2n$ linear systems. It is mainly a "return implication" used as the last member in a chain of implications (i)⇒(ii)⇒ ... ⇒(ix) ⇒(xxv)⇒(i). For a related more general result, see [23].

(xxvi)-(xxviii): The characteristic feature of these three conditions is their use of strict inequalities. They are seemingly of theoretical interest only; no application of them is known to the author so far.

(xxix): This condition brings about relationship between regularity and P-matrices (by definition, a square matrix is called a P-matrix if all its principal minors are positive). A general result published in [17], Thm. 1.2 says that *if $\mathbf{A}$ is regular, then $A_1^{-1}A_2$ is a P-matrix for each $A_1, A_2 \in \mathbf{A}$.* Here it is shown that, conversely, the P-property of finitely many matrices of the form $A_{ye}^{-1}A_{-ye}$, $y \in Y_n$, suffices to enforce regularity.

(xxx): This is an example of using a finite set of inverses for checking regularity. It is used in S. M. Rump's INTLAB function PLOTLINSOL.M [30] for verifying regularity of 2×2 interval matrices. As a necessary condition, it can be generalized: *if $\mathbf{A}$ is regular, then $(A_cA^{-1})_{ii} > \frac{1}{2}$ for each $A \in \mathbf{A}$ and each $i \in \{1, \dots, n\}$* ([17], Thm. 5.1, (C5)). The generalization to an arbitrary A is possible due to the inverse matrix representation theorem ([21], Thm. 1.1): *if $\mathbf{A}$ is regular, then for each $A \in \mathbf{A}$ there exist nonnegative diagonal matrices L_{yz}, $y, z \in Y_n$, satisfying $\Sigma_{y,z\in Y_n} L_{yz} = I$ such that $A^{-1} = \Sigma_{y,z\in Y_n} A_{yz}^{-1} L_{yz}$ holds* (a "convex combination").

(xxxi)-(xxxii): Both the conditions say that $\mathbf{A}$ is regular if and only if the determinants of all the matrices A_{yz}, $y, z \in Y_n$, are nonzero and of the same sign. The sole nonsingularity of all the matrices A_{yz}, $y, z \in Y_n$, is not sufficient to guarantee

[3]The inequality (4.10) in condition (xxviii) can be written without the absolute value as $-x < A_c^{-1}T_y\Delta x < x$.

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regularity: a counterexample is given by the interval matrix $\mathbf{A} = [-I, I]$ which is singular (it contains the zero matrix) despite nonsingularity of all the A_{yz}'s (each A_{yz} is a ± 1-diagonal matrix). Condition (xxxi), due to its simplicity, can be recommended for solving classroom examples of small sizes.

(xxxiii): This is an equivalent, but more specific form of (xxxii). It is used in the proofs of conditions (xxx) and (xl).

(xxxiv): There are several applications of this condition. First, the radius of regularity defined by $d(\mathbf{A}) = \inf\{\varepsilon \geq 0\,;\, [A_c - \varepsilon\Delta, A_c + \varepsilon\Delta] \text{ is singular}\}$ can be expressed by finite means as $d(\mathbf{A}) = 1/\max_{y,z\in Y_n} \varrho_0(A_c^{-1}T_y\Delta T_z)$ (Poljak and Rohn [14]). Second, since $\varrho_0(A_c^{-1}T_y\Delta T_z) \leq \varrho(A_c^{-1}T_y\Delta T_z) \leq \varrho(|A_c^{-1}T_y\Delta T_z|) \leq \varrho(|A_c^{-1}|\Delta)$, the condition $\varrho(|A_c^{-1}|\Delta) < 1$ implies regularity of $\mathbf{A}$ (originally proved by Beeck [5] by simpler means). This is the most often used sufficient regularity condition (the so-called strong regularity); for complexity or verifiability purposes it can also be stated in an equivalent form $(I - |A_c^{-1}|\Delta)^{-1} \geq 0$. Another sufficient regularity condition, $\sigma_{\max}(\Delta) < \sigma_{\min}(A_c)$ (singular values), is due to Rump [29]; it follows from (ii). Third, the condition (xxxiv) shows that an interval matrix of the form $[I - \Delta, I + \Delta]$ is regular if and only if $\varrho(\Delta) < 1$, cf., [19]. And fourth, it implies that an interval matrix of the form $[A_c - pq^T, A_c + pq^T]$, where p, q are nonnegative column vectors in $\mathbb{R}^n$, is regular if and only if $\|T_qA_c^{-1}T_p\|_{\infty,1} < 1$ ([17], Thm. 5.2, (R2); see [22] for the norm $\|\cdot\|_{\infty,1}$). Finally, closely related to the condition (xxxiv) is the following result ([17], Thm. 4.6): if $T_zA_{yz}^{-1}T_y \geq 0$ and $T_zA_{-yz}^{-1}T_y \geq 0$ for some $y, z \in Y_n$, then $\mathbf{A}$ is regular and, moreover, $T_zA^{-1}T_y \geq 0$ for each $A \in \mathbf{A}$. In particular, for $y = z = e$ we obtain that if $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$, then $\mathbf{A}$ is regular and $A^{-1} \geq 0$ for each $A \in \mathbf{A}$ (Kuttler [11]); moreover, in this case A^{-1} can be expanded into infinite series as $A^{-1} = (\sum_{j=0}^{\infty}(\overline{A}^{-1}(\overline{A} - A))^j)\overline{A}^{-1}$ ([16], Thm. 2).

