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ON SPECTRA PERTURBATION AND ELEMENTARY DIVISORS OF POSITIVE MATRICES*

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Abstract. A remarkable result of Guo [Linear Algebra Appl., 266:261–270, 1997] establishes that if the list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of an $n \times n$ nonnegative matrix, where λ_1 is its Perron root and $\lambda_2 \in \mathbb{R}$, then for any $t > 0$, the list $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ is also the spectrum of a nonnegative matrix. In this paper it is shown that if $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, then Guo's result holds for positive stochastic, positive doubly stochastic and positive symmetric matrices. Stochastic and doubly stochastic matrices are also constructed with a given spectrum and with any legitimately prescribed elementary divisors.

Key words. Stochastic matrix, Doubly stochastic matrix, Symmetric matrix, Spectrum perturbation, Elementary divisors.

AMS subject classifications. 15A18.

1. Introduction. The *nonnegative inverse eigenvalue problem* (hereafter *NIEP*) is the problem of determining necessary and sufficient conditions for a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be the spectrum of an $n \times n$ entrywise nonnegative matrix A. If there exists a nonnegative matrix A with spectrum Λ, we say that Λ is realizable and that A is the realizing matrix. For $n \geq 5$ the *NIEP* remains unsolved. When the possible spectrum Λ is a list of real numbers we have the *real nonnegative inverse eigenvalue problem* (*RNIEP*). A number of sufficient conditions or realizability criteria for the existence of a solution for the *RNIEP* have been obtained. For a comparison of these criteria and a comprehensive survey see [1], [7]. If we additionally require that the realizing matrix to be symmetric, we have the *symmetric nonnegative inverse eigenvalue problem* (*SNIEP*). Both problems, *RNIEP* and *SNIEP* are unsolved for $n \geq 5$. They are equivalent for $n \leq 4$ (see [4]), but are different otherwise (see [6]).

One of the most important contributions to the *SNIEP* is due to Fiedler, who proved the following result:

THEOREM 1.1. [2, Theorem 3.2] *If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative symmetric matrix and if $\epsilon > 0$, then $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is the*

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spectrum of a positive symmetric matrix.

Let $A \in \mathbb{C}^{n\times n}$ and let

$$J(A) = S^{-1}AS = \begin{bmatrix} J_{n_1(\lambda_1)} & & & \\ & J_{n_2(\lambda_2)} & & \\ & & \ddots & \\ & & 0 & J_{n_k(\lambda_k)} \end{bmatrix}$$

be the *Jordan canonical form* of A (hereafter JCF of A). The $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, 2, \ldots, k$$

are called the *Jordan blocks* of $J(A)$. Then the *elementary divisors* of A are the polynomials $(\lambda - \lambda_i)^{n_i}$, that is, the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \ldots, k$.

The *inverse elementary divisor problem* (*IEDP*) is the problem of determining necessary and sufficient conditions under which the polynomials $(\lambda-\lambda_1)^{n_1}, (\lambda-\lambda_2)^{n_2}, \ldots, (\lambda-\lambda_k)^{n_k}$, $n_1+\cdots+n_k = n$, are the *elementary divisors* of an $n\times n$ matrix A. In order that the problem be meaningful, the matrix A is required to have a particular structure. When A has to be an entrywise nonnegative matrix, the problem is called the *nonnegative inverse elementary divisor problem* (*NIEDP)* (see [8], [9]). The *NIEDP,* which is also unsolved, contains the *NIEP.*

A matrix $A = (a_{ij})_{i,j=1}^n$ is said to have *constant row sums* if all its rows add up to the same constant α, i.e. $\sum\limits_{j=1}^{n} a_{ij} = \alpha, \;\; i = 1, \ldots, n$. The set of all matrices with constant row sums equal to α is denoted by $\mathcal{CS}_\alpha$. It is clear that $\mathbf{e} = (1, 1, \ldots, 1)^T$ is an eigenvector of any matrix $A \in \mathcal{CS}_\alpha$, corresponding to the eigenvalue α. Denote by $\mathbf{e}_k$ the vector with 1 in the $k-th$ position and zeros elsewhere. A nonnegative matrix A is called stochastic if $A \in \mathcal{CS}_1$ and is called doubly stochastic if $A, A^T \in \mathcal{CS}_1$. If $A \in \mathcal{CS}_\alpha$, we shall write that A is generalized stochastic, while if $A, A^T \in \mathcal{CS}_\alpha$, we shall write that A is generalized doubly stochastic. We denote by E_{ij} the $n\times n$ matrix with 1 on the (i, j) position and zeros elsewhere.

The following result, due to Johnson [5], shows that the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum Λ.

Lemma 1.2. [5] *Any realizable list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is realized in particular by a nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$, where λ_1 is its Perron root.*

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In [3], Guo proves the following outstanding results:

THEOREM 1.3. [3] *If the list of complex numbers* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is realizable, where* λ_1 *is the Perron root and* $\lambda_2 \in \mathbb{R}$*, then for any* $t \geq 0$ *the list* $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ *is also realizable.*

COROLLARY 1.4. [3] *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a realizable list of real numbers. Let* $t_1 = \sum\limits_{i=2}^{n} |t_i|$*, with* $t_i \in \mathbb{R}$*,* $i = 2, \ldots, n$*. Then*

$$\Lambda_{t_i} = \{\lambda_1 + t_1, \lambda_2 + t_2, \ldots, \lambda_n + t_n\}$$

is also realizable.

Moreover Guo [3] sets the following question, which is of our interest in this paper: For any list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ symmetrically realizable, and $t > 0$, whether or not the list $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ is also symmetrically realizable.

