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A NOTE ON SEMISCALAR EQUIVALENCE OF POLYNOMIAL MATRICES*

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Abstract. Polynomial matrices $A(\lambda)$ and $B(\lambda)$ of size $n \times n$ over a field $\mathbb{F}$ are semiscalar equivalent if there exist a nonsingular $n \times n$ matrix P over $\mathbb{F}$ and an invertible $n \times n$ matrix $Q(\lambda)$ over $\mathbb{F}[\lambda]$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$. The aim of this article is to present necessary and sufficient conditions for the semiscalar equivalence of nonsingular matrices $A(\lambda)$ and $B(\lambda)$ over a field $\mathbb{F}$ of characteristic zero in terms of solutions of a homogenous system of linear equations.

Key words. Equivalence of matrices, Smith normal form, Similarity of matrices.

AMS subject classifications. 15A21, 15A24, 65F15, 65F30.

1. Introduction. Let $\mathbb{F}$ be a field. Denote by $M_{m,n}(\mathbb{F})$ the set of $m \times n$ matrices over $\mathbb{F}$ and by $M_{m,n}(\mathbb{F}[\lambda])$ the set of $m \times n$ matrices over the polynomial ring $\mathbb{F}[\lambda]$. A polynomial $a(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \ldots + a_k \in \mathbb{F}(\lambda)$ is said to be monic if the first nonzero term a_0 is equal to 1.

Let $A(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be a nonzero matrix and rank $A(\lambda) = r$. For the matrix $A(\lambda)$, there exist matrices $P(\lambda), Q(\lambda) \in GL(n, \mathbb{F}[\lambda])$ such that

$$P(\lambda)A(\lambda)Q(\lambda) = S_A(\lambda) = \operatorname{diag}\left(s_1(\lambda), s_2(\lambda), \ldots, s_r(\lambda), 0, \ldots, 0\right),$$

where $s_j(\lambda) \in \mathbb{F}[\lambda]$ are monic polynomials for all $j = 1, 2, \ldots, r$ and $s_1(\lambda)|s_2(\lambda)| \ldots |s_r(\lambda)$ (divides) are the invariant factors of $A(\lambda)$. The diagonal matrix $S_A(\lambda)$ is called the Smith normal form of $A(\lambda)$.

Definition 1.1. *Matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ are said to be semiscalar equivalent if there exist matrices $P \in GL(n, \mathbb{F})$ and $Q(\lambda) \in GL(n, \mathbb{F}[\lambda])$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$. ([7], Chapter 4).*

Let $A(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be nonsingular matrix over an infinite field $\mathbb{F}$. Then $A(\lambda)$ is semiscalar equivalent to the lower triangular matrix ([7])

$$S_l(\lambda) = \begin{bmatrix} s_{11}(\lambda) & 0 & \ldots & \ldots & 0 \\ s_{21}(\lambda) & s_{22}(\lambda) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ s_{n1}(\lambda) & s_{n2}(\lambda) & \ldots & s_{n,n-1}(\lambda) & s_{nn}(\lambda) \end{bmatrix},$$

with the following properties:

(a) $s_{ii}(\lambda) = s_i(\lambda)$, $i = 1, 2, \ldots, n$, where $s_1(\lambda)|s_2(\lambda)|\cdots|s_n(\lambda)$ (divides) are the invariant factors of $A(\lambda)$;
(b) $s_{ii}(\lambda)$ divides $s_{ji}(\lambda)$ for all i, j with $1 \le i < j \le n$.

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It may be noted that for a singular matrix $A(\lambda)$ the matrix $S_l(\lambda)$ does not always exist (see [7]). The matrix $A(\lambda) = \begin{bmatrix} \lambda & \lambda \\ \lambda^2+1 & \lambda^2+1 \end{bmatrix}$ is not semiscalar equivalent to the lower triangular matrix $S_l(\lambda) = \begin{bmatrix} 1 & 0 \\ * & 0 \end{bmatrix}$.

