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ON THE GERŠGORIN DISKS OF DISTANCE MATRICES OF GRAPHS*

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Abstract. For a simple connected graph G, let $D(G)$, $Tr(G)$, $D^L(G) = Tr(G) - D(G)$, and $D^Q(G) = Tr(G) + D(G)$ be the distance matrix, the diagonal matrix of the vertex transmissions, the distance Laplacian matrix, and the distance signless Laplacian matrix of G, respectively. Atik and Panigrahi [2] suggested the study of the problem: *Whether all eigenvalues, except the spectral radius, of $D(G)$ and $D^Q(G)$ lie in the smallest Geršgorin disk?* In this paper, we provide a negative answer by constructing an infinite family of counterexamples.

Key words. Distance matrix, Distance Laplacian, Distance signless Laplacian, Eigenvalues inequalities, Geršgorin disks.

AMS subject classifications. 05C50, 05C12, 15A18

1. Introduction. In this article, all graphs are connected, simple, and undirected. A graph is denoted by $G(V,E)$ (or simply by G), where V and E are its vertex and edge set. The cardinality of V is the *order* n of G and the cardinality of E is the *size* m of G. We use standard terminology, by K_n and P_n we denote the complete graph and the path graph on n vertices. For notations and definitions not given here, we refer the readers to [9].

For two vertices u and v in a graph G, the *distance* between u and v, denoted by $d(u,v)$, is the length (number of edges) of a shortest path between them. The *distance matrix* $D(G)$ of a connected graph G, is defined as $D(G) = (d(u,v))_{u,v\in V(G)}$. The matrix $D(G)$ is real symmetric and its eigenvalues are real and denoted $\rho_1, \rho_2, \ldots, \rho_n$ such that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. See [3] for a survey of results on distance spectra of graphs.

The *transmission* (or *transmission degree*) of a vertex v, denoted by $Tr_G(v)$, is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum\limits_{u\in V(G)} d(u,v)$. We note that $Tr_G(v)$ is also the v-th row (column) sum of the matrix $D(G)$. The *minimum* and *maximum transmissions* in G are $T_{min} = \min\limits_{v\in V} Tr_G(v)$ and $T_{max} = \max\limits_{v\in V} Tr_G(v)$, respectively.

Let $Tr(G)$ be the diagonal matrix of row sums of $D(G)$. The *distance Laplacian matrix* is denoted by $D^L(G)$ and is defined as $D^L(G) = Tr(G) - D(G)$ (see [4, 5]). It immediately follows that $D^L(G)$ is a real symmetric and positive semi-definite matrix. We denote its eigenvalues $\rho_1^L, \rho_2^L, \ldots, \rho_n^L$ such that $\rho_1^L \geq \rho_2^L \geq \cdots \geq \rho_n^L$.

Similarly, the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian* matrix of G (see [4, 6]). The distance signless Laplacian matrix is positive definite, we take its eigenvalues as $\rho_1^Q, \rho_2^Q, \ldots, \rho_n^Q$ such that $\rho_1^Q \geq \rho_2^Q \geq \cdots \geq \rho_n^Q$. Both the matrices $D(G)$ and $D^Q(G)$ are *irreducible*, so by *Perron–Frobenius* theorem, ρ_1 and ρ_1^Q are unique and are known as the distance spectral radius and distance signless Laplacian spectral radius of G, respectively.

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Let $\mathbb{M}_n$ denote the set of all square complex matrices of order n. The following well-known result, commonly referred to as the *Geršgorin disks theorem*, provides a relationship between the eigenvalues of a (general) matrix and its diagonal entries.

THEOREM 1.1 ([10]). *If $M = (m_{ij})_n \in \mathbb{M}_n$, let $R_i(M) = \sum_{i \neq j} |m_{ij}|$, $i = 1, 2, \dots, n$ and consider the n Geršgorin disks:*

$$\Big\{z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M)\Big\},\ i = 1, 2, \dots, n.$$

Then the eigenvalues of M are in the union of Geršgorin disks

$$\mathcal{G}(M) = \bigcup_{i=1}^{n} \Big\{z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M)\Big\}.$$

Furthermore, if the union of k of the n disks that comprise $\mathcal{G}(M)$ forms a set $\mathcal{G}_k(M)$ that is disjoint from the remaining $n-k$ disks, then $\mathcal{G}_k(M)$ contains exactly k eigenvalues of M, including their algebraic multiplicities.

