Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

ELA

ON THE INVERSE EIGENVALUE PROBLEMS: THE CASE OF SUPERSTARS*

ROSÁRIO FERNANDES†

Abstract. Let T be a tree and let x_0 be a vertex of T. T is called a superstar with central vertex x_0 if $T - x_0$ is a union of paths. The General Inverse Eigenvalue Problem for certain trees is partially answered. Using this description, some superstars are presented for which the problem of ordered multiplicity lists and the Inverse Eigenvalue Problem are not equivalent.

Key words. Eigenvalues, Tree, Graph, Symmetric matrices.

AMS subject classifications. 15A18, 05C38, 05C50.

1. Introduction. Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix. We denote by $G(A) = (X, U)$ the simple graph on n vertices, $\{1, \ldots, n\}$, such that $\{i, j\} \in U$, $i \neq j$, if and only if $a_{ij} \neq 0$. Let $A(i)$ denote the principal submatrix of A obtained by deleting row and column i.

Let $G = (X, U)$ a simple graph, where $X = \{x_1, \ldots, x_n\}$ is the vertex set of G and let $\mathcal{S}(G)$ be the set of all $n \times n$ real symmetric matrices A such that $G(A) \cong G$. One of the most important problems of Spectral Graph Theory is the General Inverse Eigenvalue Problem for $\mathcal{S}(G)$ (GIEP for $\mathcal{S}(G)$):

"What are all the real numbers $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \leq \ldots \leq \mu_{n-1}$ that may occur as the eigenvalues of A and $A(i)$, respectively, as A runs over $\mathcal{S}(G)$?"

Another important problem is the Inverse Eigenvalue Problem for $\mathcal{S}(G)$ (IEP for $\mathcal{S}(G)$);

"What are all the real numbers $\lambda_1 \leq \ldots \leq \lambda_n$ that may occur as the eigenvalues of A, as A runs over $\mathcal{S}(G)$?"

First, we remind the reader of some results concerning the GIEP.

Perhaps the most well known result on this subject is the Interlacing Theorem for Eigenvalues of Hermitian matrices:

*Received by the editors September 16, 2007. Accepted for publication July 27, 2009. Handling Editor: Stephen J. Kirkland.

†Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal (mrff@fct.unl.pt). This research was done within the activities of "Centro de Estruturas Lineares e Combinatórias".

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

THEOREM 1.1. [11] *If A is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ and if $A(i)$ has eigenvalues $\mu_1 \leq \ldots \leq \mu_{n-1}$ then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

If λ is a real number and A is an $n \times n$ real symmetric matrix, we denote by $m_A(\lambda)$ the multiplicity of λ as an eigenvalue of A. As a Corollary of Theorem 1.1 we have the following result:

PROPOSITION 1.2. [11] *Let A be an $n \times n$ Hermitian matrix and let λ be an eigenvalue of A. Then*

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1, \qquad i = 1, \ldots, n.$$

When the graph G is a path, we have the well-known fact:

PROPOSITION 1.3. *Let T be a path. If A is a matrix in $\mathcal{S}(T)$ and j is a pendant vertex of $T(A)$, all the eigenvalues of A have multiplicity 1 and the eigenvalues of $A(j)$ strictly interlace those of A.*

Several authors proved the converse of Proposition 1.3.

The solution of the GIEP for $\mathcal{S}(G)$ when G is a cycle is also well known, see [3, 4, 5, 6].

Leal Duarte generalized the converse of Proposition 1.3 to any tree, [13].

PROPOSITION 1.4. *Let T be a tree on n vertices and let i be a vertex of T. Let $\lambda_1 < \ldots < \lambda_n$ and $\mu_1 < \ldots < \mu_{n-1}$ be real numbers. If*

$$\lambda_1 < \mu_1 < \lambda_2 < \ldots < \mu_{n-1} < \lambda_n,$$

then there exists a matrix A in $\mathcal{S}(T)$, with eigenvalues $\lambda_1 < \ldots < \lambda_n$, and such that, $A(i)$ has eigenvalues $\mu_1 < \ldots < \mu_{n-1}$.

In [7], Johnson and Leal Duarte studied this problem for vertices, of a generic path T, of degree two and solved it for the particular case that occurs when A is a matrix in $\mathcal{S}(T)$ and $A(i)$ has eigenvalues of multiplicity two.

In 2003, Johnson, Leal Duarte and Saiago, [8], rewrote the GIEP for $\mathcal{S}(G)$:

"Let G be a simple connected graph G on n vertices, x_0 be a vertex of G of degree k and $G_1, \ldots, G_k$ be the connected components of $G - x_0$. Let $\lambda_1, \ldots, \lambda_n$ be real numbers, $g_1(t), \ldots, g_k(t)$ be monic polynomials having only real roots and such that

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

deg $g_i(t)$ is equal to the number of vertices of G_i. Is it possible to construct a matrix A in $\mathcal{S}(G)$ such that A has eigenvalues $\lambda_1, \dots, \lambda_n$ and the characteristic polynomial of $A[G_i]$ is $g_i(t)$ ($A[G_i]$ is the principal submatrix of A obtained by deleting rows and columns that correspond to vertices of $G - G_i$)?"

With respect to GIEP for $\mathcal{S}(G)$ as written above, the following results are proven in [8].

THEOREM 1.5. *Let T be a tree on n vertices and x_0 be a vertex of T of degree k whose neighbors are $x_1, \dots, x_k$. Let T_i be the branch (connected component) of T at x_0 containing x_i and s_i be the number of vertices in T_i, $i = 1, \dots, k$.*

Let $g_1(t), \dots, g_k(t)$ be monic polynomials having only distinct real roots, with deg $g_i(t) = s_i$, $p_1, \dots, p_s$ be the distinct roots among polynomials $g_i(t)$ and m_i be the multiplicity of root p_i in $\prod_{i=1}^k g_i(t)$.

Let $g(t)$ be a monic polynomial of degree $s + 1$.

There exists a matrix A in $\mathcal{S}(T)$ with characteristic polynomial

$$f(t) = g(t) \prod_{i=1}^{s} (t - p_i)^{m_i - 1}$$

and such that

1) *$A[T_i]$ has characteristic polynomial $g_i(t)$, $i = 1, \dots, k$,*
2) *If $1 \leq i \leq k$ and $s_i > 1$, the eigenvalues of $A[T_i - x_i]$ strictly interlace those of $A[T_i]$,*

if and only if the roots of $g(t)$ strictly interlace those of $\prod_{i=1}^s (t - p_i)$.

The statement of the previous theorem is shorter when T is a generalized tree, [8].

THEOREM 1.6. *Let T be a generalized star on n vertices with central vertex x_0, let $T_1, \dots, T_k$ be the branches of T at x_0, and let $l_0, \dots, l_k$ be the number of vertices of $T_1, \dots, T_k$, respectively.*

Let $g_1(t), \dots, g_k(t)$ be monic polynomials having only real roots, with deg $g_i(t) = l_i$, let $p_1, \dots, p_l$ be the distinct roots among polynomials $g_i(t)$ and let m_i be the multiplicity of root p_i in $\prod_{i=1}^k g_i(t)$, ($m_i \geq 1$).

Let $g(t)$ be a monic polynomial of degree $l + 1$.