In [10], in connection with the *NIEP*, Perfect showed that the matrix

$$A = P\text{diag}\{1, \lambda_2, \lambda_3, \ldots, \lambda_n\}P^{-1},$$

where $1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0$ and

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \ldots & -1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & -1 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{1.1}$$

is an $n \times n$ positive stochastic matrix with spectrum $\{1, \lambda_2, \ldots, \lambda_n\}$. This result was used in [11] to completely solve the *NIEDP* for lists of real numbers $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. That is, for proving the existence of a nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$ with legitimately arbitrarily prescribed elementary divisors. In particular, for stochastic matrices, we have the following result:

THEOREM 1.5. [11] *Let* $\Lambda = \{1, \lambda_2, \ldots, \lambda_n\}$ *with* $1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$*. There exists a stochastic matrix* A *with spectrum* Λ *and arbitrarily prescribed elementary divisors* $(\lambda - 1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}$*,* $n_2 + \cdots + n_k = n - 1$*.*

In this paper we show that $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with

$$\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0,$$

is always the spectrum of a positive stochastic, positive doubly stochastic, and positive symmetric matrix. It is also shown that Λ admits negative numbers. Moreover, we

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show that the Guo result holds for this kind of matrices, that is, $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ is also the spectrum of a positive stochastic, positive doubly stochastic, and positive symmetric matrix. The examples in section 6 show that our results are useful in the *NIEP* to decide if a given list Λ (including negative numbers) is realizable by this kind of matrices.

The paper is organized as follows: In section 2 we show how to construct positive stochastic and positive doubly stochastic matrices with a given spectrum and with arbitrarily prescribed elementary divisors. In particular, for the stochastic case, we improve, to a certain degree, the result of Theorem 1.5. The doubly stochastic case has its merit in the construction of the matrix itself. In sections 3 and 4 we prove that Theorem 1.3 holds, respectively, for positive generalized stochastic and positive generalized doubly stochastic matrices with prescribed spectrum. In section 5 we show that a list of nonnegative real numbers is always the spectrum of a positive symmetric matrix and that $\{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$, $t > 0$, is also the spectrum of a positive symmetric matrix. Finally, in section 6 we introduce examples, which show that our results are useful to decide if a given list Λ is realizable by this kind of matrices.

2. Stochastic and doubly stochastic matrices with prescribed elementary divisors. In this section we prove that Theorem 1.5 still holds for a list of real numbers $\Lambda = \{1, \lambda_1, \ldots, \lambda_{n-1}\}$ containing negative numbers. For that, let us express the matrix P given in (1.1) as:

$$P = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$$

with rows

$$F_1 \quad = (1, 1, \ldots, 1)$$

$$F_{k+2} = \Bigg(\underbrace{1, \ldots, 1,}_{n-k-1 \text{ ones}} \; -1, \underbrace{0, \ldots, 0}_{k \text{ zeros}}\Bigg), \; k = 0, 1, ..., n-2.$$

and columns

$$C_1^T = \left(\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}, \frac{1}{2^{n-2}}, \frac{1}{2^{n-3}}, \ldots, \frac{1}{2^2}, \frac{1}{2}\right)$$

$$C_j^T = \left(\frac{1}{2^{n-(j-1)}}, \frac{1}{2^{n-(j-1)}}, \frac{1}{2^{n-j}}, \frac{1}{2^{n-(j+1)}}, \ldots, \frac{1}{2^2}, -\frac{1}{2}, \underbrace{0, \ldots, 0}_{j-2 \text{ zeros}}\right),$$

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$j = 2, \ldots, n$. Then the following Lemma is straightforward.

LEMMA 2.1. *Let* $D = \text{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\} \subset \mathbb{R}$. *The entries of the matrix* $B = (b_{ij}) = PDP^{-1}$ *satisfy the following relations:*

$$\begin{aligned} &b_{11} = b_{22} \\ &b_{12} = b_{21} \\ &b_{(k+2)1} = b_{(k+2)2}, \ 1 \leq k \leq n-2 \end{aligned}$$

$$b_{1(k+2)} = b_{2(k+2)} = \cdots = b_{(k+1)(k+2)}, \ 1 \leq k \leq n-2$$

$$b_{(k+2)j} = \tfrac{1}{2^k} b_{1(k+2)}, \ j = 1, 2, \quad 1 \leq k \leq n-2$$

$$b_{(k+2)j} = \tfrac{1}{2^{k-(j-2)}} b_{1(k+2)}, \ 2 \leq k \leq n-2, \quad 3 \leq j \leq k+1.$$

LEMMA 2.2. *If* $\Lambda = \{1, \lambda_1, \lambda_2, ..., \lambda_{n-1}\} \subset \mathbb{R}$ *satisfies*

$$|\lambda_r| < \frac{1}{2^{r-1}} \left(1 + \sum_{p=1}^{r-1} 2^{p-1} \lambda_p \right), \ r = 1, \ldots, n-1; \tag{2.1}$$

with $\sum_{p=1}^{0} 2^{p-1}\lambda_p = 0$, *then* $B = PDP^{-1}$, *where* $D = \text{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\}$, *is a positive stochastic diagonalizable matrix with spectrum* Λ.

Proof. It is easy to see that

$$1 + \sum_{p=1}^{r-1} 2^{p-1} \lambda_p > 0.$$

Since $PDP^{-1}\mathbf{e} = PD\mathbf{e}_1 = P\mathbf{e}_1 = \mathbf{e}$, then B is quasi-stochastic. Hence, it only remains to show that B is positive. From Lemma 2.1, we only need to prove the positivity of

$$b_{1(k+2)} \ \text{ and } \ b_{(k+2)(k+2)}, \quad k = 0, 1, \ldots, n-2.$$

Observe that for $k = 0, 1, \ldots, n-2$, the condition (2.1) is equivalent to

$$|\lambda_{n-k-1}| < \frac{1}{2^{n-k-2}} \left(1 + \sum_{p=1}^{n-k-2} 2^{p-1} \lambda_p \right)$$

$$\Updownarrow$$

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$$-\frac{1}{2^{n-k-2}}\left(1+\sum_{p=1}^{n-k-2}2^{p-1}\lambda_p\right)<\lambda_{n-k-1}<\frac{1}{2^{n-k-2}}\left(1+\sum_{p=1}^{n-k-2}2^{p-1}\lambda_p\right)$$

$$\Updownarrow$$

$$\frac{1}{2^{n-k-2}}+\sum_{p=1}^{n-k-2}\frac{\lambda_p}{2^{n-k-p-1}}+\lambda_{n-k-1}>0 \quad \text{and}$$

$$\frac{1}{2^{n-k-2}}+\sum_{p=1}^{n-k-2}\frac{\lambda_p}{2^{n-k-p-1}}-\lambda_{n-k-1}>0$$

$$\Updownarrow$$

$$\frac{1}{2^{n-k-1}}+\sum_{p=1}^{n-k-2}\frac{\lambda_p}{2^{n-k-p}}+\frac{\lambda_{n-k-1}}{2}>0 \quad \text{and}$$

$$\frac{1}{2^{n-k-1}}+\sum_{p=1}^{n-k-2}\frac{\lambda_p}{2^{n-k-p}}-\frac{\lambda_{n-k-1}}{2}>0$$

$$\Updownarrow$$

$$\begin{aligned} b_{(k+2)(k+2)} &= F_{k+2}DC_{k+2} > 0,\\ b_{1(k+2)} &= F_1DC_{k+2} \quad > 0. \end{aligned}$$