The triangular form $S_l(\lambda)$ for nonsingular matrices over a finite field does not always exist (see the Remark following Corollary 2 of [10]). Let $\mathbb{F} = \{0,1\}$ be a field of two elements. It is easily verified that the polynomial matrix

$$A(\lambda) = \begin{bmatrix} \lambda & 0 \\ \lambda^2+1 & (\lambda^2+\lambda+1)(\lambda^2+1) \end{bmatrix},$$

over the field $\mathbb{F}$ is not semiscalar equivalent to the lower triangular matrix

$$S_l(\lambda) = \begin{bmatrix} 1 & 0 \\ * & \lambda(\lambda^2+\lambda+1)(\lambda^2+1) \end{bmatrix}.$$

Example 1.2. Let $\mathbb{F} = \mathbb{R}$ be the field of real numbers. Further, let

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^3 - 3\lambda^2 - \lambda & (\lambda^2-1)(\lambda^2-2\lambda) \end{bmatrix} \text{ and } B(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^3 - \lambda^2 - \lambda & (\lambda^2-1)(\lambda^2-2\lambda) \end{bmatrix},$$

be 2×2 matrices with entries from $\mathbb{R}[\lambda]$. For $A(\lambda)$ and $B(\lambda)$, there exist the nonsingular matrix $P = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix} \in M_{2,2}(\mathbb{R})$ and the invertible matrix $Q(\lambda) = \begin{bmatrix} 2\lambda^3 - 6\lambda^2 - 2\lambda + 9 & 2\lambda^4 - 4\lambda^3 - 2\lambda^2 + 4\lambda \\ -2\lambda^2 + 4\lambda + 4 & -2\lambda^3 + 2\lambda^2 + 2\lambda + 1 \end{bmatrix} \in M_{2,2}(\mathbb{R}[\lambda])$ such that $PA(\lambda) = B(\lambda)Q(\lambda)$.

From this example it follows, that the triangular form $S_l(\lambda)$ is not uniquely determined for a nonsingular polynomial matrix $A(\lambda)$ with respect the semiscalar equivalence (see also Example 4.1).

Dias da Silva J.A and Laffey T.J. studied polynomial matrices up to PS-equivalence.

Definition 1.3 (See [1]). *Matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ are PS-equivalent if $A(\lambda) = P(\lambda)B(\lambda)Q$ for some $P(\lambda) \in GL(n, \mathbb{F}[\lambda])$ and $Q \in GL(n, \mathbb{F})$.*

Let $\mathbb{F}$ be an infinite field. A matrix $A(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ with $\det A(\lambda) \neq 0$ is PS-equivalent to the upper triangular matrix (see [1], Proposition 2)

$$S_u(\lambda) = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) & \dots & s_{1n}(\lambda) \\ 0 & s_{22}(\lambda) & \dots & s_{2n}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & s_{nn}(\lambda) \end{bmatrix},$$

with the following properties:

(a) $s_{ii}(\lambda) = s_i(\lambda)$, $i = 1, 2, \dots, n$, where $s_1(\lambda)|s_2(\lambda)|\cdots|s_n(\lambda)$ (divides) are the invariant factors of $A(\lambda)$;
(b) $s_{ii}(\lambda)$ divides $s_{ij}(\lambda)$ for all integers i, j with $1 \le i < j \le n$;
(c) if $i \neq j$ and $s_{ij}(\lambda) \neq 0$, then $s_{ij}(\lambda)$ is a monic polynomial and
$\deg s_{ii}(\lambda) < \deg s_{ij}(\lambda) < \deg s_{jj}(\lambda)$.

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The matrix $S_u(\lambda)$ is called a near canonical form of the matrix $A(\lambda)$ with respect to PS equivalence. We note that conditions (a) and (b) for semiscalar equivalence were proved in [7]. It is evident that matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ are PS equivalent if and only if the transpose matrices $A^T(\lambda)$ and $B^T(\lambda)$ are semiscalar equivalent. It is clear that semiscalar equivalence and PS equivalence represent an equivalence relation on $M_{n,n}(\mathbb{F}[\lambda])$. On the basis of the semiscalar equivalence of polynomial matrices in [7], algebraic methods for factorization of matrix polynomials were developed.

The semiscalar equivalence of regular matrix polynomials matters for the problem of classifying linear controllable systems, when change of bases in the state and input spaces are allowed (see [11]). Each class of similar controllable linear systems can be identified up to the right equivalence with a regular matrix polynomial (see [9], [15]). Suppose that two pairs of $n \times n$ matrices (A_1, B_1) and (A_2, B_2) are controllable and unimodular $n \times n$ matrices $P_i(\lambda)$ are polynomial system representations of (A_i, B_i). According to Theorem 2.4 from [15] $(A_2, B_2) = (T^{-1}A_1T, T^{-1}B_1Q)$ for some invertible matrices T and Q if and only if $P_2(\lambda) = QP_1(\lambda)U(\lambda)$, where $U(\lambda)$ is an unimodular matrix. Put $G_i(\lambda) = (I_n\lambda - A)^{-1}B_i$, $(i = 1, 2)$ and assume that $(A_2, B_2) = (T^{-1}A_1T, T^{-1}B_1Q)$. Now we write the strictly proper rational matrix functions $G_i(\lambda)$ as irreducible matrix fractions (see [6], Sec. 6.2.3) in the forms $G_i(\lambda) = N_i(\lambda)(P_i(\lambda))^{-1}$, $i = 1, 2$. From equality $G_2(\lambda) = TG_1(\lambda)Q$, we have