Marsli and Hall [11] gave an interesting result stating that if M has the eigenvalue λ with algebraic multiplicity k, then λ lies in at least k of the n Geršgorin disks. Bárány and Solymosi [7] showed that if M is a nonnegative real matrix and λ is an eigenvalue of M with geometric multiplicity at least k, then it is in a smaller disk. More literature about the Geršgorin disk theorem can be found in [10, 15] and the references therein.

Atik and Panigrahi [2] studied the eigenvalues of $D(G)$ and $D^Q(G)$ and their relation with the Geršgorin disks. They observed that the eigenvalues of the matrices $D(G)$ and $D^Q(G)$ for regular graphs, except their spectral radii, are contained in the smallest Geršgorin disk. Atik and Mondal [1] constructed infinite families of graphs satisfying that condition.

For a nonnegative matrix $M \in \mathbb{M}_n$, we define a property $\mathcal{P}$ as:

$\mathcal{P}$: *All eigenvalues of the matrix M, except the spectral radius, lie inside the smallest Geršgorin disk of M.*

It was proven in [2] that, under some sufficient conditions, $\mathcal{P}$ holds for the signless Laplacian $D^Q(G)$. Then, Atik and Panigrahi [2] suggested the study of the following problems.

PROBLEM 1. *Whether property $\mathcal{P}$ holds for the distance matrix $D(G)$ of an arbitrary graph G?*

PROBLEM 2. *Whether property $\mathcal{P}$ holds for the distance signless Laplacian matrix $D^Q(G)$ of an arbitrary graph G?*

In [1], Atik and Mondal provided results supporting a "positive" answer to this problem. In the present study, we provide a negative answer to Problem 2 by constructing an infinite family of graphs for which property $\mathcal{P}$ does not hold for the distance signless Laplacian D^Q.

Before proceeding further, we recall the following well-known result that will be used in our proofs.

THEOREM 1.2 (Interlacing Theorem, [10]). *Let $M \in \mathbb{M}_n$ be a real symmetric matrix. Let A be a principal submatrix of M of order m, $(m \leq n)$. Then the eigenvalues of M and A satisfy the following inequalities:*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \qquad \textit{with}\ \ 1 \leq i \leq m.$$

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The rest of this article is organised as follows. In Section 2, we will find the D^Q eigenvalues of a special class of graphs and show that at least two D^Q eigenvalues of such graphs lie outside the smallest Geršgorin disk. In Section 3, we give a brief discussion of the smallest Geršgorin disk and the distance (Laplacian) matrix of graphs.

2. Geršgorin disks and the distance signless Laplacian. In this section, we consider Problem 2. We are going to construct an infinite family of graphs providing a negative answer to the problem.

The distance signless Laplacian matrix $D^Q(G)$ is real symmetric and positive definite with diagonal entries are the vertices transmissions. The corresponding Geršgorin disks are the intervals $[0, 2Tr_G(v)]$ for $v \in V$. Thus, by Theorem 1.1, the eigenvalues of $D^Q(G)$ belong to the interval $[0, 2T_{max}]$.

We need the following definition. For integers ω, l, n, with $\omega + l = n$, let $PK_{\omega,l}$ be the graph obtained from the complete graph K_ω and the path P_l by adding an edge between any vertex of K_ω and a pendent vertex of P_l. $PK_{\omega,l}$ is known as *kite graph* [8, 13, 14]. It is also a particular case of *path-complete graphs* [12]. An example of a graph $PK_{\omega,l}$, with $\omega = l = 7$ is shown in Figure 1. Also, it is clear that $PK_{\omega,0} \cong K_n$ and $PK_{0,l} \cong P_n$.