Thus, from Lemma 2.1 all entries b_{ij} are positive and B is a positive stochastic diagonalizable matrix with spectrum Λ. □

The following result shows that Lemma 2.2 contains the result of Perfect [10, Theorem 1] mentioned in section 1, and the inclusion is strict.

PROPOSITION 2.3. *If* $1>\lambda_1\geq\cdots\geq\lambda_{n-1}\geq 0$, *then*

$$|\lambda_r|<\frac{1}{2^{r-1}}\left(1+\sum_{p=1}^{r-1}2^{p-1}\lambda_p\right), \quad r=1,2,\ldots,n-1.$$

Proof. For $r=1,2,\ldots,n-1$, we have

$$\begin{aligned} &\lambda_r<1,\\ &\lambda_r\leq\lambda_p, \quad p=1,2,...,r-1. \end{aligned}$$

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Then, by adding the inequalities

$$\begin{gathered}\lambda_r < 1, \\ 2^{p-1}\lambda_r \leq 2^{p-1}\lambda_p, \quad p = 1, 2, ..., r-1.\end{gathered}$$

we obtain

$$\begin{aligned}(1+1+2+2^2+2^3+\cdots+2^{r-2})\lambda_r &< 1+\sum_{p=1}^{r-1} 2^{p-1}\lambda_p \\ (1+(2^{r-1}-1))\lambda_r &< 1+\sum_{p=1}^{r-1} 2^{p-1}\lambda_p \\ 2^{r-1}\lambda_r &< 1+\sum_{p=1}^{r-1} 2^{p-1}\lambda_p \\ \lambda_r &< \frac{1}{2^{r-1}}\left(1+\sum_{p=1}^{r-1} 2^{p-1}\lambda_p\right).\end{aligned}$$

Since $\lambda_r \geq 0$, then the result follows. $\square$

Theorem 2.4. *If* $\Lambda = \{1, \lambda_1, \ldots, \lambda_{n-1}\} \subset \mathbb{R}$ *satisfies*

$$|\lambda_r| < \frac{1}{2^{r-1}}\left(1+\sum_{p=1}^{r-1} 2^{p-1}\lambda_p\right), \quad r = 1, 2, \ldots, n-1,$$

with $\sum_{p=1}^{0} 2^{p-1}\lambda_p = 0$, *then there exists an* $n \times n$ *positive stochastic matrix* A *with spectrum* Λ *and with arbitrarily prescribed elementary divisors.*

Proof. Let $D = \operatorname{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\}$. From Lemma 2.2 there exists an $n \times n$ positive stochastic matrix $B = PDP^{-1}$ with spectrum Λ and linear elementary divisors $(\lambda - 1), \ldots, (\lambda - \lambda_{n-1})$. Let $K \subset \{2, 3, \ldots, n-1\}$ and $C = \sum_{t \in K} E_{t,t+1}$, in such a way that $D + C$ is the desired JCF. Then

$$A = PDP^{-1} + \epsilon PCP^{-1},$$

where $\epsilon > 0$ is such that $(PDP^{-1})_{ij} + \epsilon(PCP^{-1})_{ij} > 0$, $i, j = 1, \ldots, n$, is positive with spectrum Λ, and since $D + \epsilon C$ and $D + C$ are diagonally similar (with $\operatorname{diag}\{1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}\}$), then A has JCF equal to $D + C$. Since $P\mathbf{e}_1 = \mathbf{e}$ and $P^{-1}\mathbf{e} = \mathbf{e}_1$ then $PCP^{-1}\mathbf{e} = \mathbf{0}$ and $A\mathbf{e} = \mathbf{e}$. Thus $A \in \mathcal{CS}_1$ and A has the desired elementary divisors. $\square$

Remark 2.5. We observe that the real numbers $1, \lambda_1, \ldots, \lambda_{n-1}$ are considered not ordered. This allows us, if the list $1 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$ does not satisfy the

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condition (2.1) of Lemma 2.2, to arrange the list in such a way that, eventually, it satisfies the condition.

We now construct a positive doubly stochastic matrix A, with prescribed real spectrum and arbitrarily prescribed elementary divisors. Consider the $n \times n$ nonsingular matrix

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \cdots & -2 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & -(n-2) & \cdots & 0 & 0 \\ 1 & -(n-1) & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.2}$$

Then

$$R = \begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} \quad \text{and} \quad R^{-1} = \begin{bmatrix} H_1 & \cdots & H_n \end{bmatrix}$$

have, for $k = 2, \ldots, n$, respectively, rows

$$\left.\begin{aligned} G_1 &= (1, 1, \ldots, 1) \\ G_k &= \Bigg(\underbrace{1, 1, \ldots, 1}_{n-(k-1)\text{ ones}}, -(k-1), \underbrace{0, \ldots, 0}_{k-2\text{ zeros}}\Bigg) \end{aligned}\right\} \tag{2.3}$$

and columns

$$\left.\begin{aligned} H_1^T &= \left(\tfrac{1}{n}, \tfrac{1}{n(n-1)}, \tfrac{1}{(n-1)(n-2)}, \ldots, \tfrac{1}{3*2}, \tfrac{1}{2*1}\right) \\ H_k^T &= \Bigg(\tfrac{1}{n}, \tfrac{1}{n(n-1)}, \tfrac{1}{(n-1)(n-2)}, \ldots, \tfrac{1}{(k+1)k}, -\tfrac{1}{k}, \underbrace{0, \ldots, 0}_{k-2\text{ zeros}}\Bigg) \end{aligned}\right\}. \tag{2.4}$$

Then the following Lemma is straightforward:

Lemma 2.6. *Let* $D = \operatorname{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\} \subset \mathbb{R}$. *Then the entries of the matrix* $B = RDR^{-1} = (b_{ij})$ *satisfy the following relations*

$$\left.\begin{aligned} b_{11} &= b_{22} \\ b_{12} &= b_{21} \\ b_{k1} &= b_{k2} = \cdots = b_{k(k-1)} = b_{1k} = b_{2k} = \cdots = b_{(k-1)k}, \end{aligned}\right\}$$

$k = 3, \ldots, n.$

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The following result gives a sufficient condition for the existence of a positive symmetric doubly stochastic matrix with prescribed spectrum.