$$N_2(\lambda)P_2^{-1}(\lambda) = TN_1(\lambda)(P_1(\lambda))^{-1}Q = TN_1(\lambda)(Q^{-1}P_1(\lambda))^{-1}.$$

This means that $N_2(\lambda)P_2(\lambda)^{-1}$ and $TN_1(\lambda)(Q^{-1}P_1(\lambda))^{-1}$ are two irreducible matrix descriptions of $G_2(\lambda)$. In accordance with [6], there is an unimodular matrix $U(\lambda)$ such that $N_2(\lambda) = TN_1(\lambda)U(\lambda)$ and $P_2(\lambda) = QP_1(\lambda)U(\lambda)$. It is obvious that matrices $N_2(\lambda)$ and $N_1(\lambda)$ are semiscalar equivalent, and matrices $P_2(\lambda)$ and $P_1(\lambda)$ are semiscalar equivalent. Thus, any contribution to the problem of classifying the matrix polynomials according to the semiscalar equivalence is a contribution to the problem of classifying the linear controllable systems by changes of bases in the state and input spaces. [1] We note that the semiscalar equivalence and the PS equivalence were used in the study of the controllability of linear systems in [2] and [3].

The semiscalar equivalence of matrices includes the following two tasks: (1) the determination of a complete system of invariants and (2) the construction of a canonical form for a matrix with respect to semiscalar equivalence. But these tasks have satisfactory solutions only in isolated cases. The canonical and normal forms with respect to semiscalar equivalence for a matrix pencil $A_0\lambda + A_1 \in M_{n,n}(\mathbb{F}[\lambda])$, where A_0 is nonsingular, were investigated in [12] and [13]. A canonical form with respect to semiscalar equivalence for a polynomial matrix over a field is unknowns in general case. The article is organized as follows. In Section 2, we prove preparatory results of this article. Necessary and sufficient conditions under which nonsingular matrices $A(\lambda)$ and $B(\lambda)$ over a field $\mathbb{F}$ of characteristic zero are semiscalar equivalent are proposed in Section 3. In Section 4, numerical examples are also given.

2. Preparatory notations and results. To prove the main result, we need the following notations and propositions. Let $\mathbb{F}$ be a field of characteristic zero. In what follows $A^*(\lambda) = \operatorname{Adj} A(\lambda)$ means the classical adjoint matrix of $A(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$. We denote by O the zero matrix and by $\bar{0}$ the zero vector, respectively, and their dimensions are determined from the context. In the polynomial ring $\mathbb{F}[\lambda]$, we consider the operation of differentiation $\mathbf{D}$: $\mathbb{F}[\lambda] \to \mathbb{F}[\lambda]$ such that

$$\mathbf{D}(a(\lambda) + b(\lambda)) = \mathbf{D}(a(\lambda)) + \mathbf{D}(b(\lambda)) \text{ and } \mathbf{D}(a(\lambda)b(\lambda)) = \mathbf{D}(a(\lambda))b(\lambda) + a(\lambda)\mathbf{D}(b(\lambda)),$$

[1] This statement proposed by the referee.

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for all $a(\lambda), b(\lambda) \in \mathbb{F}[\lambda]$. Let $a(\lambda) = a_0\lambda^l + a_1\lambda^{l-1} + \cdots + a_{l-1}x + a_l \in \mathbb{F}[\lambda]$. Put

$$\mathbf{D}\left(a(\lambda)\right) = la_0\lambda^{l-1} + (l-1)a_1\lambda^{l-2} + \cdots + a_{l-1} = a^{(1)}(\lambda),$$

and $\mathbf{D}^k(a(\lambda)) = \mathbf{D}(a^{(k-1)}(\lambda)) = a^{(k)}(\lambda)$ for every natural $k \geq 2$. The differentiation of a matrix $A(\lambda) = \left[\begin{array}{c} a_{ij}(\lambda) \end{array}\right] \in M_{m,n}(\mathbb{F}[\lambda])$ is understood as its elementwise differentiation, i.e.,

$$A^{(1)}(\lambda) = \mathbf{D}(A(\lambda)) = [\mathbf{D}(a_{ij}(\lambda))] = [a_{ij}^{(1)}(\lambda)],$$

and $A^{(k)}(\lambda) = \mathbf{D}(A^{(k-1)}(\lambda))$ is the k-th derivative of $A(\lambda)$ for every natural $k \geq 2$.