For the graph $PK_{8,7}$, the distance signless Laplacian matrix is given by:

$$D^Q(PK_{8,7}) = \begin{pmatrix} 35 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 42 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 42 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 42 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 42 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 42 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 42 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 42 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 36 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 39 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 1 & 44 & 1 & 2 & 3 & 4 \\ 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 3 & 2 & 1 & 51 & 1 & 2 & 3 \\ 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 4 & 3 & 2 & 1 & 60 & 1 & 2 \\ 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 5 & 4 & 3 & 2 & 1 & 71 & 1 \\ 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 6 & 5 & 4 & 3 & 2 & 1 & 84 \end{pmatrix}.$$

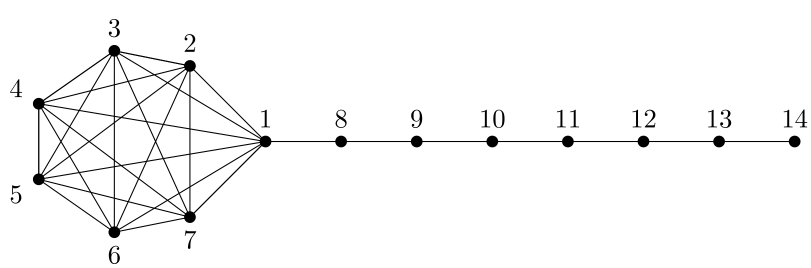

FIGURE 1. *Graph* $PK_{7,7}$.

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Clearly, the minimum transmission of the above matrix is 35, so its smallest Geršgorin disk of $D^Q(PK_{8,7})$ is the circle (interval) with center at 35 and passing through the origin: $[0, 70]$. The approximate eigenvalues (precision 10^{-4}) of $D^Q(PK_{8,7})$ are 108.3722, 75.6213, 62.2837, 51.7205, 43.5872, $41^{(6)}$, 37.8609, 34.9826, 33.8082, and 19.7637, where the superscript (6) represents the algebraic multiplicity of the eigenvalue 41. Thus, it follows that 75.6213 lies outside the smallest Geršgorin disk. Therefore, property $\mathcal{P}$ does not hold for $D^Q(PK_{8,7})$.

This example is generalized below for infinitely many values of ω and l. First, we prove the following lemma.

LEMMA 1. *Consider the graph $PK_{\omega,l}$, with $\omega \geq l$ and $n = \omega + l \geq 3$. Label the vertices set of $PK_{\omega,l}$ such that v_1 has the maximum degree, $v_2, \ldots, v_\omega$ the other vertices in the clique, and $v_{\omega+1}, \ldots, v_n$ are the vertices on the path from the neighbor of v_1 to the pendent vertex, respectively. Following this labeling, let $T_1, T_2, \ldots, T_n$ be the transmission sequence in $PK_{\omega,l}$. Then, (a) $T_{\omega+i+1} > T_{\omega+i}$ for $i = 1, 2, \ldots, l-1$; (b) the minimum transmission in $PK_{\omega,l}$ is T_1 (also $T_{\omega+1}$ if $\omega = l$); (c) the maximum transmission is T_n.*

Proof. Following the labeling defined in the statement, we have

$$T_1 = \underbrace{1 + 1 + \cdots + 1}_{\omega-1} + 1 + 2 + 3 + \cdots + l - 1 + l = \omega - 1 + \frac{l(l+1)}{2}; \tag{2.1}$$

for $j = 2, 3, \ldots, \omega$,

$$T_j = \underbrace{1 + 1 + \cdots + 1}_{\omega-1} + 2 + 3 + \cdots + l + l + 1 = \omega - 1 + l + \frac{l(l+1)}{2}; \tag{2.2}$$

and for $i = 1, 2, \ldots, l$,

$$T_{\omega+i} = (\omega - 1)(i + 1) + \sum_{k=1}^{i} k + \sum_{k=1}^{l-i} k \tag{2.3}$$

(a) From (2.3) and for $i = 1, 2, \ldots l - 1$, we have

$$\begin{aligned} T_{\omega+1+i} - T_{\omega+i} &= \left((\omega-1)(i+2) + \sum_{k=1}^{i+1} k + \sum_{k=1}^{l-i-1} k\right) - \left((\omega-1)(i+1) + \sum_{k=1}^{i} k + \sum_{k=1}^{l-i} k\right) \\ &= (\omega - 1) + (i + 1) - (l - 1) \\ &= \omega - l + 2i \ > \ 0. \end{aligned}$$

Therefore, $T_{\omega+1+i} > T_{\omega+i}$, for $i = 1, 2, \ldots, l - 1$.