LEMMA 2.7. *If* $\Lambda = \{1, \lambda_1, \ldots, \lambda_{n-1}\} \subset \mathbb{R}$ *satisfies*

$$-\frac{n-r+1}{n-r} S_{r-1}(\lambda_p) < \lambda_r < (n-r+1) S_{r-1}(\lambda_p), \tag{2.5}$$

where

$$S_{r-1}(\lambda_p) = \left(\frac{1}{n} + \sum_{p=1}^{r-1} \frac{\lambda_p}{(n-(p-1))(n-p)} \right),$$

$r = 1, 2, \ldots, n-1$, *with* $S_0(\lambda_p) = \frac{1}{n}$, *then* $B = RDR^{-1}$, *where*

$$D = \operatorname{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\},$$

is a positive symmetric doubly stochastic matrix with spectrum Λ.

Proof. Since $R\mathbf{e}_1 = \mathbf{e}$ and $R^{-1}\mathbf{e} = \mathbf{e}_1$ then $B\mathbf{e} = RDR^{-1}\mathbf{e} = \mathbf{e}$. Moreover $R^T\mathbf{e} = n\mathbf{e}_1$ and $B^T\mathbf{e} = (RDR^{-1})^T\mathbf{e} = \mathbf{e}$. So, B is a doubly quasi-stochastic matrix. In addition, from Lemma 2.6, B is also symmetric. To show the positivity of B we only need to prove that $b_{1k} > 0$, $b_{kk} > 0$, $k = 2, 3, \ldots, n$. Observe that (2.5) is equivalent to

$$-\frac{k}{k-1} S_{n-k}(\lambda_p) < \lambda_{n-k+1} < k S_{n-k}(\lambda_p), \ k = 2, 3, \ldots, n.$$

$$\Updownarrow$$

$$\frac{1}{n} + \sum_{p=1}^{n-k} \frac{\lambda_p}{(n-(p-1))(n-p)} + \frac{k-1}{k} \lambda_{n-k+1} > 0 \tag{2.6}$$

and

$$\frac{1}{n} + \sum_{p=1}^{n-k} \frac{\lambda_p}{(n-(p-1))(n-p)} - \frac{\lambda_{n-k+1}}{k} > 0, \tag{2.7}$$

$k = 2, 3, \ldots, n$. It is clear, from (2.3) and (2.4), that $b_{kk} = G_k D H_k$ is the left side of (2.6), while $b_{1k} = G_1 D H_k$ is the left side of (2.7), $k = 2, 3, \ldots n$. Hence, the result follows. ◻

THEOREM 2.8. *If* $\Lambda = \{1, \lambda_1, \ldots, \lambda_{n-1}\} \subset \mathbb{R}$ *satisfies*

$$-\frac{n-r+1}{n-r} S_{r-1}(\lambda_p) < \lambda_r < (n-r+1) S_{r-1}(\lambda_p), \tag{2.8}$$

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where

$$S_{r-1}(\lambda_p) = \left(\frac{1}{n} + \sum_{p=1}^{r-1} \frac{\lambda_p}{(n-(p-1))(n-p)} \right),$$

$r = 1, 2, \ldots, n-1$, *with* $S_0(\lambda_p) = \frac{1}{n}$, *then there exists a positive doubly stochastic matrix with spectrum* Λ *and with arbitrarily prescribed elementary divisors.*

Proof. Let $D = \operatorname{diag}\{1, \lambda_1, \ldots, \lambda_{n-1}\}$ and let R be the nonsingular matrix in (2.2). Then from Lemma 2.7, $B = RDR^{-1}$ is a positive symmetric doubly stochastic matrix with spectrum Λ. Now, from a result of Minc [9, Theorem 4], there exists an $n \times n$ positive doubly stochastic matrix A with spectrum Λ and with arbitrarily prescribed elementary divisors. □

Proposition 2.9. *If* $1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0$, *then* (2.8) *is satisfied.*

Proof. The proof is similar to the proof of Proposition 2.3. □

We observe that the real numbers $1, \lambda_1, \ldots, \lambda_{n-1}$ need not to be ordered. By a straightforward calculation we may prove the following result, which shows that Theorem 2.8 contains Theorem 2 in [9].

Corollary 2.10. *Let* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$. *Then for* $r = 1, 2, \ldots, n-1$, *equation* (2.8) *is equivalent to*

$$-\frac{1}{n-1} < \lambda_r < 1.$$

Corollary 2.11. *The list* $\Lambda = \{1, \alpha, \alpha, \ldots, \alpha\}$, $\alpha \in \mathbb{R}$, *is the spectrum of an* $n \times n$ *positive symmetric doubly stochastic matrix if and only if* $-\frac{1}{n-1} < \alpha < 1$.

Proof. Let A be an $n \times n$ positive symmetric doubly stochastic matrix with spectrum Λ. Then $1 + (n-1)\alpha > 0$ and $-\frac{1}{n-1} < \alpha$ with $|\alpha| < 1$. Therefore, $-\frac{1}{n-1} < \alpha < 1$. Now, suppose $-\frac{1}{n-1} < \alpha < 1$. Then from Corollary 2.10 and Lemma 2.7, Λ is the spectrum of an $n \times n$ positive symmetric doubly stochastic matrix. □

3. Guo perturbations for positive generalized stochastic matrices. In this section we show that if a list of real numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_1 > 0$, is the spectrum of a positive generalized stochastic matrix A, then the modified list $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$, $t \geq 0$, is also the spectrum of a positive generalized stochastic matrix with arbitrarily prescribed elementary divisors.