Let $b(\lambda) = (\lambda - \beta_1)^{k_1}(\lambda - \beta_2)^{k_2}\cdots(\lambda - \beta_r)^{k_r} \in \mathbb{F}[\lambda]$, $\deg b(\lambda) = k = k_1 + k_2 + \cdots + k_r$, and $A(\lambda) \in M_{m,n}(\mathbb{F}[\lambda])$. By analogy with [7] for the monic polynomial $b(\lambda)$ and the matrix $A(\lambda)$, we will define the matrix $M_A(b) = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_r \end{bmatrix} \in M_{mk,n}(\mathbb{F})$, where $N_j = \begin{bmatrix} A(\beta_j) \\ A^{(1)}(\beta_j) \\ \vdots \\ A^{(k_j-1)}(\beta_j) \end{bmatrix} \in M_{mk_j,n}(\mathbb{F})$, $j = 1, 2, \ldots, r$.

Proposition 2.1. *Let $b(\lambda) = (\lambda-\beta_1)^{k_1}(\lambda-\beta_2)^{k_2}\cdots(\lambda-\beta_r)^{k_r} \in \mathbb{F}[\lambda]$, where $\beta_i \in \mathbb{F}$ for all $i = 1, 2, \ldots, r$, and $A(\lambda) \in M_{m,n}(\mathbb{F}[\lambda])$ be a nonzero matrix. Then $A(\lambda)$ admits the representation*

$$A(\lambda) = b(\lambda)C(\lambda), \tag{2.1}$$

if and only if $M_A(b) = \mathrm{O}$.

Proof. Suppose that (2.1) holds. It is evident that $b(\beta_j) = b^{(1)}(\beta_j) = \cdots = b^{(k_j-1)}(\beta_j) = 0$ for all $j = 1, 2, \ldots, r$ and $A(\beta_j) = \mathrm{O}$. Differentiating equality (2.1) $(k_j - 1)$ times and substituting each time $\lambda = \beta_j$ into both sides of the obtained equalities, we finally obtain $A^{(l)}(\beta_j) = \mathrm{O}$ for all $l = 1, 2, \ldots, k_j - 1$. Thus, $N_j = \mathrm{O}$. Since $1 \leq j \leq r$, we have $M_A(b) = \mathrm{O}$.

Conversely, let $M_A(b) = \mathrm{O}$. Dividing the matrix $A(\lambda)$ by $I_n b(\lambda)$ with residue (see, for instance, Theorem 7.2.1 in the classical book by Lancaster and Tismenetski [8]), we have $A(\lambda) = b(\lambda)C(\lambda) + R(\lambda)$, where $C(\lambda), R(\lambda) \in M_{m,n}(\mathbb{F}[\lambda])$ and $\deg R(\lambda) < \deg b(\lambda)$. Thus, $M_A(b) = M_R(b) = \mathrm{O}$. Since $M_R(b) = \mathrm{O}$, then $R(\lambda) = (\lambda-\beta_i)^{k_i}R_i(\lambda)$ for all $i = 1, 2, \ldots, r$, i.e., $R(\lambda) = b(\lambda)R_0(\lambda)$. On the other hand, $\deg R(\lambda) < \deg b(\lambda)$. Thus, $R(\lambda) \equiv \mathrm{O}$. This completes the proof. ∎

Corollary 2.2. *Let $A(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be a matrix of* $\operatorname{rank} A(\lambda) \geq n - 1$ *with the Smith normal form $S(\lambda) = \operatorname{diag}(s_1(\lambda), \ldots, s_{n-1}(\lambda), s_n(\lambda))$. If $s_{n-1}(\lambda) = (\lambda - \alpha_1)^{k_1}(\lambda - \alpha_2)^{k_2}\cdots(\lambda - \alpha_r)^{k_r}$, where $\alpha_i \in \mathbb{F}$ for all $i = 1, 2, \ldots, r$; then $M_{A^*}(s_{n-1}) = \mathrm{O}$.*