There exists a matrix A in $\mathcal{S}(T)$ with characteristic polynomial

$$f(t) = g(t) \prod_{i=1}^{l} (t - p_i)^{m_i - 1}$$

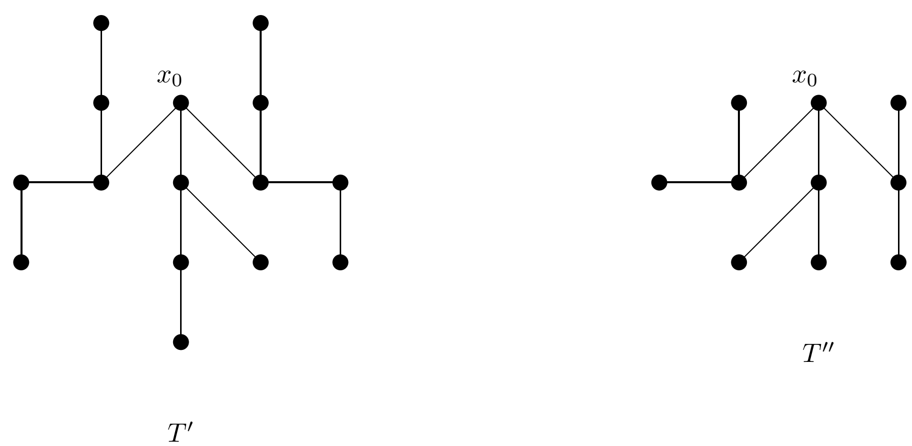
Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

and such that $A[T_i]$ *has characteristic polynomial* $g_i(t)$, $i = 1, \ldots, k$, *if and only if each* $g_i(t)$ *has only simple roots and the roots of* $g(t)$ *strictly interlace those of* $\prod_{i=1}^{l}(t-p_i)$.

In [8], the GIEP was solved for $\mathcal{S}(T)$ when T is a generalized star. Moreover, the authors of [8] proved that the IEP for $\mathcal{S}(G)$ when G is a generalized star, T, is equivalent to the determination of all possible ordered multiplicity lists of T; that is, if $A \in \mathcal{S}(T)$ has eigenvalues $\lambda_1 < \ldots < \lambda_t$ of multiplicities $m_1, \ldots, m_t$, respectively, then for any set of real numbers $\lambda'_1 < \ldots < \lambda'_t$, there exists a matrix $A' \in \mathcal{S}(T)$ having eigenvalues $\lambda'_1 < \ldots < \lambda'_t$ of multiplicities $m_1, \ldots, m_t$, respectively.

The case of double generalized stars has also been studied by Barioli and Fallat in [1].

In [2], Barioli and Fallat gave the first example of a graph (the graph T') for which the equivalence between the ordered multiplicity lists and the IEP does not occur. Another example appears in [10], the graph T''.



Bearing in mind these two graphs, we give now the following definitions.

Definition 1.7. Let T be a tree and x_0 be a vertex of T. A superstar T with central vertex x_0 is a tree such that $T - x_0$ is a union of paths.

Definition 1.8. Let T be a tree and x_0 be a vertex of T. Let T_i be a connected component of $T - x_0$ and x_i be the vertex of T_i adjacent to x_0 in T. We say that T_i is a cut branch at x_i if $T_i - x_i$ has at most two connected components.

The aforementioned trees T' and T'' are superstars. All the paths, stars and generalized stars (defined in [8]) are also superstars. Sometimes, when T is a superstar

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

with central vertex x_0, we say that T is a superstar at x_0.

More recently, ([12] Lemma 16) Kim and Shader proved a necessary condition for the GIEP to have a solution for some superstars. Notice that in [12] the k-whirls are the superstars of the present paper.

Lemma 1.9. ([12] Lemma 16) *Let T be a superstar on n vertices with central vertex x_0 whose neighbors are $x_1, \ldots, x_k$. Let T_i be the branch of T at x_0 containing x_i, $i = 1, \ldots, k$. Suppose that $T_1, \ldots, T_k$ are cut branches at $x_1, \ldots, x_r$, respectively, and let α_1^i, α_2^i be the paths of $T_i - x_i$, $i = 1, \ldots, k$.*

Let A in $\mathcal{S}(T)$ and let A' be the direct sum of $A[\alpha_j^i]$, for all $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. If n_r denotes the number of eigenvalues of A with multiplicity r, then the following holds:

(a) *$n_{k+1} \leq 1$ and $n_j = 0$ for $j \geq k + 2$;*
(b) *if λ is an eigenvalue of A and $m_A(\lambda) = k + 1$, then λ is a simple eigenvalue of $A[\alpha_j^i]$, for all $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$, and $m_{A'}(\lambda) = 2k$;*
(c) *if μ is an eigenvalue of A and $m_A(\mu) = k$, then for all $i \neq s$, j and t, μ is a simple eigenvalue of at least one of $A[\alpha_j^i]$, $A[\alpha_t^s]$, and $m_{A'}(\mu) \geq 2k - 2$; and*
(d) *$(2k - 2)n_k + (2k)n_{k+1} \leq n - (k + 1)$.*

In section 3, the methods used to prove Theorem 1.5 allows us to generalize it. As in this generalization we suppose that some branches of the tree T are cut branches we obtain a much more general result than Lemma 1.9. Using this generalization we prove in section 4:

1) there is no matrix $A' \in \mathcal{S}(T')$ (where T' is the above mentioned tree) having eigenvalues

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7),$$

but there exists a matrix $A \in \mathcal{S}(T')$ having eigenvalues

$$(1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7).$$

2) there is no matrix $A' \in \mathcal{S}(T'')$ (where T'' is the above mentioned tree) having eigenvalues

$$(-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, 2),$$

but there exists a matrix $A \in \mathcal{S}(T'')$ having eigenvalues

$$(-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5}).$$

In section 4, we also prove that the converse of Lemma 1.9 is not true.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

2. Prior Results. The key tool used to prove Proposition 1.4 and Theorem 1.5 was the decomposition of a real rational function into partial fractions. We recall here the following well known results, which will be useful for the present work.

LEMMA 2.1. [13] *Let $g(t)$ be a monic polynomial of degree n, $n > 1$, having only distinct real roots and $h(t)$ be a monic polynomial with deg $h(t) <$deg $g(t)$. Then $h(t)$ has $n-1$ distinct real roots strictly interlacing the roots of $g(t)$ if and only if the coefficients of the partial fraction decomposition (pfd) of $\frac{h(t)}{g(t)}$ are positive real numbers.*

REMARK 2.2. If $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$ are real numbers such that

$$\lambda_1 < \mu_1 < \lambda_2 < \ldots < \mu_{n-1} < \lambda_n$$

and, $g(t)$ and $h(t)$ are the monic polynomials

$$\begin{aligned} g(t) &= (t-\lambda_1)(t-\lambda_2)\ldots(t-\lambda_n), \\ h(t) &= (t-\mu_1)(t-\mu_2)\ldots(t-\mu_{n-1}), \end{aligned}$$

it is easy to show that $\frac{g(t)}{h(t)}$ can be represented in a unique way as

$$\frac{g(t)}{h(t)} = (t-a) - \sum_{i=1}^{n-1} \frac{x_i}{t-\mu_i}$$

in which $a = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i$ and x_i, $i = 1, \ldots, n-1$, are positive real numbers such that

$$x_i = -\frac{g(\mu_i)}{\prod_{j=1,j\neq i}^{n-1}(\mu_i - \mu_j)} = -\frac{\prod_{j=1}^{n}(\mu_i - \lambda_j)}{\prod_{j=1,j\neq i}^{n-1}(\mu_i - \mu_j)}.$$

If T is a tree on n vertices, $A \in \mathcal{S}(T)$ and T_i is a subgraph of T, we denote by $A[T_i]$ (respectively, $A(T_i)$) the principal submatrix of A obtained by deleting rows and columns that correspond to vertices of $T \setminus T_i$ (respectively, T_i). We will also need the expansion of the characteristic polynomial at a particular vertex of T with neighbors $x_1, \ldots, x_k$.