Remark 3.1. It is clear that a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, whose normalized version $\Lambda' = \{1, \lambda_1', \ldots, \lambda_{n-1}'\}$, with $\lambda_i' = \frac{1}{\lambda_1}\lambda_{i+1}$, $i = 1, \ldots, n-1$, satisfies the con-

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dition (2.1) of Lemma 2.2, is always the spectrum of a positive generalized stochastic matrix.

Here, our interest is to show an easy way to construct directly this kind of matrices. We shall need the following result given in [11]:

THEOREM 3.2. [11] *Let $A \in \mathcal{CS}_{\lambda_1}$ be an $n \times n$ diagonalizable positive matrix with complex spectrum. Then there exists an $n \times n$ positive matrix $B \in \mathcal{CS}_{\lambda_1}$, with the same spectrum as A and with prescribed elementary divisors.*

THEOREM 3.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of real numbers, whose normalized version satisfies the condition* (2.1) *of* Lemma 2.2 *(in particular, Λ satisfying $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0$). Then for every $t > 0$, the list $\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ is the spectrum of a positive generalized stochastic matrix B, with arbitrarily prescribed elementary divisors.*

Proof. Let P be the matrix given in (1.1) and let $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then from Lemma 2.2 and Remark 3.1, $A = PDP^{-1} \in \mathcal{CS}_{\lambda_1}$ is positive with spectrum Λ. For

$$\begin{aligned} C &= tE_{11} - tE_{22} \\ &= \begin{bmatrix} t & & & & \\ & -t & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \quad t > 0, \end{aligned}$$

we have

$$PCP^{-1} = \begin{bmatrix} & & & & & & t \\ & & & & & & t \\ & & & & & & \vdots \\ & & & & & & \vdots \\ & & & & & & t \\ & & & & & & t \\ \frac{1}{2^{n-2}}t & \frac{1}{2^{n-2}}t & \frac{1}{2^{n-3}}t & \cdots\cdots & \frac{1}{4}t & \frac{1}{2}t & 0 \end{bmatrix}.$$

Since

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-2}} = 1 - \frac{1}{2^{n-2}},$$

then the last row of PCP^{-1} sums to t, and $PCP^{-1} \in \mathcal{CS}_t$. Similarly, for $C = tE_{11} + tE_{22}$, $PCP^{-1} \in \mathcal{CS}_t$ and it is nonnegative. Let $B' = A + PCP^{-1}$. Then

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B' is positive, $B' \in \mathcal{CS}_{\lambda_1+t}$ and B' has JCF

$$P^{-1}B'P = P^{-1}AP + C = D + C.$$

Then, from Theorem 3.2 there exists a positive generalized stochastic matrix B with spectrum Λ_t and with arbitrarily prescribed elementary divisors. $\square$

4. Guo perturbations for positive generalized doubly stochastic matrices. In this section we prove alternative results to Theorem 1.3 and Corollary 1.4, for positive symmetric generalized doubly stochastic matrices. These results are useful to decide the realizability of a list Λ by this kind of matrices.

Remark 4.1. It is clear that a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, whose normalized version $\Lambda' = \{1, \lambda'_1, \ldots, \lambda'_{n-1}\}$, with $\lambda'_i = \frac{1}{\lambda_1}\lambda_{i+1}, \;\; i = 1, \ldots, n-1$, satisfies the condition (2.5) of Lemma 2.7 (in particular, Λ satisfying $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0$), is always the spectrum of a positive symmetric generalized doubly stochastic matrix.

We shall need the following result given in [9]:

Theorem 4.2. [9] *Let A be an $n \times n$ diagonalizable, positive doubly stochastic, matrix with real eigenvalues. Then there exists a positive doubly stochastic matrix with the same spectrum as A and any prescribed elementary divisors.*

Theorem 4.3. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of real numbers, whose normalized version satisfies the condition* (2.5) *of* Lemma 2.7 *(in particular, Λ satisfying $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0$). Then for*

$$0 < t < \frac{1}{n-2}(\lambda_1 + (n-1)\lambda_2), \tag{4.1}$$

the list $\Lambda_t^- = \{\lambda_1 + t, \lambda_2 - t, \lambda_3, \ldots, \lambda_n\}$ is the spectrum of a positive symmetric generalized doubly stochastic matrix, while for all

$$t > 0, \text{ the list } \Lambda_t^+ = \{\lambda_1 + t, \lambda_2 + t, \lambda_3, \ldots, \lambda_n\}$$

is also the spectrum of a positive symmetric generalized doubly stochastic matrix. In both cases, Λ_t^- and Λ_t^+ are also the spectrum of a positive generalized doubly stochastic matrix with arbitrarily prescribed elementary divisors.

Proof. Let $D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and let R be the nonsingular matrix given in (2.2). From Remark 4.1 and Lemma 2.7, there exists a positive symmetric generalized doubly stochastic matrix $A = RDR^{-1}$ $(A, A^T \in \mathcal{CS}_{\lambda_1})$. Now, we pay attention to the

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entry in position (n, n) of RDR^{-1}, which is

$$G_n D H_n = (1, -(n-1), 0, \ldots, 0) \begin{pmatrix} \frac{1}{n}\lambda_1 \\ -\frac{1}{n}\lambda_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\lambda_1 + (n-1)\lambda_2}{n}.$$

Let $C = tE_{11} - tE_{22}$, $t > 0$. Then,

$$RCR^{-1} = \begin{bmatrix} \frac{(n-2)t}{n(n-1)} & \cdots & \frac{(n-2)t}{n(n-1)} & \frac{2t}{n} \\ \frac{(n-2)t}{n(n-1)} & \cdots & \frac{(n-2)t}{n(n-1)} & \frac{2t}{n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(n-2)t}{n(n-1)} & \cdots & \frac{(n-2)t}{n(n-1)} & \frac{2t}{n} \\ \frac{2t}{n} & \cdots & \frac{2t}{n} & \frac{-(n-2)t}{n} \end{bmatrix}.$$