Proof. By inequality $\operatorname{rank} A(\lambda) \geq n - 1$, we have $A^*(\lambda) \neq \mathrm{O}$. Since $s_{n-1}(\lambda)|s_n(\lambda)$, the matrix $A^*(\lambda)$ admits the representation $A^*(\lambda) = s_{n-1}(\lambda)B(\lambda)$, where $B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$. By virtue of Proposition 2.1, $M_{A^*}(s_{n-1}) = \mathrm{O}$. The proof is completed. ∎

The Kronecker product of matrices $A = [a_{ij}]$ $(n \times m)$ and B is denoted by

$$A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \ldots & a_{nm}B \end{bmatrix}.$$

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Let nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be equivalent and $S(\lambda) = \operatorname{diag}(s_1(\lambda), \ldots, s_{n-1}(\lambda), s_n(\lambda))$ be their Smith normal form. For $A(\lambda)$ and $B(\lambda)$, we define the matrix

$$D(\lambda) = \left(\Big(s_1(\lambda)s_2(\lambda)\cdots s_{n-1}(\lambda)\Big)^{-1} B^*(\lambda)\right) \otimes A^T(\lambda) \in M_{n^2,n^2}(\mathbb{F}[\lambda]).$$

It may be noted if $S(\lambda) = \operatorname{diag}(1, \ldots, 1, s(\lambda))$ is the Smith normal form of the matrices $A(\lambda)$ and $B(\lambda)$, then $D(\lambda) = B^*(\lambda) \otimes A^T(\lambda)$.

3. Main results. It is clear that two semiscalar or PS equivalent matrices are always equivalent. The converse of the above statement is not always true. The main result of this chapter is the following theorem.

THEOREM 3.1. *Let nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be equivalent and*

$$S(\lambda) = \operatorname{diag}(s_1(\lambda), \ldots, s_{n-1}(\lambda), s_n(\lambda)),$$

be their Smith normal form. Further, let $s_n(\lambda) = (\lambda - \alpha_1)^{k_1}(\lambda - \alpha_2)^{k_2} \cdots (\lambda - \alpha_r)^{k_r}$, where $\alpha_i \in \mathbb{F}$ for all $i = 1, 2, \ldots, r$. Then $A(\lambda)$ and $B(\lambda)$ are semiscalar equivalent if and only if the homogeneous system of equations $M_D(s_n)x = \bar{0}$ has a solution $x = [v_1, v_2, \ldots, v_{n^2}]^T$ over $\mathbb{F}$ such that the matrix

$$V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \\ v_{n+1} & v_{n+2} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n^2-n+1} & v_{n^2-n+2} & \ldots & v_{n^2} \end{bmatrix},$$

is nonsingular. If $\det V \neq 0$, then $VA(\lambda) = B(\lambda)Q(\lambda)$, where $Q(\lambda) \in GL(n, \mathbb{F}[\lambda])$.

Proof. Since the matrices $A(\lambda)$ and $B(\lambda)$ are equivalent, then $\operatorname{rank} M_D(s_n) < n^2$.

Let nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ be semiscalar equivalent, i.e., $A(\lambda) = PB(\lambda)Q(\lambda)$, where $P \in GL(n, \mathbb{F})$ and $Q(\lambda) \in GL(n, \mathbb{F}[\lambda])$. From the last equality, we have

$$B^*(\lambda)P^{-1}A(\lambda) = Q(\lambda)\det B(\lambda). \tag{3.2}$$

Write $B^*(\lambda)$ in the form $B^*(\lambda) = d(\lambda)C(\lambda)$ (see the proof of Corollary 2.2) and $\det B(\lambda) = b_0 d(\lambda)s_n(\lambda)$, where $d(\lambda) = s_1(\lambda)s_2(\lambda)\cdots s_{n-1}(\lambda)$, $C(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ and b_0 is a nonzero element in $\mathbb{F}$. Thus, from equality (3.2), we obtain

$$C(\lambda)P^{-1}A(\lambda) = Q(\lambda)s_n(\lambda)b_0. \tag{3.3}$$

Put

$$P^{-1} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \\ v_{n+1} & v_{n+2} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n^2-n+1} & v_{n^2-n+2} & \ldots & v_{n^2} \end{bmatrix},$$

and

$$Q(\lambda)b_0 = W(\lambda) = \begin{bmatrix} w_1(\lambda) & w_2(\lambda) & \ldots & w_n(\lambda) \\ w_{n+1}(\lambda) & w_{n+2}(\lambda) & \ldots & w_{2n}(\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ w_{n^2-n+1}(\lambda) & w_{n^2-n+2}(\lambda) & \ldots & w_{n^2}(\lambda) \end{bmatrix},$$

