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SHORT PROOFS OF THEOREMS OF MIRSKY AND HORN ON DIAGONALS AND EIGENVALUES OF MATRICES*

ERIC A. CARLEN† AND ELLIOTT H. LIEB‡

Abstract. A theorem of Mirsky provides necessary and sufficient conditions for the existence of an N-square complex matrix with prescribed diagonal entries and prescribed eigenvalues. A simple inductive proof of this theorem is given.

Key words. Majorization, Eigenvalues, Prescribed diagonals.

AMS subject classifications. 15A42, 15A51.

Theorems of Alfred Horn [3] and Leon Mirsky [4] give necessary and sufficient conditions for the existence of an N-square matrix A that has prescribed diagonal elements and prescribed eigenvalues. In the case of Horn's Theorem, A is required to be Hermitian, and in the case of Mirsky's theorem no constraint is imposed on A. While simple inductive proofs of Horn's Theorem have been published, the only published proof of Mirsky's Theorem that we are aware of is the original, which is neither simple nor transparent. In this note, we give a simple proof of Mirsky's Theorem which we relate to Chan and Li's proof [2] of Horn's Theorem. We begin by recalling the results in [3, 4] in more detail.

Let $\Lambda = (\lambda_1, \ldots, \lambda_N)$ and $D = (d_1, \ldots, d_N)$ be two sequences of N real numbers. Recall that Λ majorizes D (we write $\Lambda \succ D$) if for each $k = 1, \ldots, N$, the sum of the k largest λ's is at least as large as the sum of the k largest d's, with equality for $k = N$. If Λ and D are the eigenvalue and diagonal sequences, respectively, of an N-square Hermitian matrix A, then $A = U[\Lambda]U^*$ for some unitary U, where $[\Lambda]$ denotes the diagonal matrix with diagonal sequence Λ. Thus,

$$d_i = \sum_{j=1}^N |U_{ij}|^2 \lambda_j \qquad \text{for} \quad 1 \le i \le N \ . \tag{1}$$

These N equations lead directly to the necessary condition $\Lambda \succ D$ found by Schur

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†Department of Mathematics, Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019. Work partially supported by U.S. National Science Foundation grant DMS 06-00037.

‡ Departments of Mathematics and Physics, Jadwin Hall, Princeton University, P. O. Box 708, Princeton, NJ 08542. Work partially supported by U.S. National Science Foundation grant PHY 06-52854.

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in 1923 [5]. Horn's Theorem [3] says that this necessary condition is also sufficient, and moreover, U can be taken to be real orthogonal. Thus, while Birkhoff showed [1] that $\Lambda \succ D$ if and only if there exists a bi-stochastic matrix S such that $D = S\Lambda$, Horn's Theorem states that S can always be chosen to lie in the smaller set of *ortho-stochastic* matrices.

Mirsky [4] gave another proof of Horn's Theorem, and also proved the following: Given two complex N-sequences Λ and D, there exists an N-square matrix whose eigenvalue sequence is Λ and whose diagonal sequence is D if and only if

$$\sum_{j=1}^{N} \lambda_j = \sum_{j=1}^{N} d_j \ , \tag{2}$$

and that if Λ and D are both real, then the matrix can be taken to be real as well.

We now give a simple inductive proof of Mirsky's Theorem. Let $[T_\Lambda]$ denote the N-square matrix that has Λ as its diagonal sequence, and has 1 in every entry immediately above the diagonal, and 0 in all remaining entries. A *unit lower triangular* matrix, is a square matrix in which every entry on the diagonal is 1, and every entry above the diagonal is 0. The unit lower triangular matrices are a group under matrix multiplication. The following result includes Mirsky's Theorem and a little more.

Theorem 1. *There exists a matrix A with eigenvalue sequence Λ and diagonal sequence D if and only if condition* (2) *is satisfied. Indeed, under this condition, there exists a unit lower triangular matrix L such that the diagonal sequence of $L^{-1}[T_\Lambda]L$ is D. If Λ and D are both real, L can be taken to be real.*

Proof. The necessity of (2) for the existence of A is obvious, and we must prove that (2) is sufficient for the existence of L. For $N = 2$, direct computation shows that, with $L = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, the diagonal sequence of $L^{-1}[T_\Lambda]L$ is $(\lambda_1 + c, \lambda_2 - c)$. Choosing $c = d_1 - \lambda_1$ yields the assertion for $N = 2$.

Assume that the assertion is proved for K-square matrices of size up through $K = N - 1$. Given N-sequences Λ and D satisfying (2), let L be the 2-square unit lower triangular matrix such that $L^{-1}[T_{(\lambda_1,\lambda_2)}]L$ has the diagonal sequence $(d_1, \widetilde{\lambda}_2)$, where

$$\widetilde{\lambda}_2 = \lambda_1 + \lambda_2 - d_1 \ . \tag{3}$$

Let $\widetilde{L}$ be the N-square matrix obtained by replacing the upper left 2-square block of $I_{N\times N}$ by L. Then $\widetilde{L}^{-1}[T_\Lambda]\widetilde{L}$ has the following properties: Its upper left 2-square block has diagonal sequence $(d_1, \widetilde{\lambda}_2)$, its lower right $(N-2)$-square block is the same as that of $[T_\Lambda]$, and $\left[\widetilde{L}^{-1}[T_\Lambda]\widetilde{L}\right]_{2,3} = 1$.