LEMMA 2.3. [9] *Let T be a tree on n vertices and let $A = [a_{ij}]$ be a matrix in $\mathcal{S}(T)$. If x_0 is a vertex of T of degree k, whose neighbors in T are $x_1, \ldots, x_k$, then*

$$p_A(t) = (t - a_{x_0x_0})p_{A[T-x_0]}(t) - \sum_{i=1}^{k} |a_{x_0x_i}|^2 p_{A[T_i - x_i]}(t) \prod_{j=1,j\neq i}^{k} p_{A[T_j]}(t), \tag{2.1}$$

with the convention that $p_{A[T_i-x_i]}(t) = 1$ whenever the vertex set of T_i is $\{x_i\}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

Since T is a tree, if A is a matrix in $\mathcal{S}(T)$ and x_0 is a vertex of degree k, we have $A(x_0) = A[T_1] \oplus \ldots \oplus A[T_k]$, where T_i is the branch of $T - x_0$ containing the neighbor x_i of x_0 in T. The following result was shown in [8].

LEMMA 2.4. [8] *Let T be a tree and A be a matrix in $\mathcal{S}(T)$. Let x_0 be a vertex of T and λ be an eigenvalue of $A(x_0)$. Let x_i be a neighbor of x_0 in T and T_i be the branch of T at x_0 containing x_i. If λ is an eigenvalue of $A[T_i]$ and*

$$m_{A[T_i - x_i]}(\lambda) = m_{A[T_i]}(\lambda) - 1,$$

then $m_{A(x_0)}(\lambda) = m_A(\lambda) + 1$.

When T is a generalized star with central vertex x_0, each branch T_i of T at x_0 is a path. Thus, if B is a matrix in $\mathcal{S}(T_i)$ then all the eigenvalues of B have multiplicity 1 and the eigenvalues of $B(x_i)$ strictly interlace those of B.

LEMMA 2.5. [8] *Let T be a generalized star with central vertex x_0. If A is a matrix in $\mathcal{S}(T)$ and λ is an eigenvalue of $A(x_0)$ then $m_{A(x_0)}(\lambda) = m_A(\lambda) + 1$.*

Let T be a tree and A be a matrix in $\mathcal{S}(T)$. A result concerning the multiplicity of the largest and smallest eigenvalues of A was proved in [9].

LEMMA 2.6. *If T is a tree, the largest and smallest eigenvalues of each matrix A in $\mathcal{S}(T)$, have multiplicity 1. Moreover, the largest or smallest eigenvalue of a matrix A in $\mathcal{S}(T)$ cannot occur as an eigenvalue of a submatrix $A(x_0)$, for any vertex x_0 in T.*

Using this result we can prove the following lemma.

LEMMA 2.7. *Let T be a tree on $n > 1$ vertices, x_0 be a vertex in T and A be a matrix in $\mathcal{S}(T)$. Then there exists an eigenvalue λ of $A(x_0)$ such that $m_A(\lambda) = m_{A(x_0)}(\lambda) - 1$.*

Proof. Suppose that for each eigenvalue λ of $A(x_0)$ we have $m_A(\lambda) \neq m_{A(x_0)}(\lambda) - 1$. Consequently, if λ is an eigenvalue of $A(x_0)$ then $m_A(\lambda) \geq m_{A(x_0)}(\lambda)$. Since $A(x_0)$ has $n-1$ eigenvalues and A has n eigenvalues, there exists at most one eigenvalue of A which is not an eigenvalue of $A(x_0)$. Using Lemma 2.6 we obtain a contradiction, and the result follows. ∎

Let T be a tree and A be a matrix in $\mathcal{S}(T)$. We have the following result, which can be obtained collectively from [14, 15]. Note that if A is a matrix in $\mathcal{S}(T)$, then $A(i)$ will be a direct sum of matrices, and we refer to the direct summands as blocks of $A(i)$.

THEOREM 2.8. *Let T be a tree, A be a matrix in $\mathcal{S}(T)$ and λ be an eigenvalue of A such that $m_A(\lambda) \geq 2$. Then there exists a vertex i in $G(A)$ such that*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

1) $m_{A(i)}(\lambda) = m_A(\lambda) + 1$;
2) λ *is an eigenvalue of at least three blocks of* $A(i)$.

3. General Inverse Eigenvalue Problem. The following theorem generalizes Theorem 1.5 and gives a partial answer of the GIEP for $\mathcal{S}(T)$ when some of the branches of T, at a fixed vertex x_0, are cut branches.

THEOREM 3.1. *Let T be a tree on n vertices and x_0 be a vertex of T of degree k whose neighbors are $x_1, \ldots, x_k$. Let T_i be the branch of T at x_0 containing x_i and s_i be the number of vertices in T_i, $i = 1, \ldots, k$.*

Let $T_1, \ldots, T_r$, with $0 \le r \le k$, be cut branches at $x_1, \ldots, x_r$, respectively, and $l_j = \min\{$number of vertices in each connected component of $T_j - x_j\}$, $j = 1, \ldots, r$.

Let $g_1(t), \ldots, g_k(t)$ be monic polynomials having only distinct real roots, with deg $g_i(t) = s_i$, $p_1, \ldots, p_s$ be the distinct roots among polynomials $g_i(t)$ and m_j be the multiplicity of root p_j in $\prod_{i=1}^k g_i(t)$.

Let $g(t)$ be a monic polynomial of degree $s+1$, $p_1, \ldots, p_l$ be the common roots of $g(t)$ and $\prod_{i=1}^s (t - p_i)$ and

$$\overline{g}(t) = \frac{g(t)}{\prod_{i=1}^l (t - p_i)}.$$

For each $1 \le j \le r$, let $q_{j1} < q_{j2} < \ldots < q_{jv_j}$, with $1 \le v_j \le l_j$, be roots of $g_j(t)$.