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where $v_j \in \mathbb{F}$ and $w_j(\lambda) \in \mathbb{F}[\lambda]$ for all $j = 1, 2, \dots, n^2$. Then we can write equality (3.3) in the form (see [8], Chapter 12)

$$(3.4) \qquad \big(C(\lambda) \otimes A^T(\lambda)\big) \cdot \begin{bmatrix} v_1, & v_2, & \dots, & v_{n^2} \end{bmatrix}^T = s_n(\lambda) \begin{bmatrix} w_1(\lambda), & w_2(\lambda), & \dots, & w_{n^2}(\lambda) \end{bmatrix}^T.$$

Note that $C(\lambda) \otimes A^T(\lambda) = D(\lambda)$. In view of equality (3.4) and Proposition 2.1, we have

$$M_D(s_n) \begin{bmatrix} v_1, & v_2, & \dots, & v_{n^2} \end{bmatrix}^T = \bar{0}.$$

Thus, the homogeneous system of equations $M_D(s_n)x = \bar{0}$ has a necessary solution.

Conversely, the homogeneous system of equations $M_D(s_n)x = \bar{0}$ has a solution $x = [v_1, v_2, \dots, v_{n^2}]^T$ over $\mathbb{F}$ such that the matrix $V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ v_{n+1} & v_{n+2} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{n^2-n+1} & v_{n^2-n+2} & \dots & v_{n^2} \end{bmatrix}$ is nonsingular. Dividing the product $C(\lambda)VA(\lambda)$ by $I_n s_n(\lambda)$ with residue, we have

$$C(\lambda)VA(\lambda) = s_n(\lambda)Q(\lambda) + R(\lambda),$$

where $Q(\lambda), R(\lambda) = [r_{ij}(\lambda)] \in M_{n,n}(\mathbb{F}[\lambda])$ and $\deg R(\lambda) < \deg s_n(\lambda)$. From the last equality we obtain

$$M_D(s_n)x_0 = M_{\mathbf{Col}\,R}(s_n) = \bar{0},$$

where $\mathbf{Col}\,R(\lambda) = \begin{bmatrix} r_{11}(\lambda) & \dots & r_{1n}(\lambda) & \dots & r_{n,n-1}(\lambda) & \dots & r_{nn}(\lambda) \end{bmatrix}^T$. In accordance with Proposition 2.1 we have $\mathbf{Col}\,R(\lambda) \equiv \bar{0}$. Thus, $R(\lambda) \equiv 0$ and

$$(3.5) \qquad C(\lambda)VA(\lambda) = s_n(\lambda)Q(\lambda).$$

Note that $\det B(\lambda) = b_0 d(\lambda) s_n(\lambda)$, where b_0 is a nonzero element in $\mathbb{F}$. Multiplying both sides of equality (3.5) by $b_0 d(\lambda)$ we have

$$(3.6) \qquad b_0 B^*(\lambda)VA(\lambda) = b_0 d(\lambda)C(\lambda)VA(\lambda) = b_0 d(\lambda)s_n(\lambda)Q(\lambda) = Q(\lambda)\det B(\lambda).$$

Hence $B(\lambda)Q(\lambda) = b_0 VA(\lambda)$ and passing to the determinants on both sides of this equality, we obtain $\det Q(\lambda) = \text{const} \neq 0$. Since $Q(\lambda) \in GL(n, \mathbb{F}[\lambda])$, we conclude that matrices $A(\lambda)$ and $B(\lambda)$ are semiscalar equivalent. This completes the proof. ∎

It may be noted that nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{F}[\lambda])$ are PS equivalent if and only if $A(\lambda)^T$ and $B(\lambda)^T$ are semiscalar equivalent. Thus, Theorem 3.1 gives the answer to the question: When are nonsingular matrices $A(\lambda)$ and $B(\lambda)$ PS equivalent?

In the future, $\mathbb{F} = \mathbb{C}$ is the field of complex numbers.