Since $RCR^{-1}\mathbf{e} = (RCR^{-1})^T\mathbf{e} = t\mathbf{e}$, then RCR^{-1} is an $n \times n$ symmetric generalized doubly stochastic matrix. Observe that RCR^{-1} has all its entries positive, except the entry in position (n, n), which is $-\frac{n-2}{n}t$. Then, from (4.1) we have

$$\frac{\lambda_1 + (n-1)\lambda_2}{n} - \frac{n-2}{n}t > 0$$

and the matrix

$$B' = RDR^{-1} + RCR^{-1}$$

is positive symmetric generalized doubly stochastic with Jordan canonical form

$$R^{-1}B'R = D + C.$$

Thus B' has spectrum Λ_t^-. For $C = tE_{11} + tE_{22}$, $t > 0$, we have that

$$RCR^{-1} = \frac{t}{n-1}\mathbf{e}\mathbf{e}^T \oplus t,$$

where $\mathbf{e} \in \mathbb{R}^{n-1}$, is an $n \times n$ nonnegative generalized doubly stochastic matrix and $B' = RDR^{-1} + RCR^{-1}$ is positive symmetric generalized doubly stochastic with Jordan canonical form $D + C$. Thus B' has spectrum Λ_t^+. Then, in both cases, from Theorem 4.2, there exists a positive generalized doubly stochastic matrix B with spectrum Λ_t^- (Λ_t^+) and with arbitrarily prescribed elementary divisors. $\square$

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COROLLARY 4.4. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a list of real numbers, whose normalized version satisfies the condition* (2.5) *of* Lemma 2.7 *(in particular, Λ satisfying $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq 0$). Then for every $t > 0$, the list*

$$\Lambda_t = \{\lambda_1 + (n-1)t, \lambda_2 - t, \lambda_3 - t, \dots, \lambda_n - t\}$$

is the spectrum of a positive symmetric generalized doubly stochastic matrix, and Λ_t is also the spectrum of a positive generalized doubly stochastic matrix with arbitrarily prescribed elementary divisors.

Proof. Let $D = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and let R be the nonsingular matrix given in (2.2). Then, from Remark 4.1 and Lemma 2.7, $A = RDR^{-1}$ is a positive symmetric generalized doubly stochastic matrix in $\mathcal{CS}_{\lambda_1}$. Let

$$C = \begin{bmatrix} (n-1)t & & & & \\ & -t & & & \\ & & -t & & \\ & & & \ddots & \\ & & & & -t \end{bmatrix}, \quad t > 0.$$

From Lemma 2.6, to compute the matrix RCR^{-1} we only need to compute the entries in positions $(1,k)$ and (k,k), $k = 2, \dots, n$. That is,

$$(RCR^{-1})_{1k} = G_1 C H_k = \frac{n-1}{n}t + \left(\frac{1}{n} - \frac{1}{k}\right)t + \frac{1}{k}t = t$$

and

$$(RCR^{-1})_{kk} = G_k C H_k = \frac{n-1}{n}t + \left(\frac{1}{n} - \frac{1}{k}\right)t - \frac{k-1}{k}t = 0,$$

Then,

$$RCR^{-1} = t\mathbf{e}\mathbf{e}^T - tI.$$

Thus, RCR^{-1} is a nonnegative symmetric generalized doubly stochastic matrix and $B' = RDR^{-1} + RCR^{-1}$ is positive symmetric generalized doubly stochastic matrix with JCF equal to $D+C$, and hence B' has the spectrum Λ_t. Moreover, from Theorem 4.2, there exists a positive generalized doubly stochastic B, with spectrum Λ_t and with arbitrarily prescribed elementary divisors. ∎

5. Guo perturbations for positive symmetric matrices. In this section we answer the question of Guo, mentioned in section 1, for positive symmetric matrices with nonnegative spectrum. That is, if the nonnegative list $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is

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realized by a positive symmetric matrix, then $\{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \dots, \lambda_n\}$, $t > 0$, is also realized by a positive symmetric matrix.

THEOREM 5.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a list of real numbers with $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$, $\lambda_1 > |\lambda_i|$, $i = 2, \dots, n$. If there exists a partition $\Lambda = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_{\frac{n}{2}}$ if n is even, or $\Lambda = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_{\frac{n+1}{2}}$ if n is odd, with*

$$\Lambda_1 = \{\lambda_1, \lambda_2\}, \quad \Lambda_k = \{\lambda_{k1}, \lambda_{k2}\}, \quad \lambda_{k1} \pm \lambda_{k2} \geq 0,$$

$k = 2, 3, \dots, \frac{n}{2}$ $(\frac{n+1}{2})$, then Λ is the spectrum of an $n \times n$ positive symmetric matrix A. Besides,

$$\Lambda_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \dots, \lambda_n\}, \quad t > 0$$

is also the spectrum of a positive symmetric matrix.

Proof. Consider the auxiliary list $\Gamma_t^- = \{\mu + t, \lambda_2 - t, \lambda_3, \dots, \lambda_n\}$, where $\mu = \lambda_1 - \epsilon$ for $0 < \epsilon \leq \lambda_1 - \max_{2 \leq i \leq n}\{|\lambda_i|\}$. The matrices

$$A_1 = \begin{bmatrix} \frac{\mu+\lambda_2}{2} & \frac{\mu-\lambda_2+2t}{2} \\ \frac{\mu-\lambda_2+2t}{2} & \frac{\mu+\lambda_2}{2} \end{bmatrix} \text{ and } A_k = \begin{bmatrix} \frac{\lambda_{k1}+\lambda_{k2}}{2} & \frac{\lambda_{k1}-\lambda_{k2}}{2} \\ \frac{\lambda_{k1}-\lambda_{k2}}{2} & \frac{\lambda_{k1}+\lambda_{k2}}{2} \end{bmatrix},$$

with $A_{\frac{n+1}{2}} = \left[\lambda_{\frac{n+1}{2}1}\right]$ if n is odd, are nonnegative symmetric with spectrum $\Lambda_1' = \{\mu + t, \lambda_2 - t\}$ and Λ_k, $k = 2, 3, \dots, \frac{n}{2}$ $(\frac{n+1}{2})$, respectively. Then the $n \times n$ matrix

$$B = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{\frac{n}{2}} \end{bmatrix} \tag{5.1}$$

if n is even, or

$$B' = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{\frac{n+1}{2}} \end{bmatrix}$$

if n is odd, is nonnegative symmetric with spectrum Γ_t. It is clear that $\Gamma_t^+ = \{\mu + t, \lambda_2 + t, \lambda_3, \dots, \lambda_n\}$ is also the spectrum of a nonnegative symmetric matrix. Now, from Theorem 1.1 we have that

$$\Lambda_t = \{\mu + t + \epsilon, \lambda_2 \pm t, \lambda_3, \dots, \lambda_n\} = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \dots, \lambda_n\}$$

is the spectrum of a positive symmetric matrix. For $t = 0$, $\Lambda_t = \Lambda$. $\square$

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The following Corollary is straightforward.