(b) For $j = 2, 3, \ldots, \omega$, from (2.1) and (2.2), we have

$$T_j - T_1 = \left(\omega - 1 + l + \frac{l(l+1)}{2}\right) - \left(\omega - 1 + \frac{l(l+1)}{2}\right) = l > 0.$$

In addition, from (2.1) and (2.3), we get

$$T_{\omega+1} - T_1 = \left(2(\omega - 1) + 1 + \sum_{k=1}^{l-1} k\right) - \left(\omega - 1 + \frac{l(l+1)}{2}\right) = \omega - l \geq 0.$$

Therefore, combining with (a), we get $T_{min} = T_1$ if $\omega > l$, and $T_{min} = T_1 = T_{\omega+1}$ if $\omega = l$.

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(c) From (a), $T_n > T_{\omega+i}$ for $i = 1, 2, \ldots, l-1$, and from (b), $T_n > T_1$.
From (2.2) and (2.3), for $j = 2, 3, \ldots, \omega$, we have

$$\begin{aligned} T_n - T_j = T_{\omega+l} - T_2 &= \left((\omega-1)(l+1) + \frac{l(l+1)}{2}\right) - \left(\omega - 1 + l + \frac{l(l+1)}{2}\right) \\ &= (\omega-1)l + l \ > \ 0. \end{aligned}$$

Therefore, $T_{max} = T_n$. ∎

THEOREM 2.1. *Let ω and l be integers with $\omega \geq l$ and $n = \omega + l$. Then at least two eigenvalues of $D^Q(PK_{\omega,l})$ lie outside the smallest Geršgorin disk whenever one of the following conditions hold:*

1. *$l = 3$ and $\omega \geq 8$;*
2. *$4 \leq l \leq 7$ and $\omega \geq 7$;*
3. *$l \geq 8$ and $\omega \geq 8$.*

Proof. Labeling the vertices of the graph $PK_{\omega,l}$ as in Lemma 1, the distance signless Laplacian matrix of $PK_{\omega,l}$ is

$$D^Q(PK_{\omega,l}) = \begin{pmatrix} T_1 & 1 & 1 & \ldots & 1 & 1 & 2 & \ldots & l-1 & l \\ 1 & T_2 & 1 & \ldots & 1 & 2 & 3 & \ldots & l & l+1 \\ 1 & 1 & T_3 & \ldots & 1 & 2 & 3 & \ldots & l & l+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & T_\omega & 2 & 3 & \ldots & l & l+1 \\ 1 & 2 & 2 & \ldots & 2 & T_{\omega+1} & 1 & \ldots & l-2 & l-1 \\ 2 & 3 & 3 & \ldots & 3 & 1 & T_{\omega+2} & \ldots & l-3 & l-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l-1 & l & l & \ldots & l & l-2 & l-3 & \ldots & T_{n-1} & 1 \\ l & l+1 & l+1 & \ldots & l+1 & l-1 & l-2 & \ldots & 1 & T_n \end{pmatrix}.$$

Combining equalities (2.1) and (2.3), we get

$$T_{n-1} = T_{\omega+l-1} = \omega l - l + \frac{(l-1)l}{2} + 1 = (\omega-2)(l-1) + T_1,$$

and

$$T_n = T_{\omega+l} = (\omega-1)(l+1) + \frac{l(l+1)}{2} = (\omega-1)l + T_1.$$

The 2×2 principal submatrix of $D^Q(PK_{\omega,l})$ corresponding to last two rows (columns) is

$$\begin{pmatrix} T_{n-1} & 1 \\ 1 & T_n \end{pmatrix} = \begin{pmatrix} (\omega-2)(l-1) + T_1 & 1 \\ 1 & (\omega-1)l + T_1 \end{pmatrix}.$$

The eigenvalues of the above matrix are

$$\lambda_{1,2} = T_1 + \frac{2l\omega - 3l - \omega + 2 \pm \sqrt{(\omega+l-2)^2 + 4}}{2}.$$

Therefore, using Theorem 1.2, we have

$$\rho_2^Q \geq T_1 + \frac{2l\omega - 3l - \omega + 2 - \sqrt{(\omega+l-2)^2 + 4}}{2}.$$