Let

$$\overline{g_j}(t) = \begin{cases} \frac{g_j(t)}{(t-q_{j1})\ldots(t-q_{jv_j})} & \text{if } 1 \le j \le r \\ g_j(t) & \text{if } r < j \le k. \end{cases}$$

and

$$m_{ij} = \begin{cases} 1 & \text{if } p_i \text{ is a root of } \overline{g_j}(t) \\ 0 & \text{otherwise.} \end{cases}$$

There exists a matrix A in $\mathcal{S}(T)$ with characteristic polynomial

$$f(t) = g(t) \prod_{i=1}^s (t - p_i)^{m_i - 1}$$

and such that

1) *$A[T_i]$ has characteristic polynomial $g_i(t)$, $i = 1, \ldots, k$,*
2) *for each $1 \le j \le r$, $q_{j1}, q_{j2}, \ldots, q_{jv_j}$ are the common eigenvalues of $A[T_j]$ and $A[T_j - x_j]$, and $m_{[T_j - x_j]}(q_{j1}) = \ldots = m_{[T_j - x_j]}(q_{jv_j}) = 2$,*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

3) *if* $r < i \leq k$ *and* $s_i > 1$, *the eigenvalues of* $A[T_i - x_i]$ *strictly interlace those of* $A[T_i]$,

if and only if

I) $s - l \geq 1$,
II) *the roots of* $\overline{g}(t)$ *strictly interlace those of* $\prod_{i=l+1}^{s}(t-p_i)$,
III) $p_1, \ldots, p_l$ *are not roots of* $\overline{g_j}(t)$ *and there exist positive real numbers* y_{ij}, $i \in \{l+1, \ldots, s\}$, $j \in \{1, \ldots, k\}$ *such that*
for each $i \in \{l+1, \ldots, s\}$

$$-\frac{\overline{g}(p_i)}{\prod_{j=l+1, j\neq i}^{s}(p_i - p_j)} = \sum_{j=1}^{k} m_{ij} y_{ij}, \tag{3.1}$$

and for each $j \in \{1, \ldots, r\}$, $u \in \{1, \ldots, v_j\}$

$$\sum_{i=l+1, m_{ij}=1}^{s} \frac{m_{ij} y_{ij}}{q_{ju} - p_i} = 0. \tag{3.2}$$

Proof. We start by proving the necessity of the stated conditions for the existence of the matrix A. Firstly notice that the characteristic polynomial of $A(x_0) = A[T_1] \oplus \ldots \oplus A[T_k]$ is $\prod_{i=1}^{k} g_i(t) = \prod_{i=1}^{s}(t-p_i)^{m_i}$. By hypothesis, $f(t) = g(t)\prod_{i=1}^{s}(t-p_i)^{m_i-1}$ is the characteristic polynomial of A. So, using Lemma 2.7 there exists $1 \leq i \leq s$ such that p_i is not a root of $g(t)$. Because $p_1, \ldots, p_l$ are roots of $g(t)$ then $s - l \geq 1$ and we have I).

As $\sum_{i=1}^{s} m_i = n-1$ then $n - s - 1 + l = \sum_{i=1}^{s}(m_i - 1) + l$ is the degree of the polynomial $\prod_{i=1}^{s}(t-p_i)^{m_i-1}\prod_{j=1}^{l}(t-p_j)$. Because $p_A(t) = f(t) = \overline{g}(t)\prod_{j=1}^{l}(t - p_j)\prod_{i=1}^{s}(t-p_i)^{m_i-1}$ is a polynomial of degree n then $\overline{g}(t)$ is a polynomial of degree $s-l+1$. By hypothesis, $p_1, \ldots, p_l$ are the common roots of $g(t)$ and $\prod_{i=1}^{s}(t-p_i)$ then $\overline{g}(t)$ must have $s-l+1$ real roots (because A is symmetric), each one different from each $p_{l+1}, \ldots, p_s$. By the interlacing theorem for eigenvalues of Hermitian matrices, the roots of $p_A(t)$ must interlace the roots of $p_{A(x_0)}(t) = \prod_{i=1}^{s}(t-p_i)^{m_i}$. Then the roots of $\overline{g}(t)$ must strictly interlace those of $\prod_{i=l+1}^{s}(t-p_i)$. So, we have II).

Now we are going to prove III). Let $j \in \{1, \ldots, k\}$. If $s_j > 1$, denote by $h_j(t)$ the characteristic polynomial of $A[T_j - x_j]$; if $s_j = 1$, define $h_j(t) = 1$. Let

$$\overline{h_j}(t) = \begin{cases} \frac{h_j(t)}{(t-q_{j1})\ldots(t-q_{jv_j})} & \text{if } 1 \leq j \leq r \\ h_j(t) & \text{if } r < j \leq k. \end{cases}$$

By the interlacing theorem for eigenvalues of Hermitian matrices, and by hypothesis 2), if $1 \leq j \leq r$ or by hypothesis 3), if $r < j \leq k$ and $s_j > 1$, then the roots of $\overline{g_j}(t)$ must strictly interlace those of $\overline{h_j}(t)$. By Lemma 2.1, if $s_j > 1$, or by construction, if

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

$s_j = 1$, there exist positive real numbers b_{ij}, with $1 \le i \le s$, $1 \le j \le k$, such that for each $j \in \{1, \ldots, k\}$,

$$\frac{h_j(t)}{g_j(t)} = \frac{\overline{h_j}(t)}{\overline{g_j}(t)} = \sum_{i=1}^{s} \frac{m_{ij} b_{ij}}{t - p_i}.$$

By hypothesis, if $1 \le i \le l$ then p_i is a root of $g(t)$. So, if $1 \le i \le l$ then $m_i = m_{A(x_0)}(p_i) \ne m_A(p_i) + 1$. Therefore, using Lemma 2.4, if $1 \le i \le l$ and p_i is a root of $A[T_j]$ then p_i is a root of $A[T_j - x_j]$. By hypothesis, it follows that $m_{A[T_j]}(p_i)$ is equal to 1 or to 0. Consequently, $p_1, \ldots, p_l$ are not roots of $\overline{g_j}(t)$ (if $r < j \le k$ and $s_j > 1$ we use hypothesis 3); if $1 \le j \le r$, we use hypothesis 2) and the fact that $g_j(t)$ has distinct real roots).

Therefore, there exist positive real numbers b_{ij}, with $l + 1 \le i \le s$, $1 \le j \le k$, such that for each $j \in \{1, \ldots, k\}$,

$$\frac{h_j(t)}{g_j(t)} = \frac{\overline{h_j}(t)}{\overline{g_j}(t)} = \sum_{i=l+1}^{s} \frac{m_{ij} b_{ij}}{t - p_i} \tag{3.3}$$

Since $A = [a_{ij}]$ is a matrix in $\mathcal{S}(T)$, we define for each $j \in \{1, \ldots, k\}$, $x_j = |a_{x_0 x_j}|^2$ and $a = a_{x_0 x_0}$. According to (2.1), the characteristic polynomial of A may be written as

$$p_A(t) = (t - a_{x_0 x_0}) \prod_{j=1}^{k} p_{A[T_j]}(t) - \sum_{j=1}^{k} |a_{x_0 x_j}|^2 p_{A[T_j - x_j]}(t) \prod_{u=1 u \ne j}^{k} p_{A[T_u]}(t)$$

$$= (t - a) \prod_{j=1}^{k} g_j(t) - \sum_{j=1}^{k} x_j \frac{p_{A[T_j - x_j]}(t)}{p_{A[T_j]}(t)} \prod_{u=1}^{k} g_u(t)$$

$$= ((t - a) - \sum_{j=1}^{k} x_j \sum_{i=l+1}^{s} \frac{m_{ij} b_{ij}}{t - p_i}) \prod_{i=1}^{s} (t - p_i)^{m_i}$$

$$= ((t - a) - \sum_{i=l+1}^{s} \frac{\sum_{j=1}^{k} x_j m_{ij} b_{ij}}{t - p_i}) \prod_{i=1}^{s} (t - p_i) \prod_{i=1}^{s} (t - p_i)^{m_i - 1}$$