6.4. Conditions (xxxv)-(xxxvii). The next three conditions are formulated in terms of properties of certain sets.

(xxxv): The set (4.11) is called the solution set of a system of interval linear equations $\mathbf{A}x = \mathbf{b}$; it is described by the Oettli-Prager theorem [13] as $\mathbf{X}(\mathbf{A}, \mathbf{b}) = \{\, x \mid |A_cx - b_c| \leq \Delta|x| + \delta \,\}$ (see [6], Thm. 2.9). The present condition makes it possible to define the interval hull $\mathbf{x}(\mathbf{A}, \mathbf{b})$ of $\mathbf{X}(\mathbf{A}, \mathbf{b})$ by $\mathbf{x}(\mathbf{A}, \mathbf{b}) = [\min \mathbf{X}(\mathbf{A}, \mathbf{b}), \max \mathbf{X}(\mathbf{A}, \mathbf{b})]$ (componentwise), i.e., as the narrowest interval vector containing it. In [25], p. 38, it is shown that using the matrices Q_y from condition (xxiv), the interval hull can be described explicitly by $\mathbf{x}(\mathbf{A}, \mathbf{b}) = [\min_{y\in Y_n}(Q_yb_c - |Q_y|\delta), \max_{y\in Y_n}(Q_yb_c + |Q_y|\delta)]$, (componentwise), where $\mathbf{b} = [b_c - \delta, b_c + \delta]$. Computing the interval hull is NP-hard [28], therefore the problem of solving interval linear equations is often formulated as that of finding an interval vector $\mathbf{x}$ (not necessarily the optimal one) containing $\mathbf{X}(\mathbf{A}, \mathbf{b})$; such an interval vector is called an enclosure of $\mathbf{X}(\mathbf{A}, \mathbf{b})$. This, however, cannot be seen a universal recipe as an enclosure may essentially overestimate the interval hull, see, e.g., [26].

(xxxvi): If $\mathbf{A}$ is singular, then the solution set $\mathbf{X}(\mathbf{A}, \mathbf{b})$ may be disconnected and consist of several disjoint components (maximal connected subsets with respect to inclusion). One would expect that at least *one* component should be unbounded in this case. An important result proved by Jansson [8] says that in fact *all* of them are

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unbounded. Therefore, it suffices to enclose a single component of $\mathbf{X}(\mathbf{A}, \mathbf{b})$ to get an enclosure of the whole of $\mathbf{X}(\mathbf{A}, \mathbf{b})$ and simultaneously to prove regularity of $\mathbf{A}$. Both known not-a-priori-exponential algorithms for checking regularity [9], [2] employ this idea. The VERSOFT [3] function VERREGSING.M [2], moreover, yields *verified* regularity or singularity.

(xxxvii): This condition was conjectured by K. Rosłaniec and proved in [15]. Indeed, the inclusion in (4.12) implies that $x_1 - x_2$ solves (4.1) and thus brings singularity.

6.5. Conditions (xxxviii)-(xli). The last four conditions are best understood when negated: then they show that a singular interval matrix $\mathbf{A}$ contains singular matrices of several specific forms.

(xxxviii): This condition has a geometric interpretation: if $\mathbf{A}$ is singular, then it contains a singular matrix $A = A_c - dT_y\Delta T_z = (1-d)A_c + d(A_c - T_y\Delta T_z) = (1-d)A_c + dA_{yz}$ which belongs to the segment in $\mathbb{R}^{n\times n}$ connecting the midpoint matrix A_c with a vertex matrix A_{yz}.

(xxxix): A singular matrix of this form is returned in case of singularity by the VERSOFT [3] function VERREGSING.M [2]. It is constructed there from a nonzero solution of the inequality (4.1), see the construction of S in the above comment on condition (ii).

(xl): This condition is most specific, but also most difficult to formulate, and seems to be the least used of the four.

(xli): Here again, we have a geometric interpretation: if $\mathbf{A}$ is singular, then it contains a singular matrix belonging to an edge of $\mathbf{A}$ when considered a rectangle in the $\mathbb{R}^{n^2}$ space. An eigenvalue version of this "edge theorem" was given in [10], Thm. 21.21, and a more general result was achieved by Hollot and Bartlett [7].

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