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So, to get ρ_2^Q outside of the smallest Geršgorin disk, it suffices to show that

$$\frac{2l\omega - 3l - \omega + 2 - \sqrt{(\omega+l-2)^2+4}}{2} > T_1.$$

or equivalently

$$\frac{2l\omega - 3l - \omega + 2 - \sqrt{(\omega+l-2)^2+4}}{2} > \omega - 1 + \frac{l(l+1)}{2},$$

which is also equivalent to

$$2l\omega - 3l - 3\omega - l(l+1) + 4 - \sqrt{(\omega+l-2)^2+4} > 0.$$

It suffices to get (since $\sqrt{(\omega+l-2)^2+4} < \omega + l - 1$)

$$2l\omega - 3l - 3\omega - l(l+1) + 4 - (\omega + l - 1) > 0,$$

or

$$2l\omega - 4l - 4\omega - l(l+1) + 5 > 0,$$

or

$$l(\omega-4) + \omega(l-4) - l(l+1) + 5 > 0. \tag{2.4}$$

Let $f(\omega, l) = l(\omega-4) + \omega(l-4) - l(l+1) + 5$, and consider different cases.

1. We have $f(w,3) = 2\omega - 19 > 0$, for $\omega \geq 10$.
2. We have $f(w,4) = 4\omega - 31 > 0$, $f(w,5) = 6\omega - 45 > 0$, $f(w,6) = 8\omega - 61 > 0$, $f(w,7) = 10\omega - 79 > 0$, for $\omega \geq 8$.
3. We have $f(w,8) = 12\omega - 99 > 0$, for $\omega \geq 9$. For $l \geq 9$, using the condition $\omega \geq l$, we get $f(w,l) \geq f(l,l) = l^2 - 9l + 5 > 0$, for $l \geq 9$.

For the small values, direct calculations using software give the following table:

(ω, l)	$\rho_2^Q(PK_{\omega,l})$	Smallest Geršgorin interval
$(8,3)$	27.0623	$[0,26]$
$(9,3)$	30.3170	$[0,28]$
$(7,4)$	33.3615	$[0,32]$
$(7,5)$	43.9782	$[0,42]$
$(7,6)$	55.6219	$[0,52]$
$(7,7)$	68.2830	$[0,68]$
$(8,8)$	90.3136	$[0,86]$
$(9,8)$	98.6554	$[0,88]$

$\rho_2^Q(PK_{\omega,l})$ for small values of ω and l.

Thus, in all these cases, property $\mathcal{P}$ does not hold for $D^Q(PK_{\omega,l})$. □

In view of the above results, Problem 2 can be restated as follows.

Problem 2′ *Characterize the graphs whose all distance signless Laplacian eigenvalues, except the spectral radius, lie in the smallest Geršgorin disk?*

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3. Geršgorin disks and the distance Laplacian matrix. In this section, we extend Problem 2 and Problem 1 to the case of the distance Laplacian matrix:

PROBLEM 3. *Whether property $\mathcal{P}$ holds for the distance Laplacian matrix $D^L(G)$ of an arbitrary graph G?*

We are going to construct an infinite family of graphs providing a negative answer to the problem.

The distance Laplacian matrix $D^L(G)$ is real symmetric and positive semi-definite with diagonal entries are the vertices transmissions. The corresponding Geršgorin disks are the intervals $[0, 2Tr_G(v)]$ for $v \in V$. Thus, by Theorem 1.1, the eigenvalues of $D^L(G)$ belong to the interval $[0, 2T_{max}]$. Note that $D^L(G)$ and $D^Q(G)$ have the same Geršgorin disks, despite the fact that their respective spectra are different.

THEOREM 3.1. *Let ω and l be integers with $\omega \geq l$ and $n = \omega + l$. Then at least two eigenvalues of $D^L(PK_{\omega,l})$ lie outside the smallest Geršgorin disk whenever one of the following conditions hold*

1. $l = 3$ *and* $\omega \geq 10$;
2. $4 \leq l \leq 7$ *and* $\omega \geq 8$;
3. $l \geq 8$ *and* $\omega \geq 9$.