Consequently,

$$g(t) = ((t - a) - \sum_{i=l+1}^{s} \frac{\sum_{j=1}^{k} x_j m_{ij} b_{ij}}{t - p_i}) \prod_{i=1}^{s} (t - p_i)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

and

$$\overline{g}(t)=\frac{g(t)}{\prod_{i=1}^{l}(t-p_i)}=(t-a)\prod_{i=l+1}^{s}(t-p_i)-\sum_{i=l+1}^{s}\sum_{j=1}^{k}(x_jm_{ij}b_{ij})\prod_{u=l+1,u\neq i}^{s}(t-p_u)$$

So, for each $i\in\{l+1,\ldots,s\}$, $\overline{g}(p_i)=-\sum_{j=1}^{k}(x_jm_{ij}b_{ij})\prod_{u=l+1,u\neq i}^{s}(p_i-p_u)$. Then

$$-\frac{\overline{g}(p_i)}{\prod_{u=l+1,u\neq i}^{s}(p_i-p_u)}=\sum_{j=1}^{k}(x_jm_{ij}b_{ij}).$$

For each $i\in\{l+1,\ldots,s\}$ and $j\in\{1,\ldots,k\}$, we denote x_jb_{ij} by y_{ij}. Since $b_{ij},\ x_j$ are positive real numbers, then y_{ij} is a positive real number and we have III)(3.1).

Let $j\in\{1,\ldots,r\}$. Since $q_{j1},\ldots,q_{jv_j}$ are the common eigenvalues of $A[T_j]$ and $A[T_j-x_j]$, and $m_{A[T_j-x_j]}(q_{j1})=\ldots=m_{A[T_j-x_j]}(q_{jv_j})=2$, by hypothesis 2), we have that $q_{j1},\ldots,q_{jv_j}$ are roots of $\overline{h_j}(t)$ but not of $\overline{g_j}(t)$. Thus, if p_i is a root of $\overline{g_j}(t)$ then $p_i\notin\{q_{j1},\ldots,q_{jv_j}\}$. Therefore, by (3.3), for each $u\in\{1,\ldots,v_j\}$,

$$0=\frac{x_j\overline{h_j}(q_{ju})}{\overline{g_j}(q_{ju})}=\sum_{i=l+1,\ m_{ij}=1}^{s}\frac{m_{ij}(x_jb_{ij})}{q_{ju}-p_i}=\sum_{i=l+1,\ m_{ij}=1}^{s}\frac{m_{ij}y_{ij}}{q_{ju}-p_i}.$$

So, we have III)(3.2).

Next, we prove the sufficiency of the stated conditions. Because of the strict interlacing between the roots of $\overline{g}(t)$ and those of $\prod_{i=l+1}^{s}(t-p_i)$ (hypothesis II) and because $\overline{g}(t)$ is a polynomial of degree $s+1-l>1$, due to Remark 2.2, we conclude the existence of a real number a and positive real numbers $x_{l+1},\ldots,x_s$ such that

$$\frac{\overline{g}(t)}{\prod_{i=l+1}^{s}(t-p_i)}=(t-a)-\sum_{i=l+1}^{s}\frac{x_i}{(t-p_i)}$$

i.e.,

$$\overline{g}(t)=\left((t-a)-\sum_{i=l+1}^{s}\frac{x_i}{(t-p_i)}\right)\prod_{i=l+1}^{s}(t-p_i)$$

$$=(t-a)\prod_{i=l+1}^{s}(t-p_i)-\sum_{i=l+1}^{s}x_i\prod_{j=l+1,j\neq i}^{s}(t-p_j).$$

Therefore, if $l+1\le i\le s$,

$$\overline{g}(p_i)=-x_i\prod_{j=l+1,j\neq i}^{s}(p_i-p_j)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

and

$$x_i = -\frac{\overline{g}(p_i)}{\prod_{j=l+1 j\neq i}^{s}(p_i - p_j)}.$$

Using hypothesis III)(3.1), there exist real positive numbers y_{ij}, with $l+1 \leq i \leq s$, $1 \leq j \leq k$ such that $x_i = \sum_{j=1}^{k} m_{ij} y_{ij}$. So,

$$\overline{g}(t) = \left((t-a) - \sum_{i=l+1}^{s} \frac{\sum_{j=1}^{k} m_{ij} y_{ij}}{(t-p_i)}\right) \prod_{i=l+1}^{s} (t-p_i)$$

$$= \left((t-a) - (\sum_{i=l+1}^{s} \frac{m_{i1} y_{i1}}{(t-p_i)} + \ldots + \sum_{i=l+1}^{s} \frac{m_{ik} y_{ik}}{(t-p_i)})\right) \prod_{i=l+1}^{s} (t-p_i).$$

By hypothesis III) then $\overline{g_j}(t) = \prod_{i=l+1}^{s}(t-p_i)^{m_{ij}}$. Notice that, when deg $\overline{g_j}(t) > 1$, $\sum_{i=l+1}^{s} \frac{m_{ij} y_{ij}}{(t-p_i)}$ is a pfd of $\frac{\overline{h_j}(t)}{\overline{g_j}(t)}$ for some polynomial $\overline{h_j}(t)$. By Lemma 2.1, the coefficients of this pfd are positive which means that deg $\overline{h_j}(t)$ =deg $\overline{g_j}(t) - 1$ and $\overline{h_j}(t)$ has only real roots strict interlacing those of $\overline{g_j}(t)$. If deg $\overline{g_j}(t) = 1$, $\sum_{i=l+1}^{s} \frac{m_{ij} y_{ij}}{(t-p_i)} = \frac{m_{uj} y_{uj}}{\overline{g_j}(t)}$, $m_{uj} y_{uj} > 0$, for some $u \in \{l+1, \ldots, s\}$. In this case, for convenience we denote $m_{uj} y_{uj}$ by $\overline{h_j}(t)$.

If $1 \leq j \leq r$, since $q_{j1}, \ldots, q_{jv_j}$ are not roots of $\overline{g_j}(t)$, using III)(3.2) we have that $q_{j1}, \ldots, q_{jv_j}$ are roots of $\overline{h_j}(t)$ but not of $\overline{g_j}(t)$. Remark that the leading coefficient of $\overline{h_j}(t)$ is the positive real number $\sum_{i=l+1}^{s} m_{ij} y_{ij}$. Let $h_j(t)$ be the monic polynomial such that $\overline{h_j}(t)(t-q_{j1}) \ldots (t-q_{ju_j}) = (\sum_{i=l+1}^{s} m_{ij} y_{ij}) h_j(t)$. So,

$$\overline{g}(t) = \left((t-a) - \sum_{j=1}^{k} (\sum_{i=l+1}^{s} m_{ij} y_{ij}) \frac{h_j(t)}{g_j(t)}\right) \prod_{i=l+1}^{s} (t-p_i).$$

By hypothesis, if $1 \leq j \leq r$ then T_j is a cut branch at x_i. Since $g_j(t) = \overline{g_j}(t) \prod_{i=1}^{v_j}(t-q_{ji})$, the roots of $\overline{h_j}(t)$ strictly interlace those of $\overline{g_j}(t)$ and $q_{j1}, \ldots, q_{jv_j}$ are roots of $\overline{h_j}(t)$, then by Theorem 1.5, there exists a matrix A_j in $\mathcal{S}(T_j)$ such that $p_{A_j}(t) = g_j(t)$ and $p_{A_j[T_j - x_j]}(t) = h_j(t)$. So, we have 2).