Corollary 5.2. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of real numbers with* $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$. *If there exists a partition* $\Lambda = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_{\frac{n}{2}}$ *if* n *is even, or* $\Lambda = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_{\frac{n+1}{2}}$ *if* n *is odd, with*

$$\Lambda_k = \{\lambda_{k1}, \lambda_{k2}\}, \quad \lambda_{k1} \pm \lambda_{k2} \geq 0, \ k = 1, 2, \ldots, \frac{n}{2} \ (\frac{n+1}{2}),$$

then Λ *is the spectrum of an* $n \times n$ *positive symmetric matrix* A.

Corollary 5.3. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of nonnegative real numbers with* $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ *and let* $t_1 \geq t_2 \geq \cdots t_{\frac{n}{2}} (t_{\frac{n+1}{2}}) \geq 0$. *The list*

$$\Lambda_t = \{\lambda_1 + t_1, \lambda_2 - t_1, \lambda_3 + t_2, \lambda_4 - t_2, \ldots, \lambda_{n-1} + t_{\frac{n}{2}}, \lambda_n - t_{\frac{n}{2}}\},$$

if n *is even, or the list*

$$\Lambda_t = \{\lambda_1 + t_1, \lambda_2 - t_1, \lambda_3 + t_2, \lambda_4 - t_2, \ldots, \lambda_{n-2} + t_{\frac{n-1}{2}}, \lambda_{n-1} - t_{\frac{n-1}{2}}, \lambda_n + t_{\frac{n+1}{2}}\}$$

if n *is odd, is the spectrum of a positive symmetric matrix.*

Proof. Consider the auxiliary list

$$\Gamma_t = \{\lambda_2 + t_1, \lambda_2 - t_1, \lambda_3 + t_2, \lambda_4 - t_2, \ldots, \lambda_{n-1} + t_{\frac{n}{2}}, \lambda_n - t_{\frac{n}{2}}\}$$

with

$$\Gamma_t^{'} = \{\lambda_2 + t_1, \lambda_2 - t_1, \ldots, \lambda_{n-2} + t_{\frac{n-1}{2}}, \lambda_{n-1} - t_{\frac{n-1}{2}}, \lambda_n + t_{\frac{n+1}{2}}\}$$

for odd n. The $n \times n$ matrix

$$B = \begin{bmatrix} \lambda_2 & t_1 & & & & & \\ t_1 & \lambda_2 & & & & & \\ & & \frac{\lambda_3+\lambda_4}{2} & \frac{\lambda_3-\lambda_4}{2} + t_2 & & & \\ & & \frac{\lambda_3-\lambda_4}{2} + t_2 & \frac{\lambda_3+\lambda_4}{2} & & & \\ & & & & \ddots & & \\ & & & & & \frac{\lambda_{n-1}+\lambda_n}{2} & \frac{\lambda_{n-1}-\lambda_n}{2} + t_{\frac{n}{2}} \\ & & & & & \frac{\lambda_{n-1}-\lambda_n}{2} + t_{\frac{n}{2}} & \frac{\lambda_{n-1}+\lambda_n}{2} \end{bmatrix}$$

if n is even, or

$$B' = \begin{bmatrix} \lambda_2 & t_1 & & & & \\ t_1 & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \frac{\lambda_{n-2}+\lambda_{n-1}}{2} & \frac{\lambda_{n-2}-\lambda_{n-1}}{2} + t_{\frac{n-1}{2}} & \\ & & & \frac{\lambda_{n-2}-\lambda_{n-1}}{2} + t_{\frac{n-1}{2}} & \frac{\lambda_{n-2}+\lambda_{n-1}}{2} & \\ & & & & & \lambda_n + t_{\frac{n+1}{2}} \end{bmatrix}$$

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if n is odd, is nonnegative symmetric with spectrum Γ_t. Thus, from Theorem 1.1, for $\epsilon = \lambda_1 - \lambda_2$,

$$\Lambda_t = \{\lambda_2 + t_1 + \epsilon, \lambda_2 - t_1, \lambda_3 + t_2, \ldots, \lambda_n - t_{\frac{n}{2}}\}$$

$$= \{\lambda_1 + t_1, \lambda_2 - t_1, \lambda_3 + t_2, \ldots, \lambda_n - t_{\frac{n}{2}}\}$$

if n is even, or

$$\Lambda_t = \{\lambda_1 + t_1, \lambda_2 - t_1, \lambda_3 + t_2, \ldots, \lambda_n + t_{\frac{n+1}{2}}\}$$

is n is odd, is the spectrum of a positive symmetric matrix. □

Remark 5.4. Our results are useful in the *SNIEP* to decide the realizability of a given list Λ of real numbers (including negative numbers) by a positive symmetric matrix (see examples 6.2 and 6.3). Moreover, we always may easily construct a realizing matrix.