Corollary 3.2. *Let nonsingular matrices $A(\lambda), B(\lambda) \in M_{n,n}(\mathbb{C}[\lambda])$ be equivalent and*

$$S(\lambda) = \operatorname{diag}(s_1(\lambda), \dots, s_{n-1}(\lambda), s_n(\lambda)),$$

be their Smith normal form. Then $A(\lambda)$ and $B(\lambda)$ are semiscalar equivalent if and only if the homogeneous system of equations $M_D(s_n)x = \bar{0}$ has a solution $x = [v_1, v_2, \dots, v_{n^2}]^T$ over $\mathbb{C}$ such that the matrix $V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ v_{n+1} & v_{n+2} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{n^2-n+1} & v_{n^2-n+2} & \dots & v_{n^2} \end{bmatrix}$ is nonsingular.

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Definition 3.3. *Two families of $n \times n$ matrices over the field $\mathbb{C}$*

$$\mathbf{A} = (A_1, A_2, \ldots, A_r) \quad \text{and} \quad \mathbf{B} = (B_1, B_2, \ldots, B_r),$$

are said to be similar if there exists a matrix $T \in GL(n, \mathbb{C})$ such that $A_i = TB_iT^{-1}$ for all $i = 1, 2, \ldots, r$.

We associate the families $\mathbf{A}$ and $\mathbf{B}$ with monic matrix polynomials

$$A(\lambda) = I_n\lambda^r + A_1\lambda^{r-1} + A_2\lambda^{r-2} + \cdots + A_r \quad \text{and} \quad B(\lambda) = I_n\lambda^r + B_1\lambda^{r-1} + B_2\lambda^{r-2} + \cdots + B_r,$$

over $\mathbb{C}$ of degree r respectively. The families $\mathbf{A}$ and $\mathbf{B}$ are similar over $\mathbb{C}$ if and only if the matrices $A(\lambda)$ and $B(\lambda)$ are semiscalar equivalent (PS equivalent) (see also [7], [1], [4], [5], [14] and references therein). From Theorem 3.1 and Corollary 3.2, we obtain the following corollary.

Corollary 3.4. *Let $n \times n$ monic matrix polynomials (of degree r) $A(\lambda) = I_n\lambda^r + \sum_{i=1}^r A_i\lambda^{r-i}$ and $B(\lambda) = I_n\lambda^r + \sum_{i=1}^r B_i\lambda^{r-i}$ over the field of complex numbers $\mathbb{C}$ be equivalent, and let*

$$S(\lambda) = \operatorname{diag}\left(s_1(\lambda), \ldots, s_{n-1}(\lambda), s_n(\lambda)\right),$$

be their Smith normal form. The families $\mathbf{A} = (A_1, A_2, \ldots, A_r)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_r)$ are similar over $\mathbb{C}$ if and only if the homogeneous system of equations $M_D(s_n)x = \bar{0}$ has a solution $x = [v_1, v_2, \ldots, v_{n^2}]^T$ over $\mathbb{C}$ such that the matrix $V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \\ v_{n+1} & v_{n+2} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n^2-n+1} & v_{n^2-n+2} & \ldots & v_{n^2} \end{bmatrix}$ is nonsingular.

If $\det V \neq 0$, then $A_i = V^{-1}B_iV$ for all $i = 1, 2, \ldots, r$.

4. Illustrative examples.

To illustrate Theorem 3.1 and Corollary 3.4, consider the following examples.

Example 4.1. Matrices $A(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^2 + a\lambda & \lambda^4 \end{bmatrix}$ and $B(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^2 + b\lambda & \lambda^4 \end{bmatrix}$ with entries from $\mathbb{C}[\lambda]$ are equivalent for all $a, b \in \mathbb{C}$ and $S(\lambda) = \operatorname{diag}(1, \lambda^4)$ is their Smith normal form. In what follows $a \neq b$.

Construct the matrix

$$D(\lambda) = B^*(\lambda) \otimes A^T(\lambda) = \begin{bmatrix} \lambda^4 & \lambda^6 + a\lambda^5 & 0 & 0 \\ 0 & \lambda^8 & 0 & 0 \\ -(\lambda^2 + b\lambda) & -(\lambda^4 + (a+b)\lambda^3 + ab\lambda^2) & 1 & \lambda^2 + a\lambda \\ 0 & -(\lambda^6 + b\lambda^5) & 0 & \lambda^4 \end{bmatrix},$$

and solve the system of equations $M_D(s_2)x = \bar{0}$. From this, it follows

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ -b & 0 & 0 & a \\ -2 & -2ab & 0 & 2 \\ 0 & -6(a+b) & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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From this we have, if $a+b \neq 0$, then $A(\lambda)$ and $B(\lambda)$ are not semiscalar equivalent. If $a+b=0$, then $b=-a$ and system of equations $M_D(s_2)x = \bar{0}$ is solvable. The vector $\begin{bmatrix} 1, & \frac{2}{a^2}, & 0, & -1 \end{bmatrix}^T$ is a solution of $M_D s_2 x = \bar{0}$ for arbitrary $a \neq 0$. Thus, the matrix $V = \begin{bmatrix} 1 & \frac{2}{a^2} \\ 0 & -1 \end{bmatrix}$ is nonsingular.