If $r+1 \leq j \leq k$, by Proposition 1.4, there exists a matrix A_j in $\mathcal{S}(T_j)$ such that $p_{A_j}(t) = g_j(t)$ and $p_{A_j[T_j - x_j]}(t) = h_j(t)$ (recall the convention that $p_{A_j[T_j - x_j]}(t) = 1$ whenever the vertex set of T_j is $\{x_j\}$). Therefore, we have 3) and 1).

Now define a matrix $A = [a_{ij}]$ in $\mathcal{S}(T)$ in the following way:

- $a_{x_0 x_0} = a$
- $a_{x_0 x_j} = a_{x_j x_0} = \sqrt{\sum_{i=l+1}^{s} m_{ij} y_{ij}}$, for $j = 1, \ldots, k$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

- $A[T_j] = A_j$, for $j = 1, \ldots, k$
- the remaining entries of A are 0.

According to (2.1), the characteristic polynomial of A may be written as

$$(t - a_{x_0x_0})p_{A[T-x_0]}(t) - \sum_{j=1}^{k} |a_{x_ox_j}|^2 p_{A[T_j-x_j]}(t) \prod_{i=1, i\neq j}^{k} p_{A[T_i]}(t).$$

Note that $A[T-x_0] = A[T_1] \oplus \ldots \oplus A[T_k]$ which implies $p_{A[T-x_0]}(t) = \prod_{j=1}^{k} p_{A[T_j]}(t)$. Moreover the characteristic polynomial of $A[T_j]$ is $g_j(t)$ and the characteristic polynomial of $A[T_j - x_j]$ is $h_j(t)$. Consequently, by hypothesis, $\prod_{j=1}^{k} g_j(t) = \prod_{i=1}^{s}(t-p_i)^{m_i}$ and

$$p_A(t) = (t-a)\prod_{i=1}^{s}(t-p_i)^{m_i} - \sum_{j=1}^{k}(\sum_{i=l+1}^{s} m_{ij}y_{ij})\frac{h_j(t)}{g_j(t)}\prod_{i=1}^{s}(t-p_i)^{m_i}$$

$$= \overline{g}(t)\prod_{i=1}^{l}(t-p_i)\prod_{i=1}^{s}(t-p_i)^{m_i-1} = f(t)$$

i.e., $p_A(t) = f(t)$. ☐

Under the conditions and following the notation of Theorem 3.1, if T is a superstar and T_j is a cut branch at x_j of T, then T_j is a path. So, if q_{ji} is a common eigenvalue of $A[T_j]$ and $A[T_j - x_j]$ then $m_{A[T_j-x_j]}(q_{ji}) = 2$. Therefore, if we suppose that T is a superstar then condition 2) of Theorem 3.1 is shorter and we have the following result.

THEOREM 3.2. *Let T be a superstar on n vertices and x_0 be a central vertex of T of degree k whose neighbors are $x_1, \ldots, x_k$. Let T_i be the branch of T at x_0 containing x_i and s_i be the number of vertices in T_i, $i = 1, \ldots, k$.*

Let $T_1, \ldots, T_r$, with $0 \leq r \leq k$, be cut branches at $x_1, \ldots, x_r$, respectively, and $l_j = \min\{$number of vertices in each connected component of $T_j - x_j\}$, $j = 1, \ldots, r$.

Let $g_1(t), \ldots, g_k(t)$ be monic polynomials having only distinct real roots, with deg $g_i(t) = s_i$, $p_1, \ldots, p_s$ be the distinct roots among polynomials $g_i(t)$ and m_j be the multiplicity of root p_j in $\prod_{i=1}^{k} g_i(t)$.

Let $g(t)$ be a monic polynomial of degree $s+1$, $p_1, \ldots, p_l$ be the common roots of $g(t)$ and $\prod_{i=1}^{s}(t-p_i)$ and

$$\overline{g}(t) = \frac{g(t)}{\prod_{i=1}^{l}(t-p_i)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

For each $1 \leq j \leq r$, let $q_{j1} < q_{j2} < \ldots < q_{jv_j}$, with $1 \leq v_j \leq l_j$, be roots of $g_j(t)$. Let

$$\overline{g_j}(t) = \begin{cases} \frac{g_j(t)}{(t-q_{j1})\ldots(t-q_{jv_j})} & \text{if } 1 \leq j \leq r \\ g_j(t) & \text{if } r < j \leq k. \end{cases}$$

and

$$m_{ij} = \begin{cases} 1 & \text{if } p_i \text{ is a root of } \overline{g_j}(t) \\ 0 & \text{otherwise.} \end{cases}$$

There exists a matrix A in $\mathcal{S}(T)$ with characteristic polynomial $f(t) = g(t)\prod_{i=1}^{s}(t-p_i)^{m_i-1}$ and such that

1) *$A[T_i]$ has characteristic polynomial $g_i(t)$, $i = 1, \ldots, k$,*
2) *for each $1 \leq j \leq r$, $q_{j1}, q_{j2}, \ldots, q_{jv_j}$ are the common eigenvalues of $A[T_j]$ and $A[T_j - x_j]$,*
3) *if $r < i \leq k$ and $s_i > 1$, the eigenvalues of $A[T_i - x_i]$ strictly interlace those of $A[T_i]$,*

if and only if

I) *$s - l \geq 1$,*

II) *the roots of $\overline{g}(t)$ strictly interlace those of $\prod_{i=l+1}^{s}(t - p_i)$,*

III) *$p_1, \ldots, p_l$ are not roots of $\overline{g_j}(t)$ and there exist positive real numbers y_{ij}, $i \in \{l+1, \ldots, s\}$, $j \in \{1, \ldots, k\}$ such that for each $i \in \{l+1, \ldots, s\}$*

$$-\frac{\overline{g}(p_i)}{\prod_{j=l+1, j\neq i}^{s}(p_i - p_j)} = \sum_{j=1}^{k} m_{ij} y_{ij},$$

and for each $j \in \{1, \ldots, r\}$, $u \in \{1, \ldots, v_j\}$

$$\sum_{i=l+1, m_{ij}=1}^{s} \frac{m_{ij} y_{ij}}{q_{ju} - p_i} = 0.$$

Remark 3.3. Under the conditions and following the notation of Theorem 3.1,

1) if we suppose that there are not common roots of the polynomials $g(t)$ and $\prod_{i=1}^{k} g_i(t)$, conditions 2), I) and III) of Theorem 3.1 may be omitted and we obtain Theorem 1.5
2) if T is a generalized star and T_j is a branch of T then $T_j - x_j$ is a path or an empty set. Then, conditions 2), 3), I) and III) of Theorem 3.1 may be omitted and we obtain Theorem 1.6.

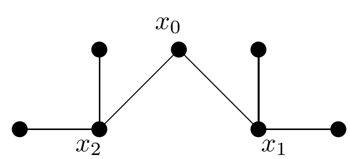
Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

Remark 3.4. Under the mentioned hypothesis and conditions 1), 2), 3) of Theorem 3.1, we have condition III) of Theorem 3.1 which implies that $p_1, \ldots, p_l$ are not roots of $\prod_{j=1}^{k} \overline{g_j}(t)$. Because $p_1, \ldots, p_l$ are roots of $\prod_{j=1}^{k} g_j(t)$ we must have

$$\{p_1, \ldots, p_l\} \subseteq \bigcup_{j=1}^{r} \{q_{j1}, \ldots, q_{jv_j}\}.$$

Example 3.5.