Theorem 5.5. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{R}$, with $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$, be the spectrum of an $n \times n$ positive symmetric matrix A. Then for all $t > 0$, there exists an $\epsilon > 0$, such that $\Gamma = \{\lambda_1 + \epsilon t, \lambda_2 \pm \epsilon t, \lambda_3, \ldots, \lambda_n\}$ is also the spectrum of a positive symmetric matrix.*

Proof. Since A is symmetric, there exists an orthogonal matrix Q such that

$$Q^T A Q = D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

Let $C = tE_{11} \pm tE_{22}, \quad t > 0$. Clearly, QCQ^T is a real symmetric matrix. Let $B = A + \epsilon QCQ^T$, with $\epsilon > 0$ small enough in such a way that B is positive. B is also symmetric and it has JCF equal to $J(B) = D + \epsilon C$. Therefore, B is positive symmetric with spectrum Γ. □

6. Examples.

Example 6.1. The list

$$\Lambda = \left\{1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{8}, -\frac{1}{8}\right\}$$

satisfies conditions of Theorem 2.8. Let $D = \text{diag}\left\{1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{8}, -\frac{1}{8}\right\}$ and let R be the 5×5 matrix given in (2.2). Then we may construct the following positive doubly

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stochastic matrices:

$$i) \quad B = RDR^{-1} = \begin{bmatrix} \frac{11}{60} & \frac{37}{120} & \frac{37}{120} & \frac{1}{10} & \frac{1}{10} \\ \frac{37}{120} & \frac{11}{60} & \frac{37}{120} & \frac{1}{10} & \frac{1}{10} \\ \frac{37}{120} & \frac{37}{120} & \frac{11}{60} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{3}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{3}{5} \end{bmatrix}$$

with spectrum Λ and linear elementary divisors

$$(\lambda - 1), \left(\lambda - \frac{1}{2}\right), \left(\lambda - \frac{1}{2}\right), \left(\lambda + \frac{1}{8}\right), \left(\lambda + \frac{1}{8}\right).$$

$$ii) \quad A_1 = B + \frac{1}{10} RE_{23}R^{-1}$$

with spectrum Λ and elementary divisors

$$(\lambda - 1), \left(\lambda - \frac{1}{2}\right)^2, \left(\lambda + \frac{1}{8}\right), \left(\lambda + \frac{1}{8}\right).$$

$$iii) \quad A_2 = B + \frac{1}{100} RE_{45}R^{-1}$$

with spectrum Λ and elementary divisors $(\lambda - 1), \left(\lambda - \frac{1}{2}\right), \left(\lambda - \frac{1}{2}\right), \left(\lambda + \frac{1}{8}\right)^2$.

$$iv) \quad A_3 = B + \frac{1}{100} R\left(E_{23} + E_{45}\right)R^{-1}$$

with spectrum Λ and elementary divisors $(\lambda - 1), \left(\lambda - \frac{1}{2}\right)^2, \left(\lambda + \frac{1}{8}\right)^2$.

The following examples show that our results are useful in the *NIEP* to decide the realizability of lists Λ (including negative numbers) by positive (nonnegative) doubly stochastic and positive (nonnegative) symmetric matrices.

Example 6.2. Is

$$\Lambda = \left\{1, \frac{11}{18}, \frac{1}{8}, -\frac{1}{16}, -\frac{1}{3}, -\frac{1}{2}\right\}$$

the spectrum of a positive symmetric matrix? Consider the partition $\Lambda = \{1, -\frac{1}{2}\} \cup \{\frac{11}{18}, -\frac{1}{3}\} \cup \{\frac{1}{8}, -\frac{1}{16}\}$. Then from Corollary 5.2 there exists a positive symmetric matrix

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A with spectrum Λ. To construct A, we consider, for $\epsilon = \frac{1}{6}$, the auxiliary list $\Lambda' = \left\{\frac{5}{6}, \frac{11}{18}, \frac{1}{8}, -\frac{1}{16}, -\frac{1}{3}, -\frac{1}{2}\right\}$. Then the matrix

$$B = \begin{bmatrix} 1/6 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/36 & 17/36 & 0 & 0 \\ 0 & 0 & 17/36 & 5/36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/32 & 3/32 \\ 0 & 0 & 0 & 0 & 3/32 & 1/32 \end{bmatrix}$$

is nonnegative symmetric with spectrum Λ' and from Theorem 1.1, we compute the positive symmetric matrix

$$A = \begin{pmatrix} \frac{29}{132} & \frac{95}{132} & \frac{13}{792}\sqrt{10} & \frac{13}{792}\sqrt{10} & \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{72}\sqrt{\frac{305}{11}} \\ \frac{95}{132} & \frac{29}{132} & \frac{13}{792}\sqrt{10} & \frac{13}{792}\sqrt{10} & \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{72}\sqrt{\frac{305}{11}} \\ \frac{13}{792}\sqrt{10} & \frac{13}{792}\sqrt{10} & \frac{41}{264} & \frac{43}{88} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} \\ \frac{13}{792}\sqrt{10} & \frac{13}{792}\sqrt{10} & \frac{43}{88} & \frac{41}{264} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} \\ \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{13}{288} & \frac{31}{288} \\ \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{72}\sqrt{\frac{305}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{1}{144}\sqrt{\frac{122}{11}} & \frac{31}{288} & \frac{13}{288} \end{pmatrix}$$

with spectrum Λ.

EXAMPLE 6.3. Is

$$\Lambda = \left\{1, 0, -\frac{1}{24}, -\frac{1}{18}, -\frac{1}{12}\right\}$$

the spectrum of a positive symmetric doubly stochastic matrix? Since

$$\Lambda' = \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{1}{12}\right\}$$

is the spectrum of a positive generalized doubly stochastic matrix by Theorem 4.3, then from Corollary 4.4, for $t = \frac{1}{6}$ we have

$$\begin{aligned} \Lambda'' &= \left\{\frac{1}{3} + 4t, \frac{1}{6} - t, \frac{1}{8} - t, \frac{1}{9} - t, \frac{1}{12} - t\right\} \\ &= \left\{1, 0, -\frac{1}{24}, -\frac{1}{18}, -\frac{1}{12}\right\} \\ &= \Lambda. \end{aligned}$$

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Thus, Λ is the spectrum of the positive symmetric doubly stochastic matrix

$$A = \begin{pmatrix} \frac{629}{4320} & \frac{989}{4320} & \frac{929}{4320} & \frac{101}{480} & \frac{1}{5} \\ \frac{989}{4320} & \frac{629}{4320} & \frac{929}{4320} & \frac{101}{480} & \frac{1}{5} \\ \frac{929}{4320} & \frac{929}{4320} & \frac{689}{4320} & \frac{101}{480} & \frac{1}{5} \\ \frac{101}{480} & \frac{101}{480} & \frac{101}{480} & \frac{27}{160} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

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