Proof. Labeling the vertices of the graph $PK_{\omega,l}$ as in Lemma 1, the distance signless Laplacian matrix of $PK_{\omega,l}$ is

$$D^L(PK_{\omega,l}) = \begin{pmatrix} T_1 & -1 & -1 & \dots & -1 & -1 & -2 & \dots & -l+1 & -l \\ -1 & T_2 & -1 & \dots & -1 & -2 & -3 & \dots & -l & -l-1 \\ -1 & -1 & T_3 & \dots & -1 & -2 & -3 & \dots & -l & -l-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & T_\omega & -2 & -3 & \dots & -l & -l-1 \\ -1 & -2 & -2 & \dots & -2 & T_{\omega+1} & -1 & \dots & -l+2 & -l+1 \\ -2 & -3 & -3 & \dots & -3 & -1 & T_{\omega+2} & \dots & -l+3 & -l+2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -l+1 & -l & -l & \dots & -l & -l+2 & -l+3 & \dots & T_{n-1} & -1 \\ -l & -l-1 & -l-1 & \dots & -l-1 & -l+1 & -l+2 & \dots & -1 & T_n \end{pmatrix}.$$

Combining equalities (2.1) and (2.3), we get

$$T_{n-1} = T_{\omega+l-1} = \omega l - l + \frac{(l-1)l}{2} + 1 = (\omega-2)(l-1) + T_1,$$

and

$$T_n = T_{\omega+l} = (\omega-1)(l+1) + \frac{l(l+1)}{2} = (\omega-1)l + T_1.$$

The 2×2 principal submatrix of $D^Q(PK_{\omega,l})$ corresponding to last two rows (columns) is

$$\begin{pmatrix} T_{n-1} & -1 \\ -1 & T_n \end{pmatrix} = \begin{pmatrix} (\omega-2)(l-1) + T_1 & -1 \\ -1 & (\omega-1)l + T_1 \end{pmatrix}.$$

The eigenvalues of the above matrix are

$$\lambda_{1,2} = T_1 + \frac{2l\omega - 3l - \omega + 2 \pm \sqrt{(\omega+l-2)^2 + 4}}{2},$$

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which are exactly the same as in the proof of Theorem 2.1. Therefore, the rest of the proof is exactly like that of Theorem 2.1. □

In view of the above results, Problem 3 can be restated as follows.

Problem 3′ *Characterize the graphs whose all distance Laplacian eigenvalues, except the spectral radius, lie in the smallest Geršgorin disk?*

4. Geršgorin disks and the distance matrix. In this section, we consider Problem 1. We are going to construct an infinite family of graphs providing a negative answer to the problem.

Since $D(G)$ is real symmetric matrix with diagonal entries zero, its corresponding Geršgorin disks are concentric circles with center at origin and radii equals the transmissions of vertices of G. Precisely, they are the intervals $[-T_G(v), T_G(v)]$ for $v \in V$. Thus, by Theorem 1.1, all the eigenvalues of $D(G)$ are contained in the interval $[-T_{max}, T_{max}]$.

The distance eigenvalues of $PK_{14,7}$ are $\{66.0144, -41.3566, -6.5884, -2.2326, -1.2834, -1^{(12)}, -0.8382, -0.6551, -0.5498, -0.5098\}$, and its minimum transmission is 41. Therefore, the smallest distance eigenvalue of G does not lie in the smallest Geršgorin interval $[-41, 41]$. So, Property $\mathcal{P}$ does not hold for $D(PK_{14,7})$, and therefore, we have a negative answer to Problem 1.

By using computer calculations, we have verified that for $PK_{\omega,l}$ with $\omega \geq 2q,\ l = q$ for some values of $q \geq 7$, the smallest distance eigenvalue lies outside the smallest Geršgorin disk. Based on those computational experimentations, we state the following conjecture.

Conjecture 1. *Let ω and l be integers with $\omega \geq 2l$, $l \leq 7$, and $n = \omega + l$. The smallest distance eigenvalue of $PK_{\omega,l}$ satisfies $\rho_n(PK_{\omega,l}) < -Tr_{\min}$, where $Tr_{\min}$ is minimum transmission of $PK_{\omega,l}$*

In view of the above computational results (and conjecture), Problem 1 can be restated as follows.

Problem 1′ *Characterize the graphs whose all distance eigenvalues, except the spectral radius, lie in the smallest Geršgorin disk?*

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