1) *Let T be the superstar*



Let $g_1(t) = t(t-1)(t-3)$, $g_2(t) = (t+2)t(t-3)$. Then $p_1 = 1,\ p_2 = -2,\ p_3 = 0,\ p_4 = 3$ are the distinct roots among polynomials $g_1(t)$ and $g_2(t)$.
Consider the root $p_1 = 1$ of $g_1(t)$ and the root $p_3 = 0$ of $g_2(t)$.
Let $g(t) = (t+3)(t+1)(t-1)(t-\frac{6}{13})(t-4)$.
Since $p_1 = 1$ is a common root of $g(t)$ and $\prod_{i=1}^{4}(t-p_i)$, let
$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$,
$\overline{g_1}(t) = t(t-3)$ *and*
$\overline{g_2}(t) = (t+2)(t-3)$.
Because

I) $s - l = 4 - 1 \geq 1$,

II) *the roots of*

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$$

strictly interlace those of

$$\prod_{i=2}^{4}(t-p_i) = (t+2)t(t-3),$$

III) *the positive real numbers $y_{22} = \frac{96}{65}$, $y_{31} = \frac{12}{13}$, $y_{41} = \frac{24}{13}$ and $y_{42} = \frac{144}{65}$ verify*
$-\frac{\overline{g}(-2)}{(-2)\times(-5)} = y_{22}$,
$-\frac{\overline{g}(0)}{(2)\times(-3)} = y_{31}$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

$-\frac{\overline{g}(3)}{(5)\times(3)} = y_{41} + y_{42}$,
$\frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} = 0$,
$\frac{y_{22}}{0+2} + \frac{y_{42}}{0-3} = 0$,
using Theorem 3.1, *there exists a matrix A in $\mathcal{S}(T)$ such that the characteristic polynomial of A is $f(t) = g(t)t(t-3)$ and, $A[T_1]$ has characteristic polynomial $g_1(t)$, $A[T_2]$ has characteristic polynomial $g_2(t)$, $p_1 = 1$ is the unique common eigenvalue of $A[T_1]$ and $A[T_1 - x_1]$, $p_3 = 0$ is the unique common eigenvalue of $A[T_2]$ and $A[T_2 - x_2]$. Moreover, following the proof of Theorem* 3.1,

$$A = \begin{bmatrix} -\frac{7}{13} & \sqrt{\frac{36}{13}} & 0 & 0 & \sqrt{\frac{48}{13}} & 0 & 0 \\ \sqrt{\frac{36}{13}} & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \sqrt{\frac{48}{13}} & 0 & 0 & 0 & 1 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}$$

2) *Let T be the superstar of the* Example 3.5, 1).
Let $g_1(t) = t(t-1)(t-3)$, $g_2(t) = (t+2)t(t-3)$.
Consider the root $p_1 = 1$ of $g_1(t)$ and let $g(t) = (t+3)(t+1)(t-1)(t-\frac{6}{13})(t-4)$.
Since $p_1 = 1$ is a common root of $g(t)$ and $\prod_{i=1}^{4}(t-p_i)$, let
$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$,
$\overline{g_1}(t) = t(t-3)$ *and*
$\overline{g_2}(t) = (t+2)t(t-3)$.
In this case, because
I) $s - l = 4 - 1 \geq 1$,
II) *the roots of*

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$$

strictly interlace those of

$$\prod_{i=2}^{4}(t-p_i) = (t+2)t(t-3),$$

III) *the positive real numbers $y_{22} = \frac{96}{65}$, $y_{31} = \frac{6}{13}$, $y_{32} = \frac{6}{13}$, $y_{41} = \frac{12}{13}$ and $y_{42} = \frac{204}{65}$ verify*
$-\frac{\overline{g}(-2)}{(-2)\times(-5)} = y_{22}$,
$-\frac{\overline{g}(0)}{(2)\times(-3)} = y_{31} + y_{32}$,
$-\frac{\overline{g}(3)}{(5)\times(3)} = y_{41} + y_{42}$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

$\frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} = 0,$
using Theorem 3.1*, there exists a matrix A in $\mathcal{S}(T)$ such that the characteristic polynomial of A is $f(t) = g(t)t(t-3)$ and, $A[T_1]$ has characteristic polynomial $g_1(t)$, $A[T_2]$ has characteristic polynomial $g_2(t)$, $p_1 = 1$ is the unique common eigenvalue of $A[T_1]$ and $A[T_1 - x_1]$. Moreover, following the proof of Theorem* 3.1*,*

$$A = \begin{bmatrix} -\frac{7}{13} & \sqrt{\frac{18}{13}} & 0 & 0 & \sqrt{\frac{66}{13}} & 0 & 0 \\ \sqrt{\frac{18}{13}} & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \sqrt{\frac{66}{13}} & 0 & 0 & 0 & \frac{14}{11} & \sqrt{\frac{643}{22} + \frac{\sqrt{273}}{1001}} & \sqrt{\frac{643}{22} - \frac{\sqrt{273}}{1001}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{643}{22} + \frac{\sqrt{273}}{1001}} & \frac{-3-\sqrt{273}}{22} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{643}{22} - \frac{\sqrt{273}}{1001}} & 0 & \frac{-3+\sqrt{273}}{22} \end{bmatrix}$$

Remark 3.6.

1) In Example 3.5, 1), we consider that the set of common roots of $g(t)$ and $g_1(t)g_2(t)$ is a subset of the set of fixed roots of $g_1(t)$ and $g_2(t)$. This is, $\{p_1, \dots, p_l\} \subset \bigcup_{j=1}^{r} \{q_{j1}, \dots, q_{jv_j}\}$.
   In 2), we consider that the set of common roots of $g(t)$ and $g_1(t)g_2(t)$ is the set of fixed roots of $g_1(t)$ and $g_2(t)$. Thus, $\{p_1, \dots, p_l\} = \bigcup_{j=1}^{r} \{q_{j1}, \dots, q_{jv_j}\}$. Following the proof of Theorem 3.1, we obtain different matrices or solution for the GIEP.
2) In Example 3.5, 1), the positive real numbers $y_{22} = \frac{96}{65}$, $y_{31} = \frac{12}{13}$, $y_{41} = \frac{24}{13}$ and $y_{42} = \frac{144}{65}$ verify conditions III)(3.1) and (3.2) of Theorem 3.1. If we consider the positive real numbers $y_{22} = \frac{96}{65}$, $y_{31} = \frac{12}{13}$, $y_{41} = \frac{132}{65}$ and $y_{42} = \frac{132}{65}$, then these integers verify condition III)(3.1) but do not verify condition III)(3.2), this is,
   $\frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} \neq 0$, and $\frac{y_{22}}{0+2} + \frac{y_{42}}{0-3} \neq 0$.
   Consequently, it is possible to find positive real numbers that verify condition III)(3.1) but do not satisfy condition III)(3.2).

4. Equivalence of ordered multiplicity lists and IEP. As we have said in section 1 (Introduction), using the mentioned tree T', Barioli and Fallat gave the first example for which the equivalence between the problem of ordered multiplicity lists and the IEP does not occur, [1]. Using the mentioned tree T'', a simpler example based on the same technique was given in [10].