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The lower right $(N-1)$-square block of $\widetilde{L}^{-1}[T_\Lambda]\widetilde{L}$ is $[T_{\widetilde{\Lambda}}]$ where $\widetilde{\Lambda}$ is the $(N-1)$-sequence $(\widetilde{\lambda}_2, \lambda_3, \ldots, \lambda_N)$. By the inductive hypothesis, there exists a unit lower triangular matrix M such that $M^{-1}[T_{\widetilde{\Lambda}}]M$ has the diagonal sequence $(d_2, \ldots, d_N)$. Now let $\widetilde{M}$ be the N-square matrix obtained by replacing the lower right $(N-1)$ square block of $I_{N\times N}$ with M. Then $(\widetilde{L}\widetilde{M})^{-1}[T_\Lambda](\widetilde{L}\widetilde{M})$ has the diagonal sequence D, and $\widetilde{L}\widetilde{M}$ is unit lower triangular. □

Our proof of Theorem 1 is related to a proof of Horn's Theorem by Chan and Li [2]. For the reader's convenience, we briefly recall their proof in our notation, and then conclude with a brief comparison of the proofs.

Proof of Horn's Theorem, after Chan and Li: Assume that Λ and D are in decreasing order. If $N = 2$, then $\Lambda \succ D$ implies $\lambda_1 \geq d_1 \geq d_2 \geq \lambda_2$. For $\lambda_1 > \lambda_2$,

$$U = (\lambda_1 - \lambda_2)^{-1/2} \begin{bmatrix} (d_1 - \lambda_2)^{1/2} & -(\lambda_1 - d_1)^{1/2} \\ (\lambda_1 - d_1)^{1/2} & (d_1 - \lambda_2)^{1/2} \end{bmatrix}$$

is a real orthogonal matrix, and the diagonal sequence of $U^*[\Lambda]U$ is $D = (d_1, d_2)$. For $\lambda_1 = \lambda_2$, $\Lambda = D$, and we may take $U = I$. This proves Horn's Theorem for $N = 2$.

We proceed inductively: Let Λ and D be two real N-sequences with $\Lambda \succ D$. There is a K with $1 \leq K < N$ such that $\lambda_K \geq d_K \geq d_{K+1} \geq \lambda_{K+1}$: Take K to be the smallest $j < N$ such that $d_{j+1} \geq \lambda_{j+1}$. Note that $(\lambda_K, \lambda_{K+1}) \succ (d_K, \lambda'_{K+1})$, where

$$\lambda'_{K+1} = \lambda_{K+1} + \lambda_K - d_K \ . \tag{4}$$

Our argument in the case $N = 2$ shows that there exists a 2-square real orthogonal matrix U such that $U^*[(\lambda_K, \lambda_{K+1})]U$ has (d_K, λ'_{K+1}) as its diagonal sequence.

We use U to construct an N-square real orthogonal matrix T as follows: Start with the identity matrix $I_{N\times N}$, and replace the 2-square diagonal block at positions $K, K+1$ by the 2-square matrix U. Then $T^*[\Lambda]T$ has the diagonal sequence Λ', which is obtained from Λ by replacing λ_K by d_K and λ_{K+1} by λ'_{K+1} in Λ. We note the important fact that $\Lambda' \succ D$.

What follows is especially simple if $K = 1$, and the reader may wish to consider that case first. Let Λ'' and D'' be the sequences of $N-1$ real numbers obtained by deleting d_K, the common Kth term in both Λ' and D. Then $\Lambda'' \succ D''$, and hence, by induction, there exists an $(N-1)$-square orthogonal matrix V such that $V[\Lambda'']V^*$ has the diagonal sequence D''. It is convenient to index the entries of V using $\{\ldots, K-1, K+1, \ldots\}$ leaving out the "deleted" index K. Promote V to an N-square orthogonal matrix by setting $V_{K,K} = 1$ and $V_{j,K} = V_{K,j} = 0$ for $j \neq K$. Then $(VT)[\Lambda](VT)^*$ has D as its diagonal sequence. □

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While Chan and Li's proof of Horn's Theorem and our proof of Mirsky's Theorem have a similar structure, there is a significant difference: The matrices whose existence is guaranteed by Horn's Theorem are all necessarily similar to the diagonal matrix $[\Lambda]$, so that $[\Lambda]$ provides a suitable starting point for the proof. This is not the case for Mirsky's Theorem: Indeed, if all entries of Λ are equal, $[\Lambda]$ is a multiple of the identity. Thus, the set of all matrices similar to $[\Lambda]$ contains only $[\Lambda]$ itself. Except in the case $D = \Lambda$, it does not contain any matrices with diagonal sequence D. However, our proof shows that in the set of all matrices similar to $[T_\Lambda]$, there is always one with diagonal sequence D provided that (2) is satisfied.

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