So, if $a \neq 0$ and $b=-a$, then the matrices $A(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^2 + a\lambda & \lambda^4 \end{bmatrix}$ and $B(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^2 - a\lambda & \lambda^4 \end{bmatrix}$ are semiscalar equivalent, i.e., $A(\lambda) = PB(\lambda)Q(\lambda)$, where $P = V^{-1} = \begin{bmatrix} 1 & \frac{2}{a^2} \\ 0 & -1 \end{bmatrix}$ and

$$Q(\lambda) = \begin{bmatrix} \frac{2\lambda^2}{a^2} + \frac{2\lambda}{a} + 1 & \frac{2\lambda^4}{a^2} \\ -\frac{2}{a^2} & -\frac{2\lambda^2}{a^2} + \frac{2\lambda}{a} - 1 \end{bmatrix} \in GL(2, \mathbb{C}[\lambda]).$$

Thus, the matrix $S_l(\lambda)$ is not uniquely determined for the nonsingular matrix $A(\lambda)$ with respect to semiscalar equivalence for arbitrary $a \neq 0$.

Example 4.2. Let

$$\mathbf{A} = \left(A_1 = \begin{bmatrix} -3 & 0 \\ -4 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right),$$

and

$$\mathbf{B} = \left(B_1 = \begin{bmatrix} 1 & 0 \\ -4 & -3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right),$$

be two families of 2×2 matrices over the field $\mathbb{C}$. Monic matrix polynomials

$$A(\lambda) = I_2\lambda^2 + A_1\lambda + A_2 = \begin{bmatrix} \lambda^2 - 3\lambda + 1 & 1 \\ -4\lambda + 1 & \lambda^2 + \lambda + 1 \end{bmatrix},$$

and

$$B(\lambda) = I_2\lambda^2 + B_1\lambda + B_2 = \begin{bmatrix} \lambda^2 + \lambda & 0 \\ -4\lambda + 1 & \lambda^2 - 3\lambda + 2 \end{bmatrix},$$

with entries from $\mathbb{C}[\lambda]$ are equivalent and $S(\lambda) = \operatorname{diag}(1, (\lambda^2-1)(\lambda^2-2\lambda))$ is their Smith normal form. It may be noted that $s_1(\lambda) = 1$ and $s_2(\lambda) = (\lambda^2-1)(\lambda^2-2\lambda))$.

Construct the matrix

$$D(\lambda) = B^*(\lambda) \otimes A^T(\lambda) = \begin{bmatrix} \lambda^2 - 3\lambda + 2 & 0 \\ 4\lambda - 1 & \lambda^2 + \lambda \end{bmatrix} \otimes \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -4\lambda + 1 \\ 1 & \lambda^2 + \lambda + 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda^2 - 3\lambda + 2) \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -4\lambda + 1 \\ 1 & \lambda^2 + \lambda + 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ (4\lambda - 1) \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -4\lambda + 1 \\ 1 & \lambda^2 + \lambda + 1 \end{bmatrix} & (\lambda^2 + \lambda) \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -4\lambda + 1 \\ 1 & \lambda^2 + \lambda + 1 \end{bmatrix} \end{bmatrix},$$

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and solve the system of equations $M_D(s_2)x = \bar{0}$. Crossing out zero rows in the matrix $M_D(s_n)$ and after elementary transformations over the rows of this matrix, we get the following system of linear equations

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 9 & 2 & 6 \\ 7 & 49 & 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this system of equations, we obtain $x_1 = -x_2 = t$, $x_3 = 0$ and $x_4 = t$. The matrix $V = \begin{bmatrix} t & -t \\ 0 & t \end{bmatrix}$ is nonsingular for nonzero $t \in \mathbb{C}$. Thus, the monic matrix polynomials $A(\lambda)$ and $B(\lambda)$ are semiscalar equivalent. Hence, the families of matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, i.e., $A_i = V^{-1}B_iV$, $i = 1, 2$.

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