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

Using Theorem 2.8 we can see that if

$$(1,2,2,3,3,3,3,4,4,5,5,5,6,6,7)$$

is the sequence of eigenvalues of a matrix $A' \in \mathcal{S}(T')$ then

$$2,3,4,5,6$$

are the eigenvalues of $A'[T_1]$ and of $A'[T_2]$, where T_1 and T_2 are the branches of $T'-x_0$ with 5 vertices and

$$2,3,4,6$$

are the eigenvalues of $A'[T_3]$, where T_3 is the branch of $T'-x_0$ with 4 vertices.

Thus,

i) $g_1(t)=(t-2)(t-3)(t-4)(t-5)(t-6)=g_2(t)$ and $g_3(t)=(t-2)(t-3)(t-4)(t-6)$,
ii) $3,\ 5$ are roots of $g_1(t)$ and of $g_2(t)$,
iii) 3 is a root of $g_3(t)$,
iv) $g(t)=(t-1)(t-3)^2(t-5)^2(t-7)$.

Consequently, $\overline{g}(t)=(t-1)(t-3)(t-5)(t-7)$. Using Theorem 3.1, it is possible to find positive real numbers y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33} such that

$$\begin{array}{l}
-\frac{\overline{g}(2)}{(2-4)\times(2-6)}=\frac{15}{8}=y_{11}+y_{12}+y_{13},\\
-\frac{\overline{g}(4)}{(4-2)\times(4-6)}=\frac{9}{4}=y_{21}+y_{22}+y_{23},\\
-\frac{\overline{g}(6)}{(6-2)\times(6-4)}=\frac{15}{8}=y_{31}+y_{32}+y_{33},\\
\frac{y_{11}}{3-2}+\frac{y_{21}}{3-4}+\frac{y_{31}}{3-6}=y_{11}-y_{21}-\frac{y_{31}}{3}=0,\\
\frac{y_{12}}{3-2}+\frac{y_{22}}{3-4}+\frac{y_{32}}{3-6}=y_{12}-y_{22}-\frac{y_{32}}{3}=0,\\
\frac{y_{13}}{3-2}+\frac{y_{23}}{3-4}+\frac{y_{33}}{3-6}=y_{13}-y_{23}-\frac{y_{33}}{3}=0,\\
\frac{y_{11}}{5-2}+\frac{y_{21}}{5-4}+\frac{y_{31}}{5-6}=\frac{y_{11}}{3}+y_{21}-y_{31}=0,\\
\frac{y_{12}}{5-2}+\frac{y_{22}}{5-4}+\frac{y_{32}}{5-6}=\frac{y_{12}}{3}+y_{22}-y_{32}=0.
\end{array}$$

Using the first six equalities we obtain a contradiction. Notice that there are positive real numbers that satisfy the first three equalities. Therefore, there is no matrix $A' \in \mathcal{S}(T')$ having eigenvalues $(1,2,2,3,3,3,3,4,4,5,5,5,6,6,7)$.

Remark 4.1. Using the sequence $(1,2,2,3,3,3,3,4,4,5,5,5,6,6,7)$, consider the matrices

$$A_1=\begin{bmatrix} 4 & 1\\ 1 & 4\end{bmatrix}$$

and

$$A_2=\begin{bmatrix} 3 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

Considering the mentioned tree T', let x_i be the vertex of T_i adjacent to x_0, $i \in \{1,2,3\}$. Let $\alpha_1^i,\ \alpha_2^i$ be the paths of $T_i - x_i$, $i \in \{1,2,3\}$, where α_2^3 is the shortest path of $T_3 - x_3$.

It is easy to prove that this sequence and the matrices

$$A[\alpha_1^1] = A[\alpha_2^1] = A[\alpha_1^2] = A[\alpha_2^2] = A[\alpha_1^3] = A_1$$

and

$$A[\alpha_2^3] = A_2$$

verify conditions (a), (b), (c), (d) of Lemma 1.9.

Thus, the converse of Lemma 1.9 is not true.

Using Theorem 3.1 it is possible to prove that

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$$

is the sequence of eigenvalues of a matrix $A \in \mathcal{S}(T')$.

Consider

i) $g_1(t) = (t-2)(t-3)(t-4)(t-5)(t-6) = g_2(t)$ and $g_3(t) = (t-2)(t-3)(t-4)(t-6)$,
ii) 3, 5 roots of $g_1(t)$ and of $g_2(t)$,
iii) 3 a root of $g_3(t)$,
iv) $g(t) = t(t-3)^2(t-5)^2(t-7)$.

It is easy to find positive real numbers that satisfy condition III) of Theorem 3.1. By this theorem, there exists a matrix $A \in \mathcal{S}(T')$ having eigenvalues

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7).$$

In the same way, we can prove the mentioned result for T'' (see Introduction).

Using this technique, it is possible to find many superstars where the problem of ordered multiplicity lists and the IEP are not equivalent.

Acknowledgment. The author thanks the referee for bringing the reference [12] to her attention and helpful comments that greatly improved the presentation.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 442-461, August 2009

ELA

REFERENCES

[1] F. Barioli and S.M. Fallat. On the eigenvalues of generalized and double generalized stars. *Linear and Multilinear Algebra*, 53:269–291, 2005.

[2] F. Barioli and S.M. Fallat. On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices. *Electronic Journal of Linear Algebra*, 11:41–50, 2004.

[3] D. Boley and G.H. Golub. A survey of matrix inverse eigenvalue problem. *Inverse Problems*, 3:595–622, 1987.

[4] M. Chu. Inverse eigenvalue problems. *Journal of Society for Industrial and Applied Mathematics*, 40:1–39, 1998.

[5] W. Ferguson. The construction of Jacobi and periodic Jacobi matrices with prescribed spectra. *Mathematics of Computation*, 35:1203–1220, 1980.

[6] R. Fernandes and C.M. da Fonseca. The inverse eigenvalue problem for Hermitian matrices whose graph is a cycle. *Linear and Multilinear Algebra*, to appear.

[7] C.R. Johnson and A. Leal Duarte. On the possible multiplicities of the eigenvalues of an Hermitian matrix whose graph is a tree. *Linear Algebra and its Applications*, 348:7–21, 2002.

[8] C.R. Johnson, A. Leal Duarte, and C.M. Saiago. Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars. *Linear Algebra and its Applications*, 373:311–330, 2003.

[9] C.R. Johnson, A. Leal Duarte, and C.M. Saiago. The Parter-Wiener theorem refinement and generalization. *SIAM Journal on Matrix Analysis and Applications*, 25:352–361, 2003.

[10] C.R. Johnson, A. Leal Duarte, and C.M. Saiago. Multiplicity Lists for the eigenvalues of symmetric matrices with a given graph, *In Handbook of Linear Algebra*. Chapman and Hall/CRC Press, Boca Raton, 2007.

[11] R.A. Horn and C.R. Johnson. *Matrix Analysis* Cambridge University Press, New York, 1985.

[12] I.-J. Kim and B.L. Shader. Smith normal form and acyclic matrices. *Journal on Algebraic Combinatorics*, 29:63–80, 2009.

[13] A. Leal Duarte. Construction of acyclic matrices from spectral data. *Linear Algebra and its Applications*, 113:173–182, 1989.

[14] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *Journal of Society for Industrial and Applied Mathematics*, 8:376–388, 1960.

[15] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra and its Applications*, 61:15–29